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Ancestor ideals of vector spaces of forms, and level algebras

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Abstract

Let $R = k[x_1, \dots, x_r]$ denote the polynomial ring in r variables over a field k, with maximal ideal $M = (x_1, \dots, x_r)$, and let $V \subset R_j$ denote a vector subspace of the space R_j of degree-j homogeneous elements of R. We study three related algebras determined by V. The first is the *ancestor algebra* $\operatorname{Anc}(V) = R/\overline{V}$ whose defining *ancestor ideal* $\overline{V}$ is the largest graded ideal of R such that $\overline{V} \cap M^j = (V)$, the ideal generated by V. The second is the level algebra $\operatorname{LA}(V) = R/L(V)$ whose defining ideal $L(V)$, is the largest graded ideal of R such that the degree-j component $L(V) \cap R_j$ is V; and third is the algebra $R/(V)$. We have that $L(V) = \overline{V} + M^{j+1}$. When $r = 2$ we determine the possible Hilbert functions H for each of these algebras, and as well the dimension of each Hilbert function stratum. We characterize the graded Betti numbers of these algebras in terms of certain partitions depending only on H, and give the codimension of each stratum in terms of invariants of the partitions. We show that when $r = 2$ and k is algebraically closed the Hilbert function strata for each of the three algebras attached to V satisfy a frontier property that the closure of a stratum is the union of more special strata. In each case the family $G(H)$ of all graded ideals of the given Hilbert function is a natural desingularization of this closure. We then solve a refinement of the simultaneous Waring problem for sets of degree-j binary forms. Key tools throughout include properties of an invariant $\tau(V)$, the number of generators of $\overline{V} \subset k[x_1, x_2]$, and previous results concerning the projective variety $G(H)$ in [Mem. Amer. Math. Soc., Vol. 10 (188), 1977].

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1. Introduction

In Section 1.1 we first define what we term the *ancestor ideal* $\overline{V}$ and *ancestor algebra* $\mathrm{Anc}(V)$ and also the *level algebra* $\mathrm{LA}(V)$ of a vector space $V \subset R_j$ of degree-j forms in the polynomial ring $R = k[x_1, \ldots, x_r]$ in r variables over a field k. We then show some initial results about the three algebras $\mathrm{Anc}(V), \mathrm{LA}(V)$ and $R/(V)$ determined by V. In Section 1.2 we state our main results about these three algebras for $r = 2$, and we give context in the literature. In Section 1.3 we show some general results about the Hilbert function strata of ancestor ideals. In Section 2 we show our main results about the three algebras of V for $r = 2$ variables. In Section 2.1 we determine the dimensions of the Hilbert function strata (Theorem 2.17); in Section 2.2 we express the codimensions of these strata in terms of partitions given by the graded Betti numbers of the three algebras attached to V (Theorem 2.24); and in Section 2.3 we determine the Zariski closure of each Hilbert function stratum when k is algebraically closed. We show that the strata for each of the three algebras satisfy the frontier property, that the closure is a union of more special strata in a natural partial order (Theorem 2.32). In Section 3.1 we study a refinement of the simultaneous Waring problem for vector spaces of degree-j forms when $r = 2$. In Section 3.2 we develop a concept of *related* vector spaces of forms, then we state some open problems.

1.1. Three algebras attached to the vector space $V \subset R_j$

We let k be an arbitrary field, and we denote by $R = k[x_1, \ldots, x_r]$ the polynomial ring over k, with maximal ideal $M = (x_1, \ldots, x_r)$, and the standard grading. For an integer $j \geqslant 0$ we denote by R_j the vector space of degree-j homogeneous elements of R. Let $j > 0$ and suppose that $V \subset R_j$ is a vector subspace of the space of degree-j homogeneous forms of R_j. We denote by (V) the ideal generated by V, and by $\overline{V}$ the largest ideal of R such that $\overline{V} \cap M^j = (V)$ (see Definition 1.1). For a form $f \in R_j$ and an integer $i \geqslant 0$ we denote by $R_i \cdot f$ the vector space

$$R_i f = \langle hf \mid h \in R_i \rangle \subset R_{i+j}.$$

For a vector space $V \subset R_j$ and an integer $i \geqslant 0$ we denote by $R_i V$ the vector space span

$$R_i V = \{hf \mid h \in R_i, f \in V\}. \tag{1.1}$$

For $0 \leqslant i \leqslant j$ we denote by $R_{-i} V$ the vector space satisfying

$$R_{-i} V = \{f \in R_{j-i} \mid f \cdot R_i \subset V\}. \tag{1.2}$$

We now define the three algebras determined by V that we study.

Definition 1.1. Let $V \subset R_j$ be a vector space of forms. The *level ideal* $L(V)$ determined by V is

$$L(V) = M^{j+1} \oplus V \oplus R_{-1} V \oplus \cdots \oplus R_{-j} V, \tag{1.3}$$

and the *level algebra* determined by V is $\mathrm{LA}(V) = R/L(V)$. The *ancestor ideal* $\overline{V}$ of V is the ideal

$$\overline{V} = (V) \oplus R_{-1}V \oplus \cdots \oplus R_{-j}V, \tag{1.4}$$

and the *ancestor algebra* determined by V is $\mathrm{Anc}(V) = R/\overline{V}$. The usual ideal determined by V is $(V) \subset R_j$, and we denote by $\mathrm{GA}(V) = R/(V)$ the graded algebra quotient.

Recall that the *s*ocle of an Artinian algebra $A = R/I$ is

$$\mathrm{Soc}(A) = (0 : M)_A = \langle f \in A \mid M \cdot f = 0\rangle.$$

The *type* of A is the vector space dimension $\dim_k(\mathrm{Soc}(A))$ of the socle.

Remark 1.2. The ancestor ideal $\overline{V}$ is the largest graded ideal of R such that $\overline{V} \cap M^j = (V)$, the ideal of R generated by V. The level ideal $L(V)$ is the largest graded ideal of R such that $L(V) \cap R_j = V$: it satisfies $L(V) = \overline{V} + M^{j+1}$; and the socle of the level algebra $\mathrm{LA}(V) = R/L(V)$ satisfies $\mathrm{Soc}(\mathrm{LA}(V)) \cong R_j/V$. The ideal (V) satisfies $(V) = \overline{V} \cap M^j$. Note, the maximality statements for the ancestor ideal $\overline{V}$ and for the level ideal $L(V)$ may appear similar, but they are quite different. The two ideals are equal only when $R_1 \cdot V = R_{j+1}$.

Proof of Remark. For $i > 0$, $R_{-i}V \subset R_{i-j}$ is the largest subset of R_{i-j} satisfying $R_i(R_{-i}V) \subset V$; and evidently $\overline{V}$ of Definition 1.1 is the largest graded ideal such that $\overline{V} \cap M^j = (V)$, the ideal generated by V. The other statements are also immediate from the relevant definitions. □

Lemma 1.3. *There are exact sequences*

$$\begin{aligned} &0 \to \overline{V}/(V) \to R/(V) \to R/\overline{V} \to 0, \text{ and} \\ &0 \to M^j/(V) \to R/\overline{V} \to R/L(R_{-1}V) \to 0. \end{aligned} \tag{1.5}$$

Proof. Immediate from the definitions. □

Example 1.4 (see [Mac1, Section 60ff], [IK, Lemma 2.14]). When the codimension of V as a vector subspace of R_j is one, then $\mathrm{LA}(V) = R/L(V)$ is a graded Artinian Gorenstein algebra, and all standard graded Artinian Gorenstein algebras quotients of R having socle degree j arise in this way. When $V = \langle xy^2 + yx^2, x^3, y^3\rangle \subset R = k[x, y]$ then $L(V) = (x^2 + xy + y^2, x^3)$ and $\mathrm{LA}(V)$ is a complete intersection of Hilbert function $H(A) = (1, 2, 2, 1)$. Here, as usual in the Gorenstein Artinian case, $\overline{V} = L(V)$; the exception is when $V = (m_p) \cap R_j$ for the maximal ideal of a point $p \in \mathbb{P}^{r-1}$, then $\overline{V} = m_p$.

Example 1.5. Let $I_{\mathfrak{Z}}$ be the defining ideal of a closed subscheme $\mathfrak{Z} \subset \mathbb{P}^{r-1}$, and let $V = I_{\mathfrak{Z}} \cap R_j$. Then $\overline{V} \subset I_{\mathfrak{Z}}$. If also $j \geqslant \sigma(Z)$, the regularity degree of $\mathfrak{Z}$, then $\overline{V} = I_{\mathfrak{Z}}$.

Recall that the saturation $\operatorname{Sat}(I)$ of a graded ideal $I \subset R$ is the ideal

$$\operatorname{Sat}(I) = I : M^{\infty} = \{f \mid \exists i \text{ with } R_i f \subset I\}. \tag{1.6}$$

Denote by $\sigma(V)$ the Castelnuovo–Mumford regularity degree of the projective scheme $\mathfrak{Z}_V = \operatorname{Proj}(R/(V)) \subset \mathbb{P}^{r-1}$. In case $(V) \supset M^{\sigma}$ but $(V) \not\supseteq M^{\sigma-1}$, when $\mathfrak{Z}(V)$ is empty, we set $\sigma(V) = \sigma$. We denote this same integer $\sigma(V)$ also by $\sigma(\operatorname{Anc}(V))$ and $\sigma(\overline{V})$.

Lemma 1.6. *Let $V \subset R_j$ be a vector subspace. For $i \geqslant 0$,*

$$R_i \cdot R_{-i} \cdot V \subset V, \quad \textit{and} \quad R_{-i} \cdot R_i V \supset V. \tag{1.7}$$

When $V \neq R_j$ we have

$$0 = \overline{R_{-j} V} \subset \cdots \subset \overline{R_{-1} V} \subset \overline{V}, \tag{1.8}$$

and

$$\overline{V} \subset \overline{R_1 V} \subset \overline{R_2 V} \subset \cdots \subset \operatorname{Sat}((V)). \tag{1.9}$$

Also, for $i \geqslant \sigma(\operatorname{Anc}(V)) - j$, we have $\overline{R_i V} = \operatorname{Sat}((V))$.

Proof. The inclusions of Eq. (1.7) are immediate from the definitions, and they imply Eqs. (1.8) and (1.9) (see also Lemma 3.6). The increasing sequence of ideals of Eq. (1.9) evidently terminates in $\operatorname{Sat}((V))$. Concerning the last claim, that $\overline{R_i V} = \operatorname{Sat}(V)$ for $i \geqslant \sigma(V) - j$ we first note that, taking $W = R_{\sigma-j} V$; that $\sigma(V) = \sigma$ implies $\sigma(W) = \sigma$. When $R_1 W = R_{\sigma+1}$ the claim is trivially satisfied; otherwise the regularity degree of $\operatorname{Proj}(R/(W))$ is σ. It follows that $W = \operatorname{Sat}((W))_{\sigma}$, and $\overline{W} = \operatorname{Sat}((W))$. This completes the proof. □

Lemma 1.7. *Let I be a graded ideal of R satisfying $H(R/I) = H$, and let $V = I_j$. Then we have*

$$I + M^{j+1} \subset \overline{V} + M^{j+1} \quad \textit{and} \quad I \cap M^j \supset (V). \tag{1.10}$$

Proof. Let $a > 0$ and $i = j - a$, then we have $V = I_j \supset R_a I_i$, hence

$$\overline{V}_i = R_{-a} \cdot V \supset R_{-a} R_a I_i \supset I_i$$

by (1.7) of Lemma 1.6. This shows $I + M^{j+1} \subset \overline{V} + M^{j+1}$. Now let $a > 0$ and $i = j + a$. We have $R_a V = R_a I_j \subset I_i$, hence $I \cap M^j \supset (V)$. □

Definition 1.8. Let $V \subset R_j$ and $W \subset R_i$. We say that V is *equivalent* to W ($V \equiv W$) if $\overline{V} = \overline{W}$. We will say that W is *simpler than* V if $W = R_{i-j} V$ and $\overline{W} \neq \overline{V}$.

The first principle behind this article is that each vector space in one of the sequences

$$V, R_{-1}V, R_{-2}V, \ldots \quad \text{or} \quad V, R_1V, R_2V, \ldots$$

should be either equivalent to or simpler than the preceding space. The complexity of a vector space $V \subset R_j$ should be measured by an invariant $\tau(V)$ that is nonincreasing along each sequence above, and where equality $\tau(V) = \tau(R_iV)$ implies $V \equiv W$. We succeed in this enterprise of measuring the complexity of V only when $r = 2$. In this case, we take $\tau(V) = \dim_k R_1V - \dim_k V$, and show that $\tau(V) = \nu(\overline{V})$, the number of generators of the ancestor ideal of V (Lemma 2.2). We show that this τ has the needed properties (Theorem 2.3). When $r \geqslant 3$ an analogous invariant with such strong properties is not possible due to an example of D. Berman (Example 3.8).

The second principle is that, fixing a degree j and vector space dimension d, the Grassmanian $\mathrm{Grass}(d, R_j)$ parametrizing d-dimensional subspaces of $V \subset R_j$ is stratified by locally closed subschemes $\mathrm{Grass}(H) = \mathrm{Grass}_H(d, j)$, parametrizing the vector spaces V for which the Hilbert function $H(R/\overline{V}) = H$ is fixed. Letting $G(H)$ be the scheme parametrizing all the graded ideals $I \subset R$ with $H(R/I) = H$, we have that $\mathrm{Grass}(H)$ is an open subscheme of $G(H)$ (Theorem 1.15). Natural questions are, when is $\mathrm{Grass}(H)$ nonempty? Is $\mathrm{Grass}(H)$ irreducible? What are the dimensions of its components? Is $\mathrm{Grass}(H)$ smooth? Describe the Zariski closure $\overline{\mathrm{Grass}(H)} \subset \mathrm{Grass}(d, R_j)$.

1.2. Background and main results

We first give the immediate background of the paper, and outline our main results, then we discuss related work of others.

Our main results are for the case $r = 2$, where we answer the above questions. We further show that $G(H)$ is a natural desingularization of $\overline{\mathrm{Grass}(H)}$ when $r = 2$, and we determine the fibre of $G(H)$ over a point in the closure of $\mathrm{Grass}(H)$.

When $r = 2$ we denote by $\mathrm{Grass}_\tau(d, R_j)$ the locally closed subscheme of $\mathrm{Grass}(d, R_j)$ parametrizing vector spaces V with $\tau(V) = \tau$. Recall that here, $\tau(V)$ is the number of generators of $\overline{V}$. Given a sequence $H = (H_0, H_1, \ldots)$ of nonnegative integers, we define the first difference sequence $E(H) = \Delta H$ by

$$E(H) = (e_1, \ldots, e_i, \ldots), \quad \text{where } e_i = H_{i-1} - H_i. \tag{1.11}$$

We let $e_0 = -1$. When $H = H(R/\overline{V})$, then $e_i = \tau(R_{i-j}V) - 1$ for $i < j$, and $e_i = \tau(R_{i-j-1}V) - 1$ for $i > j$ (Proposition 2.6). For H', H two sequences of integers that occur as Hilbert functions of ancestor algebras $\mathrm{Anc}(V)$, $V \subset R_j$, $\dim V = d$ we let (see Definition 1.14)

$$\begin{aligned} H' \geqslant_{\mathcal{P}} H \quad & \text{if for each } i \leqslant j \text{ we have } H'_i \leqslant H_i, \\ & \text{and for each } i \geqslant j \text{ we have } H'_i \geqslant H_i. \end{aligned} \tag{1.12}$$

We denote by a^+ the number a if $a \geqslant 0$ and 0 otherwise. It is well known that in two variables, the Hilbert function H of a quotient $A = R/I$ by a proper nonzero ideal (so

H is a *proper* O-sequence) satisfies, for some positive integer μ, the *order* of H (so $M^\mu \supset I, I_\mu \neq 0$)

$$H = (1, 2, \ldots, \mu, H_\mu, H_{\mu+1}, \ldots, H_i, \ldots) \quad \text{with } \mu = \min\{i \mid H_i < i+1\}, \text{ and}$$
$$\mu \geqslant H_\mu \geqslant H_{\mu+1} \geqslant \cdots \geqslant c_H \text{ and } \lim_{i\to\infty} H_i = c_H \geqslant 0. \tag{1.13}$$

Definition 1.9. Given a sequence H satisfying (1.13) with $c_H = 0$, let $\sigma = \sigma_H$ satisfy $H_{\sigma-1} \neq 0$, $H_\sigma = 0$. We denote by $G(H)$ the closed subscheme

$$G(H) \subset \prod_{\mu \leqslant i \leqslant \sigma-1} \operatorname{Grass}(i+1-H_i, R_i) \tag{1.14}$$

parametrizing graded ideals of R having Hilbert function H: here $\prod_{\mu\leqslant i\leqslant\sigma-1} \operatorname{Grass}(i+1-H_i, R_i)$ parametrizes sequences $V_\mu, V_{\mu+1}, \ldots, V_{\sigma-1}$ of vector spaces with each $V_i \subset R_i$ and $\dim V_i = i+1-H_i$; we assume $V_i = 0$ for $i < \mu$ and $V_i = R_i$ for $i > j$. The subscheme $G(H)$ is defined by the conditions $xV_i \subset V_{i+1}$ and $yV_i \subset V_{i+1}$ for $\mu \leqslant i < j$.

When $c_H > 0$, let $\sigma_H = \min\{i \mid H_{i-1} > c_H\}$. It is not hard to show that each ideal I with $H(R/I) = H$, satisfies

$$\exists f \in R_{c_H} \mid i > \sigma_H \Rightarrow I_i = (f) \cap R_i. \tag{1.15}$$

Thus, when $c_H > 0$ we may regard $G(H) \subset \prod_{\mu\leqslant i\leqslant\sigma} \operatorname{Grass}(i+1-H_i, R_i)$, in a manner similar to that above in (1.14) for the case $c_H = 0$.

We will use the following result, essentially from [I2], valid over a field k of arbitrary characteristic.

Theorem 1.10 [I2, Theorems 2.9, 2.12, 3.13, 4.3, Proposition 4.4, Eq. (4.7)]. *Let $r = 2$, and for* (1.10) *let the field k be algebraically closed. Let H be an O-sequence that is eventually constant, so H is a sequence satisfying* (1.13), *let $c = c_H$ and let $H_s = c_H$, $H_{s-1} \neq c_H$.*

(i) *Then $G(H)$ is a smooth projective variety of dimension $c + \sum_{i\geqslant\mu}(e_i+1)(e_{i+1})$. $G(H)$ has a finite cover by opens in an affine space of this dimension. If* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > s$ *then $G(H)$ has a finite cover by opens that are affine spaces.*

(ii) [I2, Theorem 4.3] *The number of generators $\nu(I)$ of a graded ideal I for which $H(R/I) = H$, satisfies $\nu(I) \geqslant \nu(H) = 1 + e_\mu + \sum_{i\geqslant\mu}(e_{i+1} - e_i)^+$.*

(iii) [I2, Proposition 4.4] *Assume that k is an infinite field. The graded ideals I with $H(R/I) = H$ and having the minimal number $\nu(H)$ of generators given by equality in* (1.10) *form an open subscheme of $G(H)$ having the dimension specified in* (1.10), *that is dense in $G(H)$ when k is algebraically closed.*

Remark on the Proof. The proof of (i) in the case R/I Artinian, so $c = 0$ is one of the main results of [I2]. The characteristic 0 case is handled in Theorems 2.9, 2.12, and the characteristic p case in Theorem 3.13 of [I2]. The proof of (i) when $c > 0$ relies on the fact that $t_s = t_{s+1} = c$ implies there is a form f of degree c such that $I_s = (f) \cap R_s$, $I_{s+1} = (f) \cap R_{s+1}$ (for a proof, see Proposition 2.3(vi)). This implies that $f \mid I_i$ for $i \leqslant s$. Thus, when $c > 0$, $I = fI'$ where I' is a graded ideal such that $H(R/I') = H'$, where H' is defined by $H'_i = H_{i+c} - c$. It follows that $G(H) \cong \mathbb{P}^c \times G(H')$. Here H' is eventually zero, so the dimension and structure of $G(H')$ is given by Theorems 2.9, 2.12, and 3.13 (see also Eq. (4.7)) of [I2]. In [I2] we defined certain subfamilies $U_H \subset G(H)$ parametrizing ideals I having "normal patterns:" such that I has a Gröbner basis with leading terms the first $i + 1 - H_i$ degree-i monomials in lexicographic order for each i. We showed that these subfamilies are affine spaces of dimension specified in (i); this result in fact requires only that k be an infinite field. However, that U_H be dense in $G(H)$ requires that k be algebraically closed.

We will show the following main results for ancestor ideals of a vector space $V \subset R_j$ of homogeneous polynomials when $r = 2$. Analogous results for level algebras and the algebras $R/(V)$ follow, and are stated in the appropriate section. Recall that we denote $\mathrm{Grass}_H(d, R_j)$ by $\mathrm{Grass}(H)$, and that we have $e_i = E(H)_i = H_{i-1} - H_i$. We denote by $c_H = \lim_{i\to\infty} H_i$. Theorem A is Theorem 2.19(ii). Theorem B is (2.34) of Theorem 2.17(B); other dimension results are in Theorems 2.17 and 2.24. Theorems C, D are the two parts of Theorem 2.32, Theorem E is Theorem 2.35. For Theorems B–E we assume that the field k is infinite, and the O-sequences H, H' belong to the set $\mathcal{H}(d, j)$ of *acceptable* sequences (Definition 2.7), which by Corollary 2.8 are those O-sequences H with d fixed satisfying the conditions of Theorem A; the partial order is that of (1.12). We denote by $\mathrm{LA}(N) = \mathrm{LA}_N(d, j) \subset \mathrm{Grass}(d, R_j)$ the scheme parametrizing those vector spaces $V \subset R_j$ whose level algebra $\mathrm{LA}(V)$ satisfies $H(\mathrm{LA}(V)) = N$; and we let $\mathrm{GA}(T) = \mathrm{GA}_T(d, j) \subset \mathrm{Grass}(d, R_j)$ parametrize graded algebras $R/(V), V \subset R_j$ satisfying $H(R/(V)) = T$. For Theorem E the set $\mathcal{P}A(d, j)$ is a certain partially ordered set of pairs of partitions (Definition 2.34).

Theorem A. *The proper O-sequence $H = (H_0, H_1, \ldots, H_j, H_{j+1}, \ldots)$ as in* (1.13) *occurs as the Hilbert function of the ancestor algebra of a proper vector subspace of R_j if and only if the first difference $E = \Delta(H)$ satisfies the conditions*

$$e_j = e_{j+1} \geqslant e_{j+2} \geqslant \cdots \geqslant e_{\sigma(V)} = 0, \tag{1.16}$$

$$e_j \geqslant e_{j-1} \geqslant e_{j-2} \geqslant \cdots \geqslant e_1 \geqslant e_0 = -1 \quad \textit{and} \tag{1.17}$$

$$\sum_{i\leqslant j}(e_i + 1) + \sum_{i>j} e_i + c_H = j + 1. \tag{1.18}$$

Each such sequence E satisfying the three conditions occurs, and for a vector space of dimension $d = \sum_{i\leqslant j}(e_i + 1)$.

Theorem B. *Let $d \leqslant j$ be positive integers, and let H be an acceptable O-sequence. The dimension of* $\mathrm{Grass}(H)$ *is $c_H + \sum_{i\geqslant\mu(H)}(e_i + 1)(e_{i+1})$.*

Theorem C (Frontier property). *Assume that* k *is algebraically closed. The Zariski closure* $\overline{\mathrm{Grass}(H)}$ *is* $\bigcup_{H' \geqslant_{\mathcal{P}} H} \mathrm{Grass}(H')$.

Theorem D. *Assume that* k *is algebraically closed. Let* d, j *be positive integers satisfying* $d \leqslant j$, *and suppose that* H *is an acceptable* O*-sequence* (*Definition* 2.7). *There is a surjective morphism* $\pi : G(H) \to \overline{\mathrm{Grass}(H)}$ *from the nonsingular variety* $G(H)$, *given by* $I \to I_j$. *The inclusion* $\iota : \mathrm{Grass}_H(d, j) \subset G(H), \iota : V \to \overline{V}$ *is a dense open immersion. For* $H' \in \mathcal{H}(d, j), H' \geqslant_{\mathcal{P}} H$, *the fibre of* π *over* $V' \in \overline{\mathrm{Grass}_H(d, j)} \cap \mathrm{Grass}_{H'}(d, j)$ *parametrizes the family of graded ideals*

$$\big\{ I \mid H(R/I) = H \text{ and } I_j = V' \big\}.$$

The schemes $\overline{\mathrm{LA}_N(d, j)}$ *and* $\overline{\mathrm{GA}_T(d, j)}$ *have desingularizations* $G(N)$ *and* $G(T)$, *respectively, with analogous properties.*

Theorem E. *There is an isomorphism* β *from the partially ordered set* $\mathcal{H}(d, j)$ *under the partial order* $\mathcal{P} = \mathcal{P}(d, j)$, *and the partially ordered set* $\mathcal{P}A(d, j)$ *under the product of the majorization partial orders* (*see Definition* 2.34). *The isomorphism is given by* $\beta(H) = (P, Q)$, $P = P(H) = A(H)^*$, $Q = Q(H) = B(H)^*$ (*see Definitions* 2.9 *and* 2.21). *This is the same order as is induced by specialization* (*closure*) *of the strata* $\mathrm{Grass}(H)$.

We show similar results to Theorems A–E for the Hilbert function strata $\mathrm{LA}_N(d, j)$ and $\mathrm{GA}_T(d, j)$. Of these results Theorems C, D—Theorem 2.32 in Section 2.3—are the deepest of the paper. The kind of frontier property shown is rare in this context of Hilbert schemes of families of ideals. The key step in the case of $R/(V)$ is the construction of an ideal I of a given Hilbert function $T = H(R/I)$ such that I contains a given ideal I' of Hilbert function $T' = H(R/I')$, where $T' \geqslant T$ termwise, and T, T' are permissible Hilbert functions $T = H(R/(V))$, $T' = H(R/(V'))$ for algebras $R/(V)$. This key step is made in Lemma 2.30, and involves constructing a sequence of intermediate ideals.

Many of the main results here, including Theorems A–D are rewritten from a youthful preprint [I1] of 1975, that was circulated then, even submitted, but not published, and is hereby retired! We have chosen to restrict the focus of the present paper to ancestor algebras, level algebras, and also the algebra $R/(V)$ determined by V, and several applications. We omit the developing of basic facts about apolarity/Macaulay's inverse systems that comprised an important part of [I1], but was both classically known, and is now well-known in recent literature in the form that we use in Section 3.1 (see, for example, [I4,EmI1,IK,G]). We give here a much-changed and clearer exposition of Theorems A–D, and their analogues for level algebras and the algebras $R/(V)$; the latter case $R/(V)$ was treated in [I2, Section 4B], but the exposition here is improved.

Several advances since 1975 have modified our exposition and influenced our results. The Persistence theorem of Gotzmann, which appeared in 1978, resolved a natural question that was open at the time of our original preprint and is a result that had been conjectured by D. Berman [Be,Go1]: see also [BrH,IKl] for further exposition of the persistence and Hilbert scheme result of G. Gotzmann, a refinement of Grothendieck's construction of the Hilbert scheme [Gro]. New here is the use of the Gotzmann results in Section 1.3 to help parametrize the Hilbert function strata of ancestor ideals, when $r > 2$ and H is not eventually zero.

Several authors have written about the restricted tangent bundle to a rational curve [GhISa,Ra,Ve], closely related to the Hilbert function strata $\mathrm{GA}_H(d, j)$. The form of the codimension results there have inspired an entirely new Section 2.2 on the minimal resolutions of the three algebras attached to V. We define partitions A, B giving the generator and relation degrees of the ancestor ideal $\overline{V}$, and depending only on the Hilbert function $H(R/\overline{V})$ (Lemma 2.23); and we find compact formulas for the codimensions of $\mathrm{Grass}_H(d, j), \mathrm{LA}_N(d, j)$ and $\mathrm{GA}_T(d, j)$ in terms of natural invariants of these partitions (Theorem 2.24). We also count level algebra and ancestor algebra Hilbert functions using the partitions (Theorem 2.19, Corollary 2.20) and as well we describe the closures of strata using them (Lemma 2.28, Theorem E). The Betti strata for more general O-sequences H—not arising from ancestor algebras—are studied in a sequel [I6].

The methods of this paper, in particular the proof of the frontier property of Theorem C for the parameter spaces $\mathrm{GA}_T(d, j)$ of the ideal (V), can be applied to show a similar frontier property for the stratification of the family of rational normal curves in $\mathbb{P}^r$ according to the decomposition of the restricted tangent bundle into a direct sum of line bundles (see [GhISa], also [Ra]). The analogous result for $\mathrm{LA}_N(d, j)$ has a similar interpretation for the stratification of such a family by the minimal rational scroll upon which they lie [I5].

In Section 3.1 we apply our results to solve a refined version of the simultaneous Waring problem for a vector space $\mathcal{W}$ of degree-j forms in $\mathcal{R} = k[X, Y]$, using apolarity or Macaulay inverse systems. The simultaneous Waring problem for a set of c general forms of specified degrees is to find a smallest integer μ such that c generic forms of these degrees may be written as linear combinations of powers of μ linear forms. It was studied classically by A. Terracini, whose approach is generalized and modernized in [DF]. Recently E. Carlini has interpreted the result concerning the generic (largest) Hilbert function for a level algebra, in terms of the simultaneous Waring problem, while making explicit the connection with secant varieties to the rational normal curve [Ca]. This well-known connection of ideals in $k[x, y]$ to secant bundles is explained in the complete intersection case related to the Waring problem for a single form in [IK, Section 1.3]. Another recent solution of the Waring problem for forms in two variables occurs in a unpublished preprint with Jacques Emsalem, a result that can be readily derived from the theory of compressed algebras [I4, Theorem 4.6C]. In the special case of equal degrees, so one considers $f \in \mathcal{W}$, for a general vector space $\mathcal{W} \subset \mathcal{R}_j$, $r = 2$ solutions are given in [CaCh, Theorem 3.1], [Ca, Theorem 3.3], and [ChGe, Theorem 3.16]; the latter result also determines the dimension of the subscheme of $\mathrm{Grass}(c, \mathcal{R}_j)$ parametrizing vector spaces $\mathcal{W}$ having a length μ simultaneous decomposition. Our refinement here is two-fold, first to consider vector spaces of degree-j forms $\mathcal{W}$ having a given *differential τ invariant*, and second, we use Theorem 2.32 to determine the closure of the relevant $\mathrm{LA}_N(d, j)$ strata (Theorem 3.4).

Section 3.2 has results from the original preprint [I1] concerning related vector spaces V, W, where $W = R_{i_k} \cdot R_{i_{k-1}} \cdots R_{i_1} V$. David Berman's article [Be] showed that a *complete Hilbert function* associated to a vector subspace of R_j, ostensibly a function from a countable set of sequences to $\mathbb{N}$, the nonnegative integers giving the dimension of each space W related to V, is determined by its restriction to a finite subset of the sequences. Here we study primarily the case $r = 2$ and we bound the number of classes $\overline{W}$ related to V (Proposition 3.9).

The results of Sections 2.1 and 2.3 in the special case of the algebras R/I where $I = (V)$ when $r = 2$ were stated and shown in Proposition 4.7–4.9 and Theorem 4.10 of [I2, Section 4B]. Our exposition here is rather more detailed and careful even in this special case. Other results of this article for the case $r = 2$ were announced in [I3, Appendix B] (the case (V), with an allusion to the ancestor ideal case), in [I4, Proposition 4.6A–C] (level algebras), in [IK, Theorem 8.1] (Gorenstein Artinian algebras), and in a note on level algebras when $r = 2$ at the end of [ChoI]. But proofs of the results of Sections 2.1 and 2.3 for ancestor ideals and level algebras, when $r = 2$ were in the original preprint [I1] and appear here for the first time.

Several authors have recently studied level algebras, but from a rather different viewpoint than taken here [ChoI,BiGe,Bj,St1]. In addition E. Carlini, and J. Chipalkatti with Tony Geramita have written about the two variable case, each determining the possible Hilbert functions for level algebras [Ca,ChGe]. E. Carlini and J. Chipalkatti have made some remarkable progress in the simultaneous resolution problem in certain other cases for $r \geqslant 3$ variables [CaCh]. J. Chilpakatti and A. Geramita give a geometric description of Hilbert function stratum $\mathrm{LA}_N(d, j)$ for level algebras in [ChGe, Propositions 3.7, 3.10]; and they draw conclusions for the simultaneous Waring problem for binary forms (ibid., Theorme 3.16). They also show that certain quite special unions of these strata are projectively normal, or arithmetically Cohen–Macaulay (ibid., Theorem 4.4): these unions are different from the closures $\overline{\mathrm{LA}_N(d, j)}$ studied here.

In higher dimensions $r > 2$, until recently only the Gorenstein case $\operatorname{cod} V = 1$ of level algebras had been extensively studied (see [IK] for results and references); also a compressed algebra case where H is maximum given the codimension of V and r had been studied [I4,FL,Bj]. The analogue for $r > 2$ of the frontier property of Theorem C does not usually hold even in the Gorenstein height three case [IK, Example 7.13], nor is $G(H)$ a desingularization of $\mathrm{Grass}(H)$ [IK, Lemma 8.3 with J. Yaméogo]. The sequences H that occur as Hilbert functions $H = H(R/\overline{V})$ are known when $r = 3$ in the Gorenstein case [BuEi,St1,Di] (see [IK, Section 5.3.1]); also in this Gorenstein case the family $\mathrm{Grass}(H)$ is irreducible and nonsingular [Di,Klp]. The question of which sequences H occur as Hilbert functions of level algebras $LA(V)$ is studied by A. Geramita, T. Harima, and Y. Shin in [GHS1] using *skew configurations* of points in $\mathbb{P}^n$. With J. Migliore they develop further results, including necessary conditions and new techniques and constructions for arbitrary socle degree and type; they also include a complete list of level Hilbert functions for $r = 3$, socle degree at most 5, of socle degree 6 and type $\operatorname{cod} V = 2$ [GHMS1]. When $r \geqslant 4$ even the set of Gorenstein sequences are unknown. However, several authors have established both minimum and maximum Hilbert functions for level algebras $\mathrm{LA}(d, j)$ in any codimension r (see [BiGe,ChoI]).

1.3. *The Hilbert function strata*

Fix r and the polynomial ring $R = k[x_1, \ldots, x_r]$. Recall that we denote by $\mathrm{Grass}(d, R_j)$ the Grassmanian parametrizing d-dimensional vector subspaces of R_j. A reader primarily interested in $r = 2$ may wish to skip over or skim this section and consult Proposition 2.5 in its place.

Definition 1.11. Let H be a sequence of nonnegative integers that occurs as the Hilbert function $H = H(R/\overline{V})$ where V is a d-dimensional vector subspace of R_j. We denote by $\mathrm{Grass}_H(d, j) \subset \mathrm{Grass}(d, R_j)$ the subscheme of the Grassmanian parametrizing vector spaces V satisfying the rank conditions

$$\operatorname{cod} R_i V = H_{i+j} \text{ in } R_{i+j}, \quad \text{for } i = -j, -j+1, \ldots. \tag{1.19}$$

When H is eventually zero, evidently Eq. (1.19) imposes a finite number of algebraic conditions on V (which we study shortly). When H is not eventually zero, we will use Gotzmann's Persistence and Hilbert scheme theorems, a refinement of the Grothendieck Hilbert scheme theorem, to show that the number of algebraic conditions imposed by (1.19) is finite.

Recall that every sequence $H = (H_0, \ldots)$ occurring as the Hilbert function $H = H(A)$ of a quotient algebra $A = R/I$ is eventually polynomial: there exists a pair $(p_H \in \mathbb{Q}[t], s = s(H) \in \mathbb{N}) \mid H_i = p_H(i)$ for $i \geqslant s(H)$. We denote by $\sigma = \sigma(p_H)$ the Gotzmann regularity degree of p_H (see [Go1,IKl]). It is easy to see that $\sigma \geqslant s(H)$. Recall that the Grothendieck Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^{r-1})$ parametrizes subschemes of $\mathbb{P}^{r-1}$ having Hilbert polynomial p [Gro]. We denote by r_i the integer $r_i = \dim_k R_i = \binom{r+i-1}{i}$, and define $q = q_H$ by $q(i) = r_i - p_H(i)$. We denote by $M(d, j)$ the vector space span of the first d monomials of degree j in R, in lexicographic order.

Theorem 1.12 (Macaulay Growth Theorem [Mac2]). *A vector space $V \in \mathrm{Grass}(d, R_j)$ satisfies*

$$\dim R_1 \cdot V \geqslant \dim R_1 \cdot M(d, j). \tag{1.20}$$

Theorem 1.13 (Gotzmann Hilbert scheme and Persistence Theorem [Go1]). *Let p be a Hilbert polynomial, and $\sigma = \sigma(p)$. The Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^{r-1})$ is the locus of pairs of vector spaces*

$$(V, V') \in \mathrm{Grass}\big(q(\sigma), R_\sigma\big) \times \mathrm{Grass}\big(q(\sigma+1), R_{\sigma+1}\big) \tag{1.21}$$

satisfying $R_1 \cdot V = V'$, or, equivalently $R_1 \cdot V \subset V'$. Such vector spaces V satisfy equality in (1.20).
(Persistence) *A vector space V occurring in such an extremal growth pair (V, V') satisfies*

$$\dim(R_{\sigma+i}/R_i V) = p(\sigma + i)\ \forall i \geqslant 0; \tag{1.22}$$

the space $R_i V$ has dimension $q(\sigma + i)$, and also satisfies equality in (1.20).

For an exposition of the persistence result over k, see [BrH, Section 4.3]; for an exposition of the Gotzmann–Grothendieck Hilbert scheme results and further references see [IKl]. One consequence of Theorem 1.13 for us is that one may suppose that $i \leqslant \max\{1, \sigma p_H + 1 - j\}$ in Eq. (1.19). Thus (1.19) defines a scheme structure on $\mathrm{Grass}_H(d, j)$ as locally closed subscheme of $\mathrm{Grass}(d, R_j)$, for all occurring sequences H.

Given such a sequence H we define a projective scheme $G(H)$ parametrizing the graded ideals $I \subset R$ that determine a quotient algebra $A = R/I$ having Hilbert function $H(A) = H$. When H is eventually zero, so $H_s = 0$, the parametrization of $G(H)$ is as a subset of $\prod_{i\leqslant s} \mathrm{Grass}(r_j - h_j, R_j)$, where $r_j = \dim_k R_j$. When H is not eventually zero, then H is eventually polynomial $H_i = p_H(i)$ for $i \geqslant s(H)$ for some polynomial $p = p_H$. As before, we take $\sigma(H)$ the regularity degree of the polynomial, and parametrize

$$G(H) \subset \left(\prod_{i<\sigma} \mathrm{Grass}(r_j - h_j, R_j)\right) \times \mathrm{Hilb}^p\left(\mathbb{P}^{r-1}\right). \tag{1.23}$$

By Theorem 1.13, we may replace the product in Eq. (1.23) by $\prod_{i\leqslant\sigma+1} \mathrm{Grass}(r_j - h_j, R)$.

Results of D. Mall (when $\mathrm{char} k = 0$ or $\mathrm{char} k > \sigma(p_H)$ and K. Pardue (for arbitrary characteristic) show that when the base field k is algebraically closed, the scheme $G(H)$ is connected [Mall,Par].

Definition 1.14. We define a partial order $\mathcal{P} = \mathcal{P}(d, j, r)$ on the set $\mathcal{H}(d, j, r)$ of Hilbert functions possible for $H(A)$, $A = R/\overline{V}$, as follows:

$$H' \geqslant_{\mathcal{P}(d,j,r)} H \quad \Leftrightarrow \quad H'_i \leqslant H_i \quad \text{for } i \leqslant j \text{ and } H'_i \geqslant H_i \text{ for } i \geqslant j. \tag{1.24}$$

When the triple (d, j, r) is obvious from context we write $H' \geqslant_{\mathcal{P}} H$ for $H' \geqslant_{\mathcal{P}(d,j,r)} H$. Recall that H *occurs* or is possible for us if it occurs as the Hilbert function of an ancestor algebra $\mathrm{Anc}(V)$ for some d-dimensional vector subspace of R_j.

Theorem 1.15. *Let H be a sequence that occurs as the Hilbert function of an ancestor algebra. The scheme* $\mathrm{Grass}_H(d, j)$ *is a locally closed subscheme of* $\mathrm{Grass}(d, R_j)$. *The condition $H' = H(R/\overline{V}) \geqslant_{\mathcal{P}} H$ is a closed condition on $V \in \mathrm{Grass}(d, R_j)$. Also the inclusion $\iota : \mathrm{Grass}_H(d, j) \to G(H)$ given by $\iota : V \to \overline{V}$ is an open immersion.*

Proof. Let $I = I_V = \overline{V}$. It is not hard to show that $\dim I_i \geqslant r_i - H_i$ is a closed condition, and $\dim I_i < r_i - H_i + 1$ is an open condition on $V \in \mathrm{Grass}(d, R_j)$, when $i \leqslant j$. Likewise, it is not hard to show that for each $i \geqslant j$ then $\dim I_i \leqslant r_i - H_i$ is a closed condition, while $\dim I_i > r_i - H_i - 1$ is an open condition. By the Gotzmann persistence and regularity theorems, if V satisfies each of these conditions for all positive integers $i \leqslant \sigma(p_H) + 1$ (which we may suppose greater than j), then $H(R/\overline{V}) = H$. Thus, we have shown that $\mathrm{Grass}_H(d, j) \subset \mathrm{Grass}(d, R_j)$ is defined by the intersection of a finite number of open and closed conditions, so it is locally closed, as claimed.

That the inclusion ι is an open immersion, follows from $I_{\geqslant j}$ being generated by I_j, and $I_i, i < j$ being $R_{i-j} I_j$. For $a > 0$ the condition that $V = I_j$ generates I_{j+a} is equivalent to the rank of the multiplication map: $R_a \otimes V \to R_i$ being greater than $\dim I_i - 1 = r_i - H_i - 1$ on $G(H)$—an open condition. Let $W = V^{\perp} \subset \mathcal{R}_j$ in the Macaulay duality. For $a > 0$ the condition that $I_{j-a} = R_{-a} V$ is equivalent to the rank of the contraction map

$R_a \times W \to R_a \circ W \subset \mathcal{R}_{j-a}$ being greater than $H_i - 1$, on $G(H)$, also an open condition. This completes the proof. $\square$

Corollary 1.16. *The Zariski closure* $\overline{\mathrm{Grass}_H(d,j)} \subset \bigcup_{H' \geqslant_{\mathcal{P}} H} \mathrm{Grass}_{H'}(d,j)$. *Similar inclusions hold for* $\overline{\mathrm{LA}_N(d,j)}$ *and for* $\overline{\mathrm{GA}_T(d,j)}$.

Remark 1.17. The partial order $\mathcal{P}(d,j,r)$ for $r \geqslant 2$ is not in general subordinate to or equal to a simple order. For $r = 2$ a simply ordered exception are the complete intersection cases $(d,j) = (d,d+1)$, where V has codimension one: see [IK, Section 1.3]. Also for $r = 2$, Example 2.36 gives a different simply ordered case, $(d,j) = (4,5)$, while Example 2.29(A) below $(d,j) = (3,5)$ and Example 2.29(B) $(d,j) = (10,12)$ illustrate the more general situation $\mathcal{P}(d,j,2)$ not a simple order, for ancestor algebras and level algebras, respectively.

2. The ancestor ideal in two variables

Throughout this section, R is the polynomial ring $R = k[x,y]$ over an arbitrary field k, and we denote by $M = (x,y)$ the homogeneous maximal ideal. The vector space R_j of degree-j forms in R satisfies, $R_j = \langle x^j, x^{j-1}y, \ldots, y^j\rangle$, of dimension $j+1$, and $V \subset R_j$ will be a vector subspace having dimension $\dim V = d$. In Section 2.1 we give our main results concerning the individual Hilbert function strata of the three algebras related to V when $r = 2$. These include a characterization of ancestor ideals (Proposition 2.11) and the dimension/structure Theorem 2.17. In Section 2.2 we give our results relating the graded Betti numbers of these three algebras to certain partitions A, B, C, D (Lemma 2.23); also we give the codimension of the Hilbert function strata in terms of the partitions A, B or C, D (Theorem 2.24). In Section 2.3 we determine the closures of the Hilbert function strata (Theorem 2.32).

2.1. *The Hilbert function strata when* $r = 2$

We first present the main tool we need, the simplicity $\tau(V)$, and a key exact sequence.

Definition 2.1. For $V \subset R_j$ we define

$$\tau(V) = \dim_k R_1 V - \dim_k V. \tag{2.1}$$

We define the sequence

$$0 \to R_{-1}V \xrightarrow{\phi} R_1 \otimes V \xrightarrow{\theta} R_1 \cdot V \to 0, \tag{2.2}$$

where $\phi : f \to y \otimes xf - x \otimes yf$, and $\theta : \sum_i \ell_i \otimes v_i \to \sum_i \ell_i v_i$, where the ℓ_i are elements of R_1 (linear forms).

For I a graded ideal of R, we denote by $\nu(I)$ the number of minimal generators for I. For a vector subspace $W \subset R_i$ we denote by $\operatorname{cod} W = i + 1 - \dim W$, the codimension of W in R_i.

Lemma 2.2. *The sequence* (2.2) *is exact. We have*

$$\tau(V) = \dim V - \dim R_{-1}V \tag{2.3}$$

$$= 1 + \operatorname{cod} R_{-1}V - \operatorname{cod} V = 1 + \operatorname{cod} V - \operatorname{cod} R_1 V \tag{2.4}$$

$$= \nu(\overline{V}). \tag{2.5}$$

Also, $\tau(V) \leqslant \min\{d, j + 2 - d\}$.

Proof. Clearly ϕ is a monomorphism, and θ is surjective, so we need only show the exactness of (2.2) in the middle. Suppose that $U \in R_1 \otimes V$ and $\theta(U) = 0$. We may suppose $U = x \otimes v_1 + y \otimes v_2$, thus $xv_1 + yv_2 = 0$, implying y divides v_1 and x divides v_2. Thus $w = v_2/x = -v_1/y \in R_{-1}V$ satisfies

$$\phi(w) = y \otimes xw - x \otimes yw = y \otimes v_2 - x \otimes (-v_1) = U. \tag{2.6}$$

This completes the proof of the exactness of (2.2). Thus, counting dimensions in (2.2) we have

$$2 \dim V = \dim R_1 \otimes V = \dim R_{-1}V + \dim R_1 V. \tag{2.7}$$

Noting the definition of τ in (2.1), we have shown (2.3). Eqs. (2.4) follow immediately. To show that $\tau(V) = \nu(\overline{V})$, we first note that applying (2.7) to $R_i V$ we have for any integer i satisfying $-j \leqslant i$,

$$\dim R_{-1}R_i V + \dim R_1 R_i V = 2 \dim R_i V. \tag{2.8}$$

When $i \leqslant 0$ we have $R_{-1}R_i V = R_{i-1}V$, so we have

$$\text{for } i \leqslant 0 \quad \dim R_1 R_i V = 2 \dim R_i V - \dim R_{i-1}V. \tag{2.9}$$

The number of generators $\nu(\overline{V})$ of the ancestor ideal of V satisfies, $\nu(\overline{V}) = \dim_k(\overline{V}/M\overline{V})$, where $M\overline{V} = R_1\overline{V}$, since $\overline{V}$ is graded. We have

$$\overline{V}/R_1\overline{V} = \bigoplus_{i=-j}^{+\infty} (R_i V/R_1 R_{i-1}V) = \bigoplus_{i=-j}^{0} (R_i V/R_1 R_{i-1}V), \tag{2.10}$$

since for $i \geqslant 0$ we have $R_1 R_{i-1}V = R_i V$. Let $d_i = \dim R_i V$. From (2.10) we have

$$\begin{aligned}
\nu(\overline{V}) &= \sum_{i=-j}^{0} \dim R_i V - \sum_{i=-j}^{0} \dim R_1 R_{i-1} V \\
&= \sum_{i=-j}^{0} d_i - \left(2 \sum_{i=-j}^{0} d_{i-1} - \sum_{i=-j}^{0} d_{i-2}\right) \quad \text{by (2.9)} \\
&= d_0 - d_{-1} \\
&= \tau(V) \quad \text{by (2.3).}
\end{aligned}$$

This completes the proof of (2.5). The upper bound on $\tau(V)$ is immediate from (2.3) and (2.4). □

Recall from Definition 1.8 that the subspace $V \subset R_j$ is *equivalent* to $W \subset R_i$ if $\overline{V} = \overline{W}$. A generalization of (iii) below is shown in Corollary 3.10.

Proposition 2.3 (Equivalence). *We assume that $V \subset R_j$; here $R = k[x, y]$.*

(i) *For $s \geqslant -j$ we have $\tau(R_s V) \leqslant \tau(V)$, with equality if and only if $\overline{R_s V} = \overline{V}$.*
(ii) *In the sequence*

$$\tau(R_{-j} V), \dots, \tau(R_{-1} V), \tau(V), \tau(R_1 V), \dots$$

the values of $\tau(R_i V)$ are monotone nondecreasing for $i \leqslant 0$, and monotone non-increasing for $i \geqslant 0$.
(iii) *For two-vector spaces $R_s V$, $R_t V$, we have*

$$\begin{aligned}
\overline{R_s V} = \overline{R_t V} \quad &\Leftrightarrow \quad R_s V = R_{s-t} R_t V \quad \textit{and} \quad R_t V = R_{t-s} R_s V \\
&\Leftrightarrow \quad \begin{cases} \textit{either } \tau(R_s V) = \tau(R_t V) = \tau(V), \\ \textit{or } \operatorname{sign}(s) = \operatorname{sign}(t) \textit{ and } \tau(R_s V) = \tau(R_t V). \end{cases}
\end{aligned}$$

(iv)

$$\overline{R_s V} = \overline{V} \quad \Leftrightarrow \quad \begin{cases} \textit{if } s > 0, & \dim R_{s+1} V = \dim V + (1+s)\tau(V); \\ \textit{if } s \leqslant 0, & \dim R_{s-1} V = \dim V - (1-s)\tau(V). \end{cases}$$

(v) *For any two-vector spaces $V \subset R_j$, $W \subset R_i$,*

$$\overline{V} = \overline{W} \quad \Leftrightarrow \quad V = R_{j-i} W \quad \textit{and} \quad \tau(V) = \tau(W).$$

(iv) $\tau(V) = 1 \Leftrightarrow V = f \cdot R_{j-c}$ *where* $\deg f = c = \operatorname{cod} V$. *Also* $\tau(V) = 0 \Leftrightarrow V = 0$.

Proof. To show (i) it suffices to prove it for $s = \pm 1$ and apply an induction. For $s = 1$ we have $\tau(R_1V) = \dim R_1V - \dim R_{-1}R_1V$, but $R_{-1}R_1V \supset V$, so $\tau(R_1V) \leqslant \dim R_1V - \dim V = \tau(V)$ with equality if and only if $R_{-1}R_1V = V$, which is equivalent to $\overline{V} = \overline{R_1V}$. For $s = -1$, we have $\tau(R_{-1}V) = \dim R_1R_{-1}V - \dim R_{-1}V \leqslant \dim V - \dim R_{-1}V = \tau(V)$ with equality if and only if $R_1R_{-1}V = V$, which is equivalent to $\overline{R_{-1}V} = \overline{V}$.

Repeated use of (i) shows the rest of the proposition. For example, we show (iv) for $s > 0$. By definition $\tau(R_iV) = \dim R_{i+1}V - \dim V$ for $i = 0, \dots, s$ so we have for $W = R_sV$,

$$\dim R_1W = \dim V + \tau(W) + \tau(R_1V) + \cdots + \tau(R_sV).$$

That $\tau(V), \tau(R_1V), \dots$ is nonincreasing shows that $\dim R_1W = \dim V + (s+1)\tau(V) \Leftrightarrow \tau(V) = \tau(R_1V) = \cdots = \tau(R_sV)$, as claimed. This completes the proof of (iv). For (vi), evidently $\tau(V) = 0 \Leftrightarrow V = 0$. When $\tau(V) = 1$, then lemma $\overline{V} = (f)$ by Lemma 2.2. Letting $c = \deg f$ we thus have $R_{c-j}V = \langle f\rangle$ and $R_{j-c}f = \overline{V}_j = V$, whence $c = \operatorname{cod} V$, as claimed. This completes the proof of (vi). □

Example 2.4. We show here the need to use the $\dim(R_{s+1}V)$ in Proposition 2.3(iv) to decide if R_sV is equivalent to V when $s > 0$, and the need for $R_{s-1}V$ when $s \leqslant 0$. Let $V = \langle x^4, x^3y, y^4\rangle \subset R_4$, then $R_{-1}V = \langle x^3\rangle$, and $\overline{V} = (x^3, y^4)$, so $\tau(V) = 2$ while $\overline{R_{-1}V} = (x^3)$, yet we have $\dim R_{-1}V = \dim(V) - \tau(V)$. Thus, the dimension of $W = R_sV$ is not enough to test the equivalence of W and V. Here $\dim R_{-2}V = 0 \neq \dim V - 2\tau(V)$, corresponding to $\overline{V} \neq \overline{R_{-1}V}$. Here $\overline{V} = \overline{R_1V}$, and $\dim R_1V = 5 = \dim V + \tau(V)$, $\dim R_2V = \dim V + 2\tau(V)$, but $R_2V = R_6$ so $\overline{V} \neq \overline{R_2V}$. Here $j = 4$, $\overline{V}$ is a complete intersection, satisfying $H(\operatorname{Anc}(V)) = (1,2,3,3,2,1)$, $E(H) = \Delta H = (-1,-1,-1,0,e_4 = 1,1,1)$. As in Proposition 2.6 (2.14) the subsequence $(-1,-1,-1,0,1 = e_4)$ of $E(H)$ is nondecreasing, while the subsequence $(1 = e_4, 1, 1)$ is nonincreasing, and $\tau(V) = 2 = e_4 + 1 = e_5 + 1$ (see Proposition 2.6 (2.17)).

We define the greatest common divisor $\operatorname{GCD}(V)$ as the principal ideal in $k[x,y]$ with a generator of highest degree, such that $\operatorname{GCD}(V)$ contains V (the generator divides each element of V). We will now show directly for $R = k[x,y]$ that $\lim_{i\to\infty} \overline{R_iV} = \operatorname{GCD}(V)$, a special case of $\lim_{i\to\infty} \overline{R_iV} = \operatorname{Sat}(V)$ in Lemma 1.6.

Proposition 2.5. *Assume that $H = H(R/\overline{V})$ satisfies $\lim_{i\to\infty} H_i = c$. Then we have*

$$\sum_{i\geqslant 0}\bigl(\tau(R_iV) - 1\bigr) = \operatorname{cod} V - c = (j+1-d) - c, \tag{2.11}$$

$$\sum_{i\leqslant 0}\tau(R_i\cdot V) = \dim V = d. \tag{2.12}$$

The degree $\deg\operatorname{GCD}(V) = c$. *For $i \geqslant \operatorname{cod} V - \tau(V) + 2$, we have*

$$\tau(R_i\cdot V) = 1 \text{ and } \overline{R_i\cdot V} = \operatorname{GCD}(V), \tag{2.13}$$

Proof. Let $k \geqslant 0$ satisfy $H_{k+j} = c$; then evidently $\tau(R_k \cdot V) = 1$ and by Proposition 2.3(ii) we have $c = \deg \mathrm{GCD}(R_k \cdot V)$ and evidently since $k \geqslant 0$, we have $\mathrm{GCD}(R_k \cdot V) = \mathrm{GCD}(V)$. Now, Eq. (2.11) is a consequence of (2.4), and Eq. (2.12) follows from (2.3). We now turn to the explicit bound on i for achieving $\tau(R_i \cdot V) = 1$. Suppose on the contrary that for an integer $i \geqslant 2$ we have $\tau(R_i \cdot V) \geqslant 2$. Proposition 2.3(ii) shows that the sequence $\tau(V), \tau(R_1 \cdot V), \ldots$ is montone, hence we have from (2.11),

$$\tau(V) - 1 + i \leqslant \bigl(\tau(V) - 1\bigr) + \bigl(\tau(R_1 \cdot V) - 1\bigr) + \cdots + \bigl(\tau(R_i \cdot V) - 1\bigr) \leqslant \operatorname{cod} V,$$

implying $i \leqslant \operatorname{cod} V - (\tau(V) - 1)$. Thus we have the explicit bound $\tau(R_i V) = 1$ for $i \geqslant \operatorname{cod} V - \tau(V) + 2$, as claimed. By Lemma 2.2 we have for such i, $\overline{R_i \cdot V} = (f)$. As above we conclude by Proposition 2.3(vi) that for such i, we have $f = \mathrm{GCD}(R_i \cdot V) = \mathrm{GCD}(V)$. $\square$

Recall that when $H = H(R/I)$ is the Hilbert function of a graded quotient of R, we denote by $E(H)$ the first difference sequence $E(H) = \Delta H = (e_0 = -1, e_1, \ldots, e_i, \ldots)$ where $e_i = (\Delta H)_i = H_{i-1} - H_i$. We set $\mu(H) = \min\{i \mid H_i < i + 1\}$, which is the order of any ideal $I \subset R$ with $H(R/I) = H$. Recall that since H is an O-sequence with $H_1 \leqslant 2$, H must satisfy (1.13), so $0 \leqslant H_i \leqslant i + 1$, and for $I_i \neq 0$, $H_{i+1} \leqslant H_i$. Thus, $H \neq H(R)$ (or $I \neq 0$) implies $\lim_{i \to \infty} H_i = c_H \geqslant 0$ with c_H a non-negative constant. When $H = H(R/\overline{V})$ we have by Proposition 2.5, $c_H = \deg \mathrm{GCD}(V)$.

Proposition 2.6. *Let $V \subset R_j$ be a vector subspace satisfying $\dim V = d$, and let $H = H(R/\overline{V})$ as above be the Hilbert function of the ancestor algebra of V, and let $c = c_H$. The first difference sequence $E(H)$ satisfies*

$$e_i \leqslant e_{i+1} \text{ for } i \leqslant j, \quad \text{and} \quad e_i \geqslant e_{i+1} \quad \text{for } i \geqslant j; \tag{2.14}$$

$$\text{also} \quad \sum_{i \leqslant j} (e_i + 1) = d \quad \text{and} \quad \sum_{i > j} e_i = (j + 1 - d) - c. \tag{2.15}$$

Let $V \subset R_j$ and let $H = H(R/\overline{V})$. Then $\tau(R_{i-j} \cdot V)$ satisfies

$$\tau(R_{i-j} \cdot V) = \begin{cases} e_i + 1 = \nu(\overline{R_{i-j} \cdot V}) = \#\{\text{generators of } \overline{V} \text{ of degree } \leqslant i\} & \text{if } i \leqslant j, \\ e_{i+1} + 1 & \text{if } i \geqslant j. \end{cases} \tag{2.16}$$

We have $e_j = \tau(V) - 1$ and

$$0 \leqslant e_j = e_{j+1} \leqslant \min\{j + 1 - d, d - 1\}, \tag{2.17}$$

with equality $e_j = d - 1$ if and only if $R_{-1} V = 0$. Also, $e_{j+1} = \operatorname{cod} V$ if and only if $R_1 V = R_{j+1}$.

Proof. By applying the first part of Eq. (2.4) to $R_{i-j} \cdot V$ when $i < j$, we obtain

$$\tau(R_{i-j} \cdot V) = \operatorname{cod} R_{i-j-1} \cdot V - \operatorname{cod} R_{i-j} \cdot V + 1 = e_i + 1$$

which is the first part of Eq. (2.16). For any i we have by Lemma 2.2 $\tau(R_{i-j} \cdot V) = \nu(\overline{R_{i-j} \cdot V})$; when $i \leqslant j$ we have also the second part of Eq. (2.16) since

$$\begin{aligned} \nu(\overline{R_{i-j} \cdot V}) &= \sum_{u \leqslant i} (\dim R_{u-j} \cdot V - \dim R_1 \cdot R_{u-j-1} \cdot V) \\ &= \#\{\text{generators of } \overline{V} \text{ having degree } \leqslant i\}. \end{aligned}$$

By applying the second part of Eq. (2.4) to $R_{i-j} \cdot V$ when $i \geqslant j$ we obtain

$$\tau(R_{i-j} \cdot V) = \operatorname{cod} R_{i-j} \cdot V - \operatorname{cod} R_{i-j+1} \cdot V + 1 = e_{i+1} + 1,$$

which is the last part of Eq. (2.16). Eq. (2.14) now follows from Proposition 2.3(ii), and Eq. (2.15), follows from the definition of $E(H)$ as a first difference of H. Eq. (2.17) and remaining claims follow from (2.16). □

Definition 2.7. Let d, j be positive integers satisfying $d \leqslant j$. We say that a proper O-sequence H (a sequence H satisfying (1.13)) is *acceptable* for an ancestor algebra in two variables of a d-dimensional subspace of R_j if H satisfies (2.14), (2.15), and (2.17) of Proposition 2.6.

The sequence $H = 0$ occurs for $V = R_j$, and $H = H(R) = (1, 2, \ldots)$ occurs for $V = 0$, but we will omit these cases henceforth.

Corollary 2.8. *Let j be a positive integer. A proper O-sequence H of* (1.13) *is acceptable for an ancestor ideal of a degree-j vector space iff the first difference $E = \Delta(H)$ satisfies*

$$e_j = e_{j+1} \geqslant e_{j+2} \geqslant \cdots \geqslant e_{\sigma(V)} = 0, \tag{2.18}$$

$$e_j \geqslant e_{j-1} \geqslant e_{j-2} \geqslant \cdots \geqslant e_1 \geqslant e_0 = -1, \quad \textit{and} \tag{2.19}$$

$$\sum_{i \leqslant j} (e_i + 1) + \sum_{i > j} e_i + c_H = j + 1. \tag{2.20}$$

Proof. Immediate from Definition 2.8, and (2.14), (2.15), (2.17). Here $d = \sum_{i \leqslant j} (e_i + 1)$. □

In the following definition we use *partition* of n in the usual sense of $n = n_1 + n_2 + \cdots + n_u, n_1 \geqslant n_2 \geqslant \cdots \geqslant n_u > 0$. Part of the reason for our choice of P, Q is that we later show they are the duals of the pair of partitions (A, B) determined by the generator degrees, and the relation degrees of ancestor algebras $\operatorname{Anc}(V)$ satisfying $H(\operatorname{Anc}(V)) = H$ (Lemma 2.23). Recall that the order $\mu(H)$ of an O-sequence is the smallest integer i such that $H_i \neq i+1$. We let $s(H) = \min\{i \mid H_i = c(H)\}$. Also given j, H, with H acceptable, we define $\tau(H) = H_{j+1} - H_j + 1 = e_{j+1} + 1 = e_j + 1$.

Definition 2.9. Given positive integers d, j with $d \leqslant j$ and an acceptable O-sequence H as in Definition 2.7, and letting $\tau = \tau(H) = e_j(H) + 1$, we define a pair of partitions $(P = P(H), Q = Q(H))$ of $(d, j+1-d-c(H))$ as follows. Let V satisfy $H(R/\overline{V}) = H$. Then $P(H), Q(H)$ satisfy

$$P(H) = \bigl(\tau, \tau(R_{-1} \cdot V) = e_{j-1}(H) + 1, \tau(R_{-2} \cdot V), \ldots,$$
$$\tau(R_{\mu-j} V) = e_\mu(H) + 1\bigr), \tag{2.21}$$
$$Q(H) = \bigl(\tau - 1 = e_{j+1}(H), e_{j+2}(H), e_{j+3}(H), \ldots, e_s(H)\bigr). \tag{2.22}$$

Recall from Definition 1.14 that $\mathcal{H}(d, j, 2)$ is the set of sequences possible for the Hilbert function of $\mathrm{Anc}(V)$, V a d-dimensional subspace of R_j, $R = k[x, y]$; understanding that $r = 2$ we will denote this set by $\mathcal{H}(d, j)$. We will likewise denote by $\mathcal{P}(d, j)$ the partial order $\mathcal{P}(d, j, 2)$ on $\mathcal{H}(d, j, 2)$ from Definition 1.14. We will denote by $\mathcal{H}(d, j)_\tau$ the subset of $\mathcal{H}(d, j)$ for which $e_j = \tau - 1$.

We will shortly show that the O-sequences that are acceptable in the sense of Definition 2.7 are exactly those that occur as the Hilbert function of an ancestor algebra (Theorem 2.19). So each pair (P, Q) of partitions described in the lemma below actually occurs as $P = P(H)$, $Q = Q(H)$ for some acceptable H.

Lemma 2.10. *For* (i), (ii) *below we suppose that the O-sequence H is proper and acceptable, as in Definition* 2.7*, and let $\tau = \tau(H)$. Then*

(i) *The partition $P = P(H)$ of Definition* 2.9 *is a partition of d having largest part τ. The partition $Q = Q(H)$ is a partition of $j+1-d-c$ having largest part $\tau - 1$.*
(ii) *Let $(\mu(H), s(H)) = (\mu, s)$. Then $P(H)$ has $j+1-\mu$ parts, and $Q(H)$ has $s-j$ parts.*
(iii) *H is uniquely determined by $(j, P(H), Q(H))$.*
(iv) *Let d, j be positive integers, with $d \leqslant j$. There is a one-to-one onto correspondence $H \to (P(H), Q(H))$ between the subset of acceptable O-sequences H satisfying $(\mu(H), s(H)) = (\mu, s)$ and $c(H) = c$, and the set of pairs of partitions (P, Q) satisfying* (i) *and* (ii)*. There are similar one-to-one correspondences between the set of partitions P and the set of sequences $N = N_H$, and also between the set of partitions Q and the set of sequences $T = T_H$* (*Definiton* 2.16)*.*

Proof. The claim in (i) that P partitions d is (2.12). That the parts of P are less than τ follows from Proposition 2.3(ii). That Q partitions $j+1-d-c$ follows from (2.15); that $e_{j+1} = \tau - 1$ is (2.17). That the parts of Q are no greater than $\tau - 1$ follows as before from Proposition 2.3(ii). The claim of (ii) is immediate from the definitions, counting the nonzero parts of P, Q. For (iii), we note that the triple (P, Q, j) determines (P, Q, τ) so determines $E(H)$, and also d, j, hence $c = c(H)$; then $H_i = c + \sum_{i<k} e_k$ determines H. The proof of (iv) is also immediate. □

The following proposition and corollary describe which ideals are ancestor ideals, in terms of the degrees of the generators and relations. In a related result, we determine the graded Betti numbers of the ancestor algebra $\mathrm{Anc}(V)$ in terms of the Hilbert function $H(\mathrm{Ann}(V))$ (Lemma 2.23).

Proposition 2.11 (Ancestor ideals). *Let I be a graded ideal of $R = k[x, y]$. The following are equivalent*

(i) *I is the ancestor ideal of I_j.*
(ii) *I is homogeneously generated by elements of degree no greater than j, and for each i satisfying $0 \leqslant i \leqslant j$ we have $\tau(I_i) = \#\{$generators of I having degree less or equal $i\}$.*
(iii) *I is generated by forms of degree at most j, and with relations of degrees at least $j+1$.*
(iv) *I has a generating set $f_1, \ldots, f_\nu$ of degrees at most j and*

$$I_{j+1} = \bigoplus_{1\leqslant i\leqslant \nu} R_{j+1-\deg f_i} f_i. \tag{2.23}$$

(v) *$H(R/I)$ satisfies Eq.* (2.14)*, and I has the minimum possible number of generators for a graded ideal defining a quotient R/I of Hilbert function H, namely*

$$\nu(I) = e_j + 1 = H_{j-1} - H_j + 1 = H_j - H_{j+1} + 1 = e_{j+1} + 1. \tag{2.24}$$

Proof. We show first that (i)–(iv) are equivalent, and then (i), (ii) ⇔ (v). That (i) ⇒ (ii) is from Eq. (2.16). Assume (ii). Then we have for $i \leqslant j$,

$$\begin{aligned}\operatorname{cod} R_{-1}I_i - \operatorname{cod} I_i &= \tau(I_i) - 1\\ &= \tau(I_{i-1}) - 1 + \#\{\text{generators of degree } i\}\\ &= \operatorname{cod}(I_{i-1}) - \operatorname{cod}(R_1 \cdot I_{i-1}) + \dim I_i - \dim(R_1 \cdot I_{i-1})\\ &= \operatorname{cod} I_{i-1} - \operatorname{cod} I_i,\end{aligned}$$

hence $\operatorname{cod} R_1 \cdot I_i = \operatorname{cod} I_{i-1}$. Since always $R_{-1} \cdot I_i \supset I_{i-1}$ the equality of dimensions shows $R_{-1} \cdot I_i = I_{i-1}$ for $i \leqslant j$: this and I generated by degree j shows that I is the ancestor ideal of I_j, so (ii) implies (i). Suppose $i \leqslant j$. We have

$$\begin{aligned}\dim I_{i+1} &= \dim I_i + \nu(I_{\leqslant i+1}) - \#\{\text{relations of } I \text{ in degrees } \leqslant i+1\}\\ \tau(I_i) &= \nu(I_{\leqslant i}) - \#\{\text{relations of } I \text{ in degrees } \leqslant i+1\},\end{aligned}$$

hence we have (ii) ⇔ (iii). The condition (iii) is evidently equivalent to (iv). We have shown (i)–(iv) equivalent.

Assuming (i), (v) is a consequence of Proposition 2.6, Eq. (2.14) and Theorem 1.10(ii). Assuming (v) we have that I has a generating set of degrees no greater than j, and for $i \leqslant j+1$,

$$\dim R_i - \dim R_1 \cdot I_{i-1} = \#\{\text{generators of degree } i\},$$

implying (ii). This completes the proof. □

Corollary 2.12. *The ideal $I \subset k[x, y]$ is an ancestor ideal if and only if the highest degree β_1 of any generator and the lowest degree β_2 of any relation satisfy $\beta_1 + 2 \leqslant \beta_2$. Then I is the ancestor ideal of I_j for each j satisfying $\beta_1 \leqslant j \leqslant \beta_2 - 2$.*

Proof. The corollary is immediate from (i) ⇔ (iii) in Proposition 2.11. □

Example 2.13. Let $H = (1, 2, 3, 3, 2, 1)$ and let $I = (x^3, y^4) \subset k[x, y]$. Then I is a complete intersection, with a single relation in degree 7. It follows from Corollary 2.12 that I is an ancestor ideal both for $I_4 = \langle x^4, x^3y, y^4\rangle$ and for I_5.

We will need the following well-known result [Mac1,I2].

Corollary 2.14. *Let $I \subset R = k[x, y]$ be an ideal satisfying $H(R/I) = T$, $\lim_{i\to\infty} T_i = c$ where $c = c_T > 0$. Then $I = f \cdot I'$ where the common factor f satisfies $\deg f = c$, and where R/I' is an Artinian quotient of Hilbert function $T : c$, where*

$$(T : c)_i = T_{i+c} - c. \tag{2.25}$$

Proof. Let $T_s = c$, $T_{s-1} > c$, and suppose $\mu = \mu(T) = \min\{i \mid T_i \neq i + 1\}$ be the order of any ideal I of R having Hilbert function $H(R/I) = T$ (so $I_\mu \neq 0$, $I_{\mu-1} = 0$). Then we have

$$\bar{I}_1 \subset \bar{I}_2 \subset \cdots \subset \bar{I}_i \subset \cdots \subset \bar{I}_s = (f), \quad f = \mathrm{GCD}(I_s). \tag{2.26}$$

Here $\bar{I}_s = (f)$ since evidently $\tau(I_s) = \operatorname{cod} I_s - \operatorname{cod} I_{s+1} + 1 = 1$, and we have $f \mid I$. The corollary follows. □

We turn now to characterizing the Hilbert functions of level algebras and the algebras $R/(V)$.

Lemma 2.15. *The Hilbert function N of a level algebra* $\mathrm{LA}(V)$ *determined by the vector subspace $V \subset R_j$, $\dim V = d$ satisfies*

$$\tau(V) \leqslant \min\{d, j + 2 - d\},\ N_j = j + 1 - d, N_i = 0 \quad \textit{for } i > j, \quad \textit{and}$$
$$e_{j+1}(N) = j + 1 - d \geqslant e_j(N) = \tau(V) - 1 \geqslant e_{j-1}(N) \geqslant \cdots. \tag{2.27}$$

The Hilbert function $T = H(R/(V))$ for the algebra $R/(V)$ determined by the vector subspace $V \subset R_j, \dim V = d$ satisfies

$$\tau(V) \leqslant \min\{d, j+2-d\}, T_j = j+1-d, T_i = i+1 \quad \text{for } i < j, \quad \text{and}$$
$$e_j(T) = d-1 \geqslant e_{j+1}(T) = \tau(V)-1 \geqslant e_{j+2}(T) \geqslant \cdots. \tag{2.28}$$

Proof. Immediate from the definitions of $\mathrm{LA}(V), \mathrm{GA}(V)$ and Proposition 2.6, Eq. (2.14). □

Definition 2.16. Let d, j be positive integers satisfying $d \leqslant j$. Let H be an acceptable O-sequence as in Definition 2.7. The *nose* N_H is the sequence

$$N_H = (H_0, \ldots, H_{j-1}, H_j = j+1-d, 0), \tag{2.29}$$

and the *tail* T_H (the Hilbert function is looking to the left!) is the sequence

$$T_H = (1, 2, \ldots, j, H_j = j+1-d, H_{j+1}, \ldots, H_i, \ldots). \tag{2.30}$$

A pair of sequences $(N, T), N = (1, \ldots, N_j, 0)$, $T = (1, 2, \ldots, j, T_j, T_{j+1}, \ldots)$ is *compatible* for (d, j), if $N_{j-1} - N_j = \tau - 1 = T_j - T_{j+1}$, and each of N, T can arise as above from acceptable O-sequences H, H': $N = N_H, T = T_{H'}$. For (N, T) compatible, we define $H(N, T)$ by

$$H(N,T) = \begin{cases} N_i & \text{for } i \leqslant j, \\ T_i & \text{for } i \geqslant j. \end{cases} \tag{2.31}$$

We let $\mathrm{LA}_N(d, j)$ parametrize all level algebras $\mathrm{LA}(V), V \subset R_j, \dim V = d$, as a subscheme of $\mathrm{Grass}(d, R_j)$. We define $\mathrm{GA}_T(d, j) \subset \mathrm{Grass}(d, R_j)$ similarly as the parameter variety for all graded algebras $\mathrm{GA}(V) = R/(V), V \subset R_j, \dim V = d$, having Hilbert function $H(\mathrm{GA}(V)) = T$. As we shall see, the maps $V \to \mathrm{LA}(V)$ and $V \to \mathrm{GA}(V)$ give open dense immersions from $\mathrm{LA}_N(d, j)$ to $G(N)$, the projective variety paremetrizing graded ideals I of Hilbert functions $H(R/I) = N$, and from $\mathrm{GA}_T(d, j)$ to $G(T)$ (Theorem 2.17(A)).

Remark. Suppose that H satisfies $H = H(\mathrm{Anc}(V))$; then $\mathrm{LA}(V)$, $\mathrm{GA}(V)$, respectively, have Hilbert functions N_H, T_H, respectively. Also, we have $H(N_H, T_H) = H$ in the sense of Eq. (2.31).

Recall that $\mathrm{Grass}_\tau(d, j)$ denotes the subfamily of $\mathrm{Grass}(d, R_j)$ parametrizing d-dimensional vector subspaces $V \subset R_j$ with $\tau(V) = \tau$. We will later show that $\mathrm{Grass}_\tau(d, j)$ is irreducible. We let $\mathrm{rem}(a, b) = b - \lfloor b/a \rfloor \cdot a$. For an integer τ satisfying $1 \leqslant \tau \leqslant \min(d, j+2-d)$, we define $H_\tau(d, j)$ as the Hilbert function corresponding to the pair of partitions $(P_\tau(d, j), Q_\tau(d, j))$ of $(d, j+1-d)$ for which P has at most one of its parts different from τ, Q has at most one part different from $\tau - 1$. Thus,

$$\begin{aligned} P_\tau(d,j) &= \bigl(\tau,\dots\tau,\operatorname{rem}(\tau,d)\bigr),\\ Q_\tau(d,j) &= \bigl(\tau-1,\dots,\tau-1,\operatorname{rem}(\tau-1,j+1-d)\bigr). \end{aligned} \tag{2.32}$$

Here $P_\tau(d,j)$ has $\lfloor d/\tau\rfloor$ parts of size τ, and if $\operatorname{rem}(\tau,d)\neq 0$ one further part; likewise the partition $Q_\tau(d,j)$ has $\lfloor (j+1-d)/(\tau-1)\rfloor$ parts of size $\tau-1$ and at most one further part. We have, letting $a=j+1-d$,

$$H_\tau(d,j)_i = \begin{cases} \min\{i+1, a+(\tau-1)(j-i)\} & \text{for } i\leqslant j,\\ \max\{0, a-(\tau-1)(i-j)\} & \text{for } i>j. \end{cases} \tag{2.33}$$

We now show our main result characterizing the Hilbert function strata of the three algebras attached to V. In each of Eqs. (2.35), (2.36), (2.38), (2.39), below the term on the far right has the same form as the terms in the sum enclosed in parentheses; we have broken out the single term for clarity, since, for example, $e_{j+1}(N)=j+2-d-\tau\neq e_{j+1}(H)=\tau-1$. In the equations below $e_i=E(H)_i=H_{i-1}-H_i$ throughout. We will show analogous equations for the codimensions of the strata in terms of the graded Betti numbers in Section 2.2, Theorem 2.24. Note that the dimension Eqs. (2.34)–(2.36) are written essentially in terms of the partitions P, Q which are determined by $E(H)$.

Theorem 2.17. *Let $r=2$, let k be an infinite field, and fix positive integers $d\leqslant j$. Let H be a proper acceptable O-sequence in the sense of Definition* 2.7. *Then*

(A) *Assume k is algebraically closed. Each of the schemes* $\operatorname{Grass}_H(d,j)$, $\operatorname{LA}_N(d,j)$, $\operatorname{GA}_T(d,j)$ *has an open cover by opens in affine spaces of the given dimension. Each such scheme is irreducible, rational and smooth. Each is an open dense subscheme of the corresponding scheme $G(H)$, $G(N)$, or $G(T)$ parametrizing all graded ideals of the given Hilbert function.*

(B) *Let* $\lim_{i\to\infty} H_i=c_H$. *The dimensions of* $\operatorname{Grass}_H(d,j)$, *and of the related varieties satisfy*

$$\dim \operatorname{Grass}_H(d,j) = c_H + \sum_{i\geqslant \mu(H)} (e_i+1)(e_{i+1}), \tag{2.34}$$

$$\dim \operatorname{LA}_N(d,j) = \left(\sum_{\mu(N)\leqslant i<j} (e_i+1)(e_{i+1})\right) + (e_j+1)(j+1-d), \tag{2.35}$$

$$\dim \operatorname{GA}_T(d,j) = c_T + \left(\sum_{i\geqslant j+1} (e_i+1)(e_{i+1})\right) + d(e_{j+1}). \tag{2.36}$$

(C) *The codimension of* $\operatorname{Grass}_H(d,j)$ *and of related varieties in* $\operatorname{Grass}(d,R_j)$ *satisfy*

$$\begin{aligned} \operatorname{cod}\operatorname{Grass}_H(d,j) &= \operatorname{cod}\operatorname{LA}_N(d,j)+\operatorname{cod}\operatorname{GA}_T(d,j)\\ &\quad - \operatorname{cod}\operatorname{Grass}_\tau(d,j), \end{aligned} \tag{2.37}$$

$$\operatorname{cod} \mathrm{LA}_N(d, j) = \left(\sum_{\mu(N) \leqslant i < j} (e_{i+1} - e_i)(i - N_{i-1}) \right) + (d - \tau)(j + 2 - d - \tau), \tag{2.38}$$

$$\operatorname{cod} \mathrm{GA}_T(d, j) = (2d - 2 - j)c_T + \left(\sum_{i \geqslant j+1} (e_i - e_{i+1})(T_{i+1}) \right) + (d - \tau)(j + 2 - d - \tau), \tag{2.39}$$

$$\begin{aligned} \operatorname{cod} \mathrm{Grass}_\tau(d, j) &= (\dim V - \tau)\bigl(\operatorname{cod} V - (\tau - 1)\bigr) \\ &= (d - \tau)(j + 2 - d - \tau). \end{aligned} \tag{2.40}$$

Proof. That each such H occurs as $H(R/\overline{V})$ for some such V is a consequence of Proposition 2.11(i) equivalent to (v), and Theorem 1.10(iii). That each scheme has a cover by opens in affine spaces of the given dimension, and the dimension formulas themselves also follow from Theorem 1.10, applied to the relevant Hilbert functions H, N, or T, respectively. In each case the schemes parametrize those ideals of the given Hilbert function having the minimum possible number of generators, hence when k is algebraically closed, they are by Theorem 1.10 open dense subschemes of the schemes $G(H)$, $G(N)$, or $G(T)$, respectively, that parametrize all graded ideals of the Hilbert function (not just those that are $\overline{V}$, $L(V)$, or (V), respectively with $V = I_j$). The codimension formulas are consequences of the dimension formulas, as we will now show. We begin by verifying (2.38), whose right side we denote by $L(N)$. Since for $I = \overline{V} \mid H(R/I) = H$ we have by Proposition 2.11(ii), (iii) there are no relations among the generators in degrees less or equal $j + 1$, we have

$$i - N_{i-1} = \dim I_{i-1} = \tau(I_{i-1}) + \tau(I_{i-2}) + \cdots = (e_{i-1} + 1) + (e_{i-2} + 1) + \cdots.$$

We have, noting that $\sum_{i<j}(e_i + 1) = \dim I_{j-1} = d - \tau$,

$$\begin{aligned} \dim \mathrm{LA}_N + L(N) &= \sum_{i<j} (e_{i+1} - e_i)\bigl((e_{i-1} + 1) + (e_{i-2} + 1) + \cdots\bigr) \\ &\quad + \sum_{i<j} (e_i + 1)e_{i+1} + (e_j + 1)(j + 1 - d) + (d - \tau)(j + 2 - d - \tau) \\ &= \sum_{i<j} e_j(e_i + 1) + (e_j + 1)(j + 1 - d) + (d - \tau)(j + 2 - d - \tau) \\ &= (\tau - 1)(d - \tau) + \tau(j + 1 - d) + (d - \tau)(j + 2 - d - \tau) \\ &= d(j + 1 - d) = \dim \mathrm{Grass}(d, R_j). \end{aligned}$$

It follows that $L(N) = \operatorname{cod} \mathrm{LA}(N)$, which is (2.38).

We now show (2.39), first when $c_T = \lim_{i \to \infty} T_i = 0$. Letting $L(T)$ denote the right side of (2.39), with the last term on the right included in the sum (here $e_j(T) = j - (j + 1 - d) = d - 1$), and noting that since $c_T = 0$, $T_{i+1} = e_{i+2} + e_{i+3} + \cdots$, we have in this case

$$\dim \mathrm{GA}_T(d,j)+L(T)=\sum_{i\geqslant j+2}\bigl(e_j(T)+1\bigr)\cdot e_i+d(e_{j+1})$$
$$=d(T_{j+1})+d(\tau-1)=d(j+1-d)=\dim \mathrm{Grass}(d,R_j),$$

thus we have $L(T)=\operatorname{cod}\mathrm{GA}_T(d,j)$ when $c_T=0$. When $c_T>0$, the formula results from a comparison with the same sums for $T'=T:c$ (see Corollary 2.14).

We now show the formula (2.40) for $\operatorname{cod}\mathrm{Grass}_\tau(d,j)$. Since $\mathrm{Grass}_\tau(d,j)=\bigcup_{\tau(H)=\tau}\mathrm{Grass}_H(d,j)$, we will need to use that its largest-dimensional stratum is $\mathrm{Grass}_{H_\tau}(d,j)$, where $H_\tau=H_\tau(d,j)$ is defined above in Eq. (2.33). Although this fact can be seen from Eq. (2.34), it is more readily apparent from (2.32) and the codimension formula (2.57) in terms of the partitions $(A,B)=(P^*,Q^*)$ of Theorem 2.23; it is also, of course, a consequence of the irreducibility of $\mathrm{Grass}_\tau(d,j)$, with $\mathrm{Grass}_{H_\tau}(d,j)$ being a dense open subscheme, shown below for k algebraically closed in Corollary 2.33. We have by (2.34) and (2.32),

$$\dim \mathrm{Grass}_{H_\tau}(d,j)=\sum_{i<j}(e_i+1)(e_{i+1})+\sum_{i\geqslant j}(e_i+1)e_{i+1}$$
$$=\sum_{i<j}(e_i+1)\cdot(\tau-1)+\tau\cdot\sum_{i\geqslant j}e_{i+1}$$
$$=(d-\tau)(\tau-1)+\tau(j+1-d)=\tau(j+2-\tau)-d,\quad (2.41)$$

whence we have $\operatorname{cod}\mathrm{Grass}_{H_\tau}(d,j)=(d-\tau)(j+1-d-(\tau-1))$, which is (2.40), with, as mentioned, the dense open subscheme $\mathrm{Grass}_{H_\tau}(d,j)$ in place of $\mathrm{Grass}_\tau(d,j)$.

We now show (2.37), which is equivalent to the analogous equation with dimension replacing codimension. We have evidently from (2.34), (2.35), and (2.36), since $e_j(H)=e_{j+1}(H)=\tau-1$,

$$\dim \mathrm{LA}_N(d,j)+\dim \mathrm{GA}_T(d,j)$$
$$=\dim \mathrm{Grass}_H(d,j)+(e_j+1)(j+1-d)+d(e_{j+1})-(e_j+1)(e_{j+1})$$
$$=\dim \mathrm{Grass}_H(d,j)+\tau\bigl(j+1-d-(\tau-1)\bigr)+d(\tau-1)$$
$$=\dim \mathrm{Grass}_H(d,j)+\dim \mathrm{Grass}_\tau(d,j),$$

using (2.41). This completes the proof of Theorem 2.17. □

Corollary 2.18. *Let d,j,τ be positive integers with $d\leqslant j$, and let H be an acceptable O-sequence in $\mathcal{H}(d,j)_\tau$. Let $N=N_H$, $T=T_H$ be the sequences of Eqs.* (2.29), (2.30) *or Definition* 2.16. *Then $\mathrm{LA}_N(d,j)$ and $\mathrm{GA}_T(d,j)$ intersect properly in $\mathrm{Grass}_\tau(d,j)$, $\tau=e_j+1$, and $\mathrm{LA}_N(d,j)\cap\mathrm{GA}_T(d,j)=\mathrm{Grass}_H(d,j)$.*

Theorem 2.19. *Let d,j be positive integers with $d\leqslant j$. Let (P,Q) be a pair of partitions satisfying* (i) *and* (ii) *of Lemma* 2.10.

(i) *The set of proper O-sequences H as in Eq.* (2.8) *that are acceptable for (d, j) as in Definition* 2.7*, is identical with $\mathcal{H}(d, j) = \mathcal{H}(d, j, 2)$, the set that occur as the Hilbert functions $H(\mathrm{Anc}(V))$ for some d-dimensional vector space $V \subset R_j$.*

(ii) *All proper O-sequences H satisfying the conditions of Corollary* 2.8 *occur as the Hilbert function of an ancestor algebra of a proper vector subspace $V \subset R_j$.*

(iii) *Fix $\tau = \tau(H)$. The pairs of partitions (P, Q) of $(d, j+1-d-c)$ where $c \leqslant j+1-d-\tau$, satisfying the condition of Lemma* 2.10(i) (*that P has at least one part τ and no larger parts, and Q has at least one part $\tau - 1$ and no larger parts*) *are exactly the pairs that occur as the partitions $P(H)$, $Q(H)$ for those Hilbert functions $H \in \mathcal{H}(d, j)$ satisfying $\tau = e_j + 1$ fixed and $c_H = c$.*

Proof. Corollary 2.18 is immediate from Theorem 2.17. Theorem 2.19(i) follows from Proposition 2.6 and (2.34): the lowest value for $\dim \mathrm{Grass}_H(d, j), H$ acceptable is one, which occurs only for $d = j, H = (1, 1, \ldots)$. Theorem 2.19(ii), (iii) follow from Theorem 2.19(i) and Lemma 2.10. $\square$

We now use our results to count the number of level algebra and related Hilbert functions, given (d, j). We first define the q-binomial series, a power series in q

$$\binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \frac{(q^{a+b}-1)(q^{a+b}-q)\cdots(q^{a+b}-q^{b-1})}{(q^b-1)(q^b-q)\cdots(q^b-q^{b-1})}. \tag{2.42}$$

Recall that the number $p(a, b, n)$ of partitions of n into at most b parts, each less or equal to a is given by the coefficient of q^n in the q-binomial series $\binom{\mathbf{a}+\mathbf{b}}{\mathbf{b}}$ [St2, Proposition 1.3.19]. We denote by $p(n)$ the number of partitions of n, and by $p_k(n)$ the number of partitions of n into exactly k parts (or, equivalently, partitions of n with a largest part equal to k). Evidently, there are $p(a-1, b-1, n-a-(b-1))$ partitions of n into exactly b parts, with largest part a.

Corollary 2.20. *Let d, j be positive integers with $d \leqslant j$. We assume $V \subset R_j$, $\dim V = d$.*

(A) *The level algebra Hilbert functions N of socle degree j with $N_j = j+1-d$, $\tau(I_j) = \tau$ correspond one to one as in* (2.21) *with the $p_\tau(d)$ partitions P of d with largest part τ. Here τ runs through all integers less or equal* $\min\{d, j+2-d\}$.

(B) *The level algebra Hilbert functions N of socle degree j with $N_j = j+1-d$, $\tau(I_j) = \tau$ having order $\mu(N) = \mu$ correspond one to one as in* (2.21) *with the $p(\tau-1, j-\mu, d-\tau-(j-\mu))$ partitions of d into exactly $j+1-\mu$ nonzero parts with largest part τ. There are $p(\tau, j+1-\mu, d)$ level algebra Hilbert functions N with $(\tau(N) \leqslant \tau, \mu(N) \geqslant \mu)$, and fixed (d, j).*

(C) *The Hilbert functions T for Artinian algebras $A = R/(V)$, $\tau(V) = \tau$ correspond one to one as in* (2.22) *to the $p_{\tau-1}(j+1-d)$ partitions Q of $j+1-d$ having largest part $\tau - 1$.*

(D) *The Hilbert functions T for Artinian algebras $A = R/(V)$, $\tau(V) = \tau$, where $T_{s-1} \neq 0$ but $T_s = 0$ correspond one to one as in* (2.22) *to the $p(\tau-1, s-j-1, j+1-d-\tau-(s-j-1))$ partitions of $j+1-d$ into $s-j$ parts, with largest part $\tau - 1$. There* *are* $p(\tau-1, s-j, j+1-d)$ *such Hilbert functions* T *with* $(\tau(T) \leqslant \tau, s(T) \leqslant s)$ *and fixed* (d, j).

(E) *There are* $p_\tau(d) \cdot p_{\tau-1}(j+1-d-c)$ *acceptable Hilbert functions* H *as in Definition* 2.7*, having* $\tau(H) = \tau, c_H = c$*. This is the subset of* $\mathcal{H}(d, j)$ *delimited in Theorem* 2.19(iii)*.*

Proof. The corollary follows immediately from Theorem 2.19, and Lemma 2.10. □

2.2. Minimal resolutions of the three algebras of V*, and partitions*

In this section we relate the sets of graded Betti numbers of the ancestor algebra $\mathrm{Anc}(V)$, the level algebra $\mathrm{LA}(V)$, and the usual graded algebra $\mathrm{GA}(V)$ determined by a vector space of degree-j homogeneous elements of R. These depend on several partitions A, B derived from the Hilbert function $H(\mathrm{Anc}(V))$—from the generator and relation degrees of the ancestor ideal $\overline{V}$. We also give further codimension formulas for the Hilbert function strata, in terms of the graded Betti numbers, or natural invariants of the partitions. The following results were not in the original preprint [I1]. They are inspired by the special case (2.59), a formula for $\operatorname{cod}\mathrm{GA}_T(d, j)$ in [GhISa], which arose from a geometric tradition in studying the restricted tangent bundle from projective space to an embedded rational curve (see also [Ra,Ve]). We will suppose that $V \subset R_j$ satisfies $H(R/\overline{V}) = H$; unless otherwise stated we will suppose also that $\lim_{i\to\infty} H_i = 0$. Then, as we shall see in Lemma 2.23, the ancestor algebra $\mathrm{Anc}(V) = R/\overline{V}$, the algebra $\mathrm{GA}(V) = R/(V)$ and the level algebra $\mathrm{LA}(V)$ determined by V have graded Betti numbers given by certain sequences/partitions A, B as follows,

$$0 \to \sum_{i=1}^{\tau-1} R(-j-1-b_i) \to \sum_{i=1}^{\tau} R(-j-1+a_i) \to R \to R/\overline{V} \to 0, \tag{2.43}$$

$$0 \to R(-j-2)^{j+1-d} \to \sum_{i=1}^{\tau} R(-j-1+a_i) \oplus R(-j-1)^{j+2-d-\tau} \to R \to \mathrm{LA}(V) \to 0, \tag{2.44}$$

$$0 \to \sum_{i=1}^{\tau-1} R(-j-1-b_i) \oplus R(-j-1)^{d-\tau} \to R(-j)^d \to R \to R/(V) \to 0, \tag{2.45}$$

where we assume that the sequences $A = (a_1, \dots, a_\tau)$ and $B = (b_1, \dots, b_{\tau-1})$ defined by (2.43) are listed in decreasing order $a_1 \geqslant \cdots \geqslant a_\tau$ and $b_1 \geqslant \cdots \geqslant b_{\tau-1}$.

Definition 2.21. When $\lim_{i\to\infty} H_i = 0$, we define partitions A, B given V by (2.43); we will show that they depend only on H, and evidently they are the same that occur in (2.44) and (2.45) (see Lemma 2.23). By $A + \underline{1}$ we mean the partition whose parts are $A + \underline{1} = (a_1 + 1, a_2 + 1, \dots)$. We denote by C the partition of $j+2$ having $j+2-d$ parts given by $(A+\underline{1}) \cup [1^{j+2-d-\tau}]$, namely $A + \underline{1}$ with $j+2-d-\tau$ parts of size one adjoined;

and we denote by D the partition of j having $d-1$ parts given by $(B+\underline{1})\cup[1^{d-\tau}]$, namely $B+\underline{1}$ with $d-\tau$ ones adjoined.

When $\lim_{i\to\infty} H_i = c_H \geqslant 0$ we define A, B from the minimal resolution of $\overline{V:f}$, where $f = \mathrm{GCD}(V)$; then A, B depend only on $H : c_H$ (see (2.25)). We define C, D in this case as above from A, B; here C again partitions $j+2$, but D partitions $j+2-d-\tau-c$.

Evidently, the generator degrees of the ideal $L(V)$ defining $\mathrm{LA}(V)$ in (2.44) are $\underline{j+2} - C$ and the relation degrees of (V) in (2.45) are $\underline{j} + D$. We have chosen A and B, then C and D in a symmetric fashion so that they partition integers depending only on d and j; this allows application of Lemma 2.27 later. As we shall see, the partitions A, C depend only on $N = N_H$, determined by $H_{\leqslant j}$; and B, D depend only on $T = T_H$, determined by $H_{\geqslant j}$ (see Definition 2.16). To describe this dependence simply, we use the dual partition.

Definition 2.22. Let $A = (a_1, \ldots, a_k),\ a_1 \geqslant a_2 \geqslant \cdots$ be a partition of $a = \sum a_i$ into k nonnegative parts (some may be zero). Recall that the *Ferrers graph* $\mathfrak{F}(A)$ of A has k rows, the ith row of length a_i. We denote by $A^* = (a_1^*, a_2^*, \ldots)$ the *dual partition* of a, whose Ferrers graph is obtained by switching rows and columns in the Ferrers graph $\mathfrak{F}(A)$. Here also, a_i^* is the number of parts of A of length greater or equal i.

Lemma 2.23. *Let d, j be positive integers satisfying $d \leqslant j$, and let H be an acceptable O-sequence as in Definition* 2.7, *and suppose that $c_H = \lim_{i\to\infty} H_i = 0$. Then the algebras* $\mathrm{Anc}(V), \mathrm{LA}(V)$, *and $R/(V)$ have minimal resolutions whose graded Betti numbers are given by* (2.43)–(2.45). *We have*

$$\sum_{1=1}^{\tau} a_i = d; \tag{2.46}$$

A satisfies $a_i \geqslant 1$, and A has dual partition $A^* = P = (\tau, \tau(R_{-1}\cdot V), \tau(R_{-2}V), \ldots)$ of d, and

$$a_i^* = \tau(R_{-i+1}\cdot V) = e_{j+1-i}(H) + 1. \tag{2.47}$$

Also

$$\sum_{i=1}^{\tau-1} b_i = j+1-d; \tag{2.48}$$

B satisfies $b_i \geqslant 1$, and B has dual $B^* = Q = (e_{j+1}(H), \ldots)$ of $j+1-d$, and $b_i^* = e_{j+i}$. We have for $i \geqslant 0$

$$\dim I_{j-i} = \sum_u |a_u - i|^+, \tag{2.49}$$

$$H_{j+i} = \sum_u |b_u - i|^+. \tag{2.50}$$

Likewise, the partition C has dual the partition $(E(N)_{j+1}+1, E(N)_j+1, \dots)$ of $j+2$

$$C^* = \bigl(j+2-d, \tau(V), \tau(R_{-1}V), \tau(R_{-2}V), \dots\bigr), \tag{2.51}$$

and D has dual the partition $E(T)_{\geqslant j}$ of j

$$D^* = (d-1, e_{j+1}, e_{j+2} \dots). \tag{2.52}$$

When $\lim_{i\to\infty} H_i = c_H > 0$, *then A from Definition* 2.21 *satisfies all the statements above, including* (2.46), (2.47), (2.49); *and B is a partition of $j+1-d-c_H$ into $\tau-1$ parts. Also, B^* satisfies the same condition above, and $H_{j+i} = c_H + \sum_u |b_u - i|^+$ in place of* (2.50). *Also, C^* satisfies* (2.51), *and D^* satisfies* (2.52).

Proof. We first assume $\lim_{i\to\infty} H_i = 0$. The definition of $\overline{V}$ shows that it is generated in degrees less or equal j, and Proposition 2.11 shows that $\overline{V}$ has no relations in degrees less or equal $j+1$. Thus, Eq. (2.43) defines ordinary partitions A and B, with nonzero parts. Given the definition of A, B in (2.43), the graded Betti numbers shown in (2.44), (2.45) for the level algebra $\mathrm{LA}(V)$ and the algebra $\mathrm{GA}(V) = R/(V)$ follow immediately from the definitions of these algebras from $\overline{V}$ in Definition 1.1, and the relations among them given in Remark 1.2. For example, since the ideal $L(V)$ defining the level algebra $\mathrm{LA}(V)$ satisfies $L(V) = \overline{V} + M^{j+1}$ one obtains $L(V)$ it by adding $H_{j+1} = (j+1-d-(\tau-1)) = j+2-d-\tau$ generators of degree $j+1$, and evidently all the relations are in degree $j+2$, since the socle of $R/L(V)$ lies solely in degree j; this shows (2.44).

Proposition 2.6 shows that for $i \geqslant 0$, $\tau(R_{-i}\cdot V) = e_{j-i}(H)+1$, so $\tau(R_{-i}\cdot V)$ depends only on initial portion N_H of H. We have from Proposition 2.11(iii), and the definition of A^* that for $i \geqslant 1$,

$$\tau(R_{-i+1}V) = \#\{u \mid a_u \geqslant i\} = a_i^*.$$

It follows from (2.12) that $\sum a_i = \sum_{i=1} a_i^* = d$, which is (2.46).

Concerning B, we have from (2.43), that for $i \geqslant 0$

$$\begin{aligned} H_{j+i} &= H_j - (\tau-1)i + \sum_{u \mid b_u \leqslant i-1} (i+1-b_u); \quad \text{thus} \\ e_{j+i} &= \tau - 1 - \sum_{u \mid b_u \leqslant i-1} (-1) = \tau - 1 - \bigl(\#\{\text{relations}\} - b_i^*\bigr) \\ &= b_i^*. \end{aligned}$$

Thus we have

$$\sum b_i = \sum b_i^* = \sum_{u\geqslant 1} e_{j+u} = H_j = j+1-d,$$

which is (2.48). It remains to show (2.49) and (2.50). We have for $i \geqslant 0$,

$$\begin{aligned} H_{j+i} &= H_j - (e_{j+1} + \cdots + e_{j+i}) \\ &= j + 1 - d - \left(b_1^* + \cdots + b_i^*\right) = b_{i+1}^* + b_{i+2}^* + \cdots \\ &= \sum |b_u - i|^+, \end{aligned} \tag{2.53}$$

which is (2.50). Since $\overline{V}$ has no relations in degrees less or equal $j+1$, we have for $i \geqslant 0$,

$$\dim I_{j-i} = \sum_{a_u \geqslant i+1} (a_u - i) = \sum_{u=1}^{\tau} |a_u - i|^+,$$

which is (2.49). This completes the proof in the case $\lim_{i\to\infty} H_i = 0$.

When $\lim_{i\to\infty} H_i = c_H > 0$, the assertions at the end of the lemma follow from Definition 2.21 of A, B in this case that uses $V : \mathrm{GCD}(V)$, Corollary 2.14 and the lemma for $V : \mathrm{GCD}(V)$. $\square$

We denote by $|n|^+$ the integer n if $n \geqslant 0$, or 0 otherwise. We will denote by $\underline{n}$ the sequence $(n, n, \ldots)$ of appropriate length. For a partition $A = (a_1, \ldots), a_1 \geqslant a_2 \geqslant \cdots$ we denote by $\ell(A)$ the sum

$$\ell(A) = \sum_{u \leqslant v} |a_u - a_v - 1|^+. \tag{2.54}$$

Recall from (2.40) that $\operatorname{cod} \mathrm{Grass}_\tau(d, j)$ in $\mathrm{Grass}(d, R_j)$ satisfies

$$\operatorname{cod}\bigl(\mathrm{Grass}_\tau(d, j)\bigr) = (d - \tau)(j + 2 - d - \tau) = (\dim V - \tau)\bigl(\operatorname{cod} V - (\tau - 1)\bigr),$$

for any V satisfying $\tau(V) = \tau$. This is a term in Eq. (2.60).

Theorem 2.24. *Let d, j be positive integers with $d \leqslant j$. Let H be an acceptable O-sequence, and let $\lim_{i\to\infty} H_i = c_H$, and let $N = N_H, T = T_H$ be the sequences of Definition* 2.16*, where $c_T = c_H$. The codimensions of the families* $\mathrm{LA}_N(d, j)$, $\mathrm{GA}_T(d, j)$, *and* $\mathrm{Grass}_H(d, j)$ *in* $\mathrm{Grass}_\tau(d, j)$ *satisfy*

$$\operatorname{cod}_\tau \mathrm{LA}_N = \ell(A), \tag{2.55}$$

$$\operatorname{cod}_\tau \mathrm{GA}_T = \ell(B) + (d - 1)c_T, \tag{2.56}$$

$$\operatorname{cod}_\tau \mathrm{Grass}_H(d, j) = \ell(A) + \ell(B) + (d - 1)c_T. \tag{2.57}$$

The codimensions of these families in $\mathrm{Grass}(d, R_j)$ *satisfy*

$$\operatorname{cod} \mathrm{LA}_N = \ell(C), \tag{2.58}$$

$$\operatorname{cod} \mathrm{GA}_T = \ell(D) + (d - 1)c_T, \tag{2.59}$$

$$\operatorname{cod} \mathrm{Grass}_H(d, j) = \ell(C) + \ell(D) + (d - 1)c_H - (d - \tau)(j + 2 - d - \tau) \tag{2.60}$$

$$= \ell(C) + \ell(B) + (d - 1)c_H. \tag{2.61}$$

Proof. We first note that (2.55) $\Leftrightarrow$ (2.61), and (2.56) $\Leftrightarrow$ (2.59); evidently (2.57) is a consequence of (2.55) and (2.56), and similarly for (2.60). Assume first that $c_H = 0$. We have

$$\begin{aligned}\ell(C) - \ell(A) &= \Bigl(\sum(a_i)\Bigr)(j+2-d-\tau)\\ &= (d-\tau)(j+2-d-\tau) = \operatorname{cod}\operatorname{Grass}_\tau(d,j).\end{aligned}$$

Likewise,

$$\begin{aligned}\ell(D) - \ell(B) &= (d-\tau)\left(\sum_{i=1}^{\tau-1}(b_i-2)\right)\\ &= (d-\tau)\bigl((j-(d-\tau)-2(\tau-1)\bigr)\\ &= \operatorname{cod}\operatorname{Grass}_\tau(d,j).\end{aligned}$$

We now show (2.56) when $c_H = 0$. Since $\lim_{i\to\infty} T_i = 0$, by Theorem 2.17, Eq. (2.39) we have

$$\operatorname{cod}\mathrm{GA}_T = \sum_{i\geqslant j+1}(e_i - e_{i+1})(T_{i+1}) + (d-1-e_{j+1})(T_{j+1}),$$

whence, subtracting $\operatorname{cod}\operatorname{Grass}_\tau(d,j) = (d-\tau)T_{j+1}$ and noting that we specify $E(H)$ below, as $e_j(H)$ is different from $e_j(T)$, we find,

$$\begin{aligned}\operatorname{cod}_\tau \mathrm{GA}_T &= \sum_{i\geqslant j+1}(e_i - e_{i+1})(T_{i+1}) + (d-1-e_{j+1})(T_{j+1}) - (d-\tau)T_{j+1}\\ &= \sum_{i\geqslant j}\bigl(e_i(H) - e_{i+1}(H)\bigr)(H_{i+1}) = \sum_{u\geqslant 0}(e_{j+u} - e_{j+u+1})H_{j+u+1}\\ &= \sum_{u\geqslant 0}\bigl(b_u^* - b_{u+1}^*\bigr)H_{j+u+1} \quad \text{by Lemma 2.23,}\\ &= \sum_{u=1}^{\tau-1} H_{j+b_u+1}\\ &= \ell(B) \quad \text{by (2.50).}\end{aligned}$$

We now show (2.55). By Theorem 2.17, Eq. (2.38), taking into account that the last term on the right is $\operatorname{cod}\operatorname{Grass}_\tau(d,j)$, and by (2.47) we have

$$\operatorname{cod}_\tau \mathrm{LA}(N) = \sum_{\mu(N)\leqslant u<j}(e_{u+1} - e_u)(\dim I_{u-1}) = \sum_{1\leqslant i}(e_{j-(i-1)} - e_{j-i})(\dim I_{j-(i+1)})$$

$$= \sum_{1 \leqslant i} \big(a_i^* - a_{i+1}^*\big) \left(\sum_u |a_u - (i+1)|^+ \right) \quad \text{by Lemma 2.23 and (2.49)}$$
$$= \sum (\#\{a_v = i\}) \left(\sum_u |a_u - i - 1|^+ \right)$$
$$= \ell(A).$$

The adjustment of adding $(d-1)c_H$ for the case $\lim_{i\to\infty} H_i = c_H$ comes from a comparison with the Hilbert function T': $T_i' = T_{i+c} - c$, $c = c_H$. The partitions B, D are the same for T and for T', and $\dim \mathrm{GA}(T) = c + \dim \mathrm{GA}(T')$, so the codimension of $\mathrm{GA}(T)$ in $\mathrm{Grass}(d, R_j)$ satisfies

$$\begin{aligned} \operatorname{cod} \mathrm{GA}(T) &= \operatorname{cod} \mathrm{GA}(T') + \dim \mathrm{Grass}(d, R_j) - \dim \mathrm{Grass}(d, R_{j-c}) - c \\ &= \ell(D) + (d-1)c_H. \end{aligned}$$

This completes the proof. □

Example 2.25. We take $(d, j) = (9, 14)$ and $\tau = 4$, then

$$\dim \mathrm{Grass}(9, R_{14}) = \dim \mathrm{Grass}(9, 15) = 9 \cdot 6 = 54,$$

and

$$\operatorname{cod} \mathrm{Grass}_4(9, 14) = (9-4)\big(6 - (4-1)\big) = 15,$$

so $\dim \mathrm{Grass}_4(9, 14) = 39$. Consider

$$H = (1, \ldots, 12, 11, 9, 6, 3, 0) \quad \text{with } H_{14} = 6.$$

Here the sequence

$$A^* = \big(\tau, \tau(R_{-1} \cdot V), \tau(R_{-2} \cdot V), \ldots\big) = (\tau, e_{13} + 1, e_{12} + 1) = (4, 3, 2),$$

whose dual partition is $A = (3, 3, 2, 1)$, with $\ell(A) = 2$ while $B^* = (2, 2, 2)$, $B = (3, 3)$, for which $\ell(B) = 0$. By (2.43) the generator degrees of $\overline{V}$ are $(j + 1 - a_1, j + 1 - a_2, \ldots) = (\underline{j+1} - A)$. Here the generator degrees are $(\underline{15} - A) = (15 - 3, 15 - 3, 15 - 2, 15 - 1) = (12, 12, 13, 14)$. The codimension of $\mathrm{Grass}_H(9, 14)$ in $\mathrm{Grass}_4(9, 14)$ is by Eq. (2.57) $\ell(A) + \ell(B) = 2 + 0 = 2$, so $\dim \mathrm{Grass}_H(9, 14) = 39 - 2 = 37$. The formula (2.34) that $\dim \mathrm{Grass}_H(9, 14) = \sum (e_i + 1)(e_{i+1})$ when applied to $E(H)_{\geqslant 13} = (1, 2, 3, 3, 3)$ also gives 37. Here the partition $C = (4, 4, 3, 2, 1, 1, 1)$ and $\ell(C) = 17$, and $\operatorname{cod}(\mathrm{Grass}_H(9, 14)) = \ell(C) + \ell(B) = 17$ in $\mathrm{Grass}(9, R_{14})$ by (2.61).

Consider now $H' = (1, \ldots, 12, 11, 9, 6, 3, 2, 1)$. Here $A' = A$, but $B' = (4, 1, 1)$, the dual partition to $(e_{15}, \ldots) = (3, 1, 1, 1)$, $\ell(B') = 4$, and we have $\operatorname{cod}_4 \mathrm{Grass}_{H'}(9, 14) = \ell(A') + \ell(B') = 6$ in $\mathrm{Grass}_4(9, 14)$, giving $\dim \mathrm{Grass}_{H'}(9, 14) = 33$.

2.3. Closure of the Hilbert function strata

We now determine the Zariski closure of $\mathrm{Grass}_H(d, j)$ when $r = 2$, and we show that the family $G(H)$ of graded algebra quotients of A having Hilbert function H is a natural desingularization of $\overline{\mathrm{Grass}_H(d, j)}$ (Theorem 2.32). This is one of our main results, and certainly the deepest.

We show that the closure of a stratum $\mathrm{Grass}_H(d, j)$ is the union of the more special strata $\mathrm{Grass}_{H'}(d, j)$, for $H' \leqslant_{\mathcal{P}} H$, where $\mathcal{P}$ is the partial order on acceptable sequences given in Definition 1.14. Evidently the partial order $\mathcal{P}$ determines related partial orders on the sequences N possible for level algebras, and to the sequences T possible for graded ideals (V). For the case $r = 2$ we interpret these latter partial orders as majorization partial orders on sets of partitions (Lemma 2.28). This result was suggested by an application to the restricted tangent bundle in [GhISa]. We show that the partially ordered set $\mathcal{H}(d, j)$ of acceptable Hilbert functions under the partial order $\mathcal{P}$—the same order as that determined by Zariski closure of the varieties $\mathrm{Grass}_H(d, j)$—is equivalent to a partially ordered set $\mathcal{P}A(d, j)$ of certain pairs of partitions, under the product of majorization partial orders (Theorem 2.35).

The proof of our main result depends on a key construction. Suppose that we are given two acceptable Hilbert functions $H, H' \in \mathcal{H}(d, j)$, with $H' \geqslant H$ (more special) in the partial order $\mathcal{P}(d, j)$, and let V' be a point of $\mathrm{Grass}_{H'}(d, j)$. We build a graded ideal I of Hilbert function H, that is related as in (1.10) to the ancestor ideal $I' = \overline{V}'$ (Lemma 2.30). This ideal I determines a point of $G(H)$ lying over the given point V' of $\mathrm{Grass}_{H'}(d, j)$ (Theorem 2.32(B)).

Definition 2.26. The *length* $|D|$ of a partition D is the sum of its parts. We recall the *majorization* partial order on partitions (see [GreK]). Let D, D' be two partitions $D = (d_1, d_2, \ldots, d_s) \mid d_1 \geqslant d_2 \geqslant \cdots$ and $D' = (d'_1, d'_2, \ldots, d'_{s'}) \mid d'_1 \geqslant d'_2 \geqslant \cdots$. We say $D' \geqslant D$ if $|D'| \geqslant |D|$ and

$$\sum_{u \leqslant i} d'_u \geqslant \sum_{u \leqslant i} d_u \quad \text{for all } i \mid 1 \leqslant i \leqslant \min\{s, s'\}. \tag{2.62}$$

Let D have r_i parts of size $v_i, v_1 > v_2 > \cdots > v_k$. We define for each $s, 1 \leqslant s \leqslant k$ the partition D_s with r_i parts of size $v_i, 1 \leqslant i \leqslant s$, and no other parts. The *polygon* of D is the convex graph with vertices $(0, 0)$ and

$$\left(\sum_{i=1}^{s} r_i, \sum_{i=1}^{s} r_i v_i\right), \quad 1 \leqslant s \leqslant k, \tag{2.63}$$

the height of the sth vertex being the length $|D_s|$ of D_s. We define the Harder–Narasimham partial order [HN] on partitions having the same number of parts, by $D' \geqslant_{HN} D$ if and only if the polygon of D' is never below the polygon of D.

The Harder–Narasimham order as stated above is a special case for bundles of the form $\bigoplus \mathcal{O}_{\mathbb{P}^1}(v_i)^{r_i} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$ over $\mathbb{P}^1$ of an order defined more generally by Harder–

Narasimham (see [HN]). This is relevant since the partition C corresponds to the generator degrees of the ideal $L(V)$ defining the level algebra $\mathrm{LA}(V)$, and D corresponds to the relation degrees of the ideal (V) determining $\mathrm{GA}(V)$. The latter corresponds to the decomposition into a direct sum of line bundles of the "restricted tangent bundle" to the rational curve X in $\mathbb{P}^{r-1}$ determined by V, studied in [GhISa,Ra,Ve]; the former corresponds to the decomposition of another natural bundle over X, of rank $j+2-d$. It is a general result that specialization in a family $\mathcal{V}(t), t \neq t_0$ of vector bundles having fixed Harder–Narasimham polygon over X yields a bundle $V(t_0)$ of equal or higher Harder–Narasimham polygon [BrPV]. Both L. Ramella and F. Ghione et al. show a converse for the restricted tangent bundle, related to Theorem 2.32(A) for the closure of $\mathrm{GA}_T(d, j)$.

We need a preparatory result, before giving some equivalent versions of the partial order $\mathcal{P}(d, j)$.

Lemma 2.27. *If D, D' are two partitions of the same integer n, then*

$$D' \geqslant D \quad \Leftrightarrow \quad D'^* \leqslant D^*. \tag{2.64}$$

Proof. It suffices to consider adjacent partitions $D' > D$ in the partial order: then D' is obtained from D by increasing a part of D by one and decreasing the next smaller-or-equal block by one. A basic case is $D = (d_1, \dots, d_{s+1}) = (a, 1, \dots, 1)$ and $D' = (d'_1, \dots, d'_s) = (a+1, 1, \dots, 1)$. Then $D^* = (s+1, 1, \dots, 1)$ with $a-1$ ones, and $D'^* = (s, 1, \dots, 1)$ with a ones, whence we have $D'^* < D$. The general case has $s+1$ relevant parts for D, $(d_i, \dots, d_{i+s}) = (k+a, k+1, \dots, k+1)$ with $d_{i-1} > d_i$, and $s+1$ relevant parts for D', $(d'_{i+1}, \dots, d'_{i+s}) = (k+a+1, k+1, \dots, k+1, k)$; then D^* has relevant parts $(d^*_{k+1}, \dots, d^*_{k+a+1}) = (i+s, i+1, \dots, i+1, i)$ and D'^* has corresponding parts $(i+s-1, i+1, \dots, i+1, i+1)$, whence $D'^* < D^*$. □

We say a Hilbert function sequence $T' \geqslant T$ if for each i, $T'_i \geqslant T_i$. Recall from Definition 1.14 the partial order $\mathcal{P} = \mathcal{P}(d, j)$ on $\mathcal{H}(d, j)$:

$$H' \geqslant_{\mathcal{P}} H \quad \Leftrightarrow \quad H'_i \leqslant H_i \quad \text{for } i \leqslant j \text{ and } H'_i \geqslant H_i \text{ for } i \geqslant j. \tag{2.65}$$

Recall from Definiton 2.16 that $(N_H)_i = H_i$ for $i \leqslant j$ and 0 otherwise, and $(T_H)_i = H_i$ for $i \geqslant j$ and $(T_H)_i = i+1$ for $i < j$. In terms of the pair N_H, T_H we thus have

$$H' \geqslant_{\mathcal{P}} H \quad \Leftrightarrow \quad N' \leqslant N \quad \text{and} \quad T' \geqslant T,$$

where $N' \leqslant N$ and $T' \geqslant T$ in the termwise partial order on sequences.

We now determine the analogues of the partial order $\mathcal{P}(d, j)$, for the pairs of partitions (P, Q) from Definition 2.9, and the pairs (A, B) or (C, D) from Definition 2.21. In the lemma below $H', N', A', B', \dots$ are more special than $H, N, A, B, \dots,$ as we shall show in Theorem 2.32. The implications $T' \geqslant T \Leftrightarrow D' \geqslant D \Leftrightarrow D(T') \geqslant D(T)$ from Lemma 2.28(B) are shown for $c(T) = c(T') = 0$ in [GhISa]. Recall that we showed $P = A^*$ and $Q = B^*$ in Lemma 2.23.

Lemma 2.28. *We fix positive integers d, j with $d \leqslant j$. We treat separately the Hilbert functions for the level algebra* $\mathrm{LA}(V)$, *graded algebra* $\mathrm{GA}(V) = R/(V)$ *and ancestor algebra* $\mathrm{Anc}(V)$.

(A) *The following are equivalent*:
 (i) $N' \leqslant N$ (*note, N' is more special!*),
 (ii) $A(N') \geqslant A(N)$, *or equivalently* $C(N') \geqslant C(N)$,
 (iii) $P(N') \leqslant P(N)$ (*i.e.,* $A'^* \leqslant A^*$), *or equivalently* $C(N') \geqslant_{HN} C(N)$.
(B) *The following are equivalent*;
 (i) $T' \geqslant T$ (*note, T' is more special!*),
 (ii) (*only when* $c(T) = c(T')$) $B(T') \geqslant B(T)$, *or, equivalently* $D(T') \geqslant D(T)$,
 (iii) $Q(T') \leqslant Q(T)$ (*i.e.,* $B'^* \leqslant B^*$), *or equivalently* $D(T') \geqslant_{HN} D(T)$.
(C) *The following are equivalent*;
 (i) $H' \geqslant_{\mathcal{P}} H$; *meaning both* $N'_H \leqslant N_H$ *and* $T'_H \geqslant T_H$,
 (ii) $P(H') \leqslant P(H)$ *and* $Q(H') \leqslant Q(H)$, (*i.e., both* $A'^* \leqslant A^*$ *and* $B'^* \leqslant B^*$),
 (iii) (*only when* $c_H = c_{H'}$) $A(H') \geqslant A(H)$ *and* $B(H') \geqslant B(H)$,
 (iv) (*only when* $c_H = c_{H'}$) $C(H') \geqslant_{HN} C(H)$ *and* $D(H') \geqslant_{HN} D(H)$.

Proof. We first show (A.i) $\Leftrightarrow$ (A.ii) $\Leftrightarrow$ (A.iii) and (B.i) $\Leftrightarrow$ (B.ii) $\Leftrightarrow$ (B.iii). From Eq. (2.47) that $a_i^* = e_{j+1-i}(H) + 1$ we have for $i \geqslant 1$

$$H_{j-i} = j + 1 - d + \left(a_1^* - 1\right) + \cdots + \left(a_i^* - 1\right) = j + 1 - d - i + \sum_{u=1}^{i} a_u^*, \quad (2.66)$$

whence we have N_H satisfies, using (2.64)

$$N_{H'} \leqslant N_H \quad \Leftrightarrow \quad A^*(N') \leqslant A^*(N) \quad \Leftrightarrow \quad A(N') \geqslant A(N). \quad (2.67)$$

Since $A' \geqslant A \Rightarrow \tau' = a'^*_1 \leqslant a^*_1 = \tau$, we have $C' = \underline{1} + A' \cup 1^{(j+2-d-\tau)'} \geqslant C = \underline{1} + A \cup 1^{(j+2-d-\tau)}$. From Lemma 2.23 we have that $b_i^* = e_{j+i}$, and as in (2.53)

$$H_{j+i} = j + 1 - d - \sum_{u=1}^{i} b_{i-1}^*,$$

whence we have using (2.64)

$$T' \geqslant T \quad \Leftrightarrow \quad B'^* \leqslant B^* \quad \Leftrightarrow \quad B' \geqslant B. \quad (2.68)$$

This completes the proof of the lemma except for the equivalences involving $\geqslant_{HN}$, which we now show. Note that for the partitons C or D both the number of parts and sum are fixed by the triple (d, j, τ). That (C.iii) $\Rightarrow$ (C.iv) follows, since, considering D, the vertices of the polygon of D are a subset of the vertices of the graph of the sum function $\sum D$ of D, used in comparing D and D': thus $D' \geqslant D \Rightarrow D' \geqslant_{HN} D$. The converse follows from the extremality of the vertices of the graph of $\sum D$ chosen as vertices of the Harder–Narasimham polygon. □

Example 2.29. $\mathcal{P}(d, j)$ is not a simple order on $\mathcal{H}(d, j)$.

(A) Let $d = 3$, $j = 5$, so $H_5 = j + 1 - d = 3$. Let $H = (1, 2, 3, 4, 4, 3, 2, 1, 0)$, where $\tau = 1$, and $\mu(H) = 4$, $H' = (1, 2, 3, 4, 5, 3, 1, 1, 1, \ldots)$ where $\tau = 2$ and $\mu(H') = 5$. Then H and H' are incomparable in the order $\mathcal{P}(3, 5)$ since $H_6 > H'_6$ but $H_8 < H'_8$. Neither stratum is in the Zariski closure of the other. The two strata are *geometrically incomparable* in the sense that no element of either stratum can be in the closure of a subfamily of the other stratum, by Corollary 1.16. This example essentially involves just the *tail* of H, namely $T(V) = H(R/(V))$, with (V) the ideal generated by V (see Definition 2.16).

(B) We give an example of similar behavior for the level algebra strata $\mathrm{LA}_N(d, j)$—the family of level algebras of socle degree j and type d having Hilbert function N. Here N is the *nose* of H as in Definition 2.16. To create the example, we begin with two partitions P: $10 = 4 + 2 + 2 + 2$ and P': $10 = 3 + 3 + 3 + 1$, that are incomparable in the *majorization* partial order of Definition 2.26. Thus, their associated sum sequences $\sum P = (4, 6, 8, 10)$, $\sum P' = (3, 6, 9, 10)$ are incomparable in the termwise order on sequences. By Definition 2.9 the corresponding sequences $E = \Delta N$, $E' = \Delta(N')$ are $(3, 1, 1, 1)$ and $2, 2, 2, 0$, respectively, and by Lemma 2.10(i) the dimension d satisfies $d = |P| = 10$. By (2.17) the simplest such case satisfies $j + 1 - d = p_1 - 1 = 4 - 1 = 3$, where p_1 is the largest part of P, so we have $(d, j) = (10, 12)$, $\mu(N) = \mu(N') = 9$, $N = (1, 2, \ldots, 8, 9, 8, 7, 6, 3, 0)$ and $N' = (1, 2, \ldots, 8, 9, 9, 7, 5, 3, 0)$. Thus, N and N' are incomparable in the partial order $\mathcal{P}_N(10, 12)$ on the set of nose sequences $\{N_H \mid H \in \mathcal{H}(10, 12)\}$ induced from the partial order $\mathcal{P}(10, 12)$ on acceptable O-sequences H. Again Corollary 1.16 implies that $\mathrm{LA}_N(10, 12)$ and $\mathrm{LA}_{N'}(10, 12)$ are *geometrically incomparable* in the sense that no subfamily of either stratum can have as limit a space V in the other stratum. This example illustrates (Lemma 2.28(A)).

The following lemma is the crux of the proof that the morphism $\pi : G(H) \to \overline{\mathrm{Grass}_H(d, j)}$ is surjective (Theorem 2.32). The proof we give is basically that of the original preprint, but we have supplied further details and made an improvement. Note that although the Hilbert functions H, H' that occur are acceptable, the ideals I, I' are *not* assumed to be ancestor ideals. Thus in the proof we are rather careful about how we use previous results. In particular, a key step, the last in the section concerning N is to show in Eq. (2.72) that $\operatorname{cod} R_1 \cdot I(1)_{u-1}$ satisfies a certain inequality (a similar step for T occurs in (2.76)); the apparent clumsiness—or perhaps we should say, subtlety—of the argument here is in part due to I' not being an ancestor ideal!

Lemma 2.30. *Let d, j be positive integers satisfying $d \leqslant j$, Assume that H and H' are acceptable O-sequences for the pair (d, j) (Definition 2.7) satisfying $H' \geqslant_{\mathcal{P}(d,j)} H$. When $c_H = c_{H'}$ let k be an arbitrary field; otherwise assume k is algebraically closed. Let I' be* *a graded ideal of Hilbert function $H(R/I') = H'$. Then there is a graded ideal I of Hilbert function $H(R/I) = H$, satisfying $I_j = V'$, or, equivalently, by Lemma* 1.7*, satisfying*

$$I + M^{j+1} \subset I' + M^{j+1} \quad \textit{and} \quad I \cap M^j \supset I' \cap M^j. \tag{2.69}$$

Let N and N' satisfy the condition (2.27) *of Lemma* 2.15 *for a fixed pair (d, j) and let I' be an ideal of Hilbert function $H(R/I') = N'$; then there is an ideal I of Hilbert function $H(R/I) = N$ satisfying $I \subset I'$. Likewise, let T, T' satisfy the condition* (2.28) *of Lemma* 2.15 *and let I' be an ideal of Hilbert function $H(R/I') = T$, then there is an ideal I satisfying $H(R/I) = T$, and such that $I \supset I'$.*

Proof. Since $\dim I_j = \dim I'_j$ we have $I_j = I'_j$; thus we may prove the result for H by proving that for N and T separately. Our overall method is to construct a sequence of ideals $I' = I(0), I(1), \ldots, I(s) = I$ of different Hilbert functions $H(R/I(u)) = H(u) \in \mathcal{H}(d, j)$ between $H' = H(0)$ and $H = H(s)$, using the properties of the τ invariant.

We begin by considering a pair of Hilbert functions $N \leqslant N'$, each satisfying the condition relevant to N in Lemma 2.15, and a given graded ideal I' satisfying $H(R/I') = N'$. We will construct an element of $G(N)$, a graded ideal of Hilbert function N satisfying $I \subset I'$. We may assume that all the ideals considered contain M^{j+1}. We first prepare to choose a Hilbert function $N(1)$ of $R/(I(1))$ differing from N' in the highest possible degree. Then we will determine the ideal $I(1) \subset I'$. Let $t < j$ be the largest integer, such that there is a permissible sequence $N(1)$ for a level algebra in the sense of Lemma 2.15, such that $N(1)_t \neq N'_t$ and satisfying both

$$\begin{aligned} &N' \leqslant N(1) \leqslant N: \quad \text{that is } \forall i \leqslant j,\ N'_i \leqslant N(1)_i \leqslant N_i, \quad \text{and} \\ &N(1)_i = N'_i \quad \forall i \mid t < i \leqslant j. \end{aligned} \tag{2.70}$$

Let $E' = \Delta(N')$ be the difference sequence, and let a be the largest nonnegative integer such that

$$e'_t = e'_{t-1} = \cdots = e'_{t-a}.$$

Claim A. The sequence $N(1)$, defined by

$$N(1)_i = \begin{cases} N'_i & \text{unless } t - a \leqslant i \leqslant t, \\ n'_i + 1 & \text{for } t - a \leqslant i \leqslant t, \end{cases} \tag{2.71}$$

is a permissible sequence, in the sense that $N(1)$ satisfies (2.27) of Lemma 2.15. Also, let $N'' \geqslant N'$ termwise (so $N'' \leqslant N'$ is a permissible sequence for which $\exists k, t - a \leqslant k \leqslant t$ with $N''_k \neq N'_k$). Then $N''_i \geqslant N(1)$.

Proof of Claim A. Because e'_i is nonincreasing as $i \leqslant j$ decreases, the integer t identifies the largest part $e'_{t+1} \neq e_{t+1}$, and we have $e'_{t+1} < e_{t+1}$. By the definition of $N(1)$ we have $e(N(1))_i = e'_i$ unless $i = t + 1$ or $i = t - a$. We have

$$e\big(N(1)\big)_{t+1} = e'_{t+1} + 1 \leqslant e_{t+1} \leqslant e_{t+2} = e'_{t+2} = e\big(N(1)\big)_{t+2}$$

and

$$e\big(N(1)\big)_{t-a} = e'_{t-a} - 1 \geqslant e'_{t-a-1} = e_{t-a-1}.$$

Since both N and N' are permissible, the above inequalities shows that $N(1)$ also is a permissible Hilbert function satisfying the condition (2.27) of Lemma 2.15.

Suppose by way of contradiction that N'' is a permissible sequence for $\mathrm{LA}(d, j)$ satisfying $N'' \geqslant N'$ termwise, but not satisfying $N'' \geqslant N(1)$, and let u be the smallest integer, $t - a \leqslant u \leqslant t$ such that $N''_u = N'_u$. If $t - a < u < t$ the difference $e''_u > e'_u = e'_{u+1} \geqslant e''_{u+1}$, contradicting the assumption that N'' is permissible for $\mathrm{LA}(d, j)$. This completes the proof of the Claim A.

We now choose an ideal $I(1) \subset I'$ with $H(R/I(1)) = N(1)$. Clearly $I(1)_i = I'_i$ unless $t - a \leqslant i \leqslant t$, so we need only choose $I(1)_{t-a}, \ldots, I(1)_t$. We construct $I(1)$ beginning with *lower* degrees. Suppose that u satisfies $t - a \leqslant u \leqslant t$ and $I(1)_0, \ldots, I(1)_{u-1}$ have been chosen so that (here we regard $I(1)_u \subset R_u$)

$$R_1 \cdot I(1)_{v-1} \subset I'_v,\ I(1)_v \subset I'_v, \quad \text{and} \quad \operatorname{cod}\big(I(1)_v\big) = N(1)_v \quad \text{for } v < u.$$

Now $R_1 \cdot I_{u-1} \subset R_1 \cdot I'_{u-1} \subset I'_u$, the first inclusion by assumption, and the second since I' is an ideal. We need to choose a vector space $I(1)_u$ between $R_1 \cdot I(1)_{u-1}$ and I'_u, having codimension $N(1)_u$ in R_u. This is possible if and only if $\operatorname{cod}(R_1 \cdot I(1)_{u-1}) \geqslant N(1)_u$. We have

$$\begin{aligned}\dim R_1 \cdot I(1)_{u-1} - \dim I(1)_{u-1} &= \tau\big(I(1)_{u-1}\big) = \dim I(1)_{u-1} - \dim R_{-1} \cdot I(1)_{u-1}\\ &\leqslant \dim I(1)_{u-1} - \dim I(1)_{u-2} \quad \text{by (1.7)}\\ &= 1 + e_{u-1}\big(N(1)\big)\\ &\leqslant 1 + e_u\big(N(1)\big), \quad \text{since } N(1) \text{ is permissible.}\end{aligned}$$

Thus

$$\begin{aligned} u + 1 - \dim R_1 \cdot I(1)_{u-1} &\geqslant u - \dim I(1)_{u-1} - e_u\big(N(1)\big),\\ \operatorname{cod} R_1 \cdot I(1)_{u-1} &\geqslant N(1)_{u-1} - e_u\big(N(1)\big) = N(1)_u \end{aligned} \tag{2.72}$$

by our choice of $N(1)$. Therefore, we may choose $I(1)_u$ such that $I'_u \supset I(1)_u \subset R_1 \cdot I(1)_{u-1}$, satisfying $\operatorname{cod} I(1)_u = \operatorname{cod} I'_u + 1$. Continuing this process, we may choose an ideal $I(1) \subset I(0) = I'$ of Hilbert function $H(R/I(1)) = N(1)$, as claimed. Continuing in this manner, we eventually construct $I(s)$ of Hilbert function $H(R/I(s)) = N(s) = N$, and satisfying $I(s) \subset I'$, as claimed. This completes the proof of the lemma for the pair (N, N').

We now turn to choosing an ideal I of Hilbert function $H(R/I) = T$ given I' satisfying $H(R/I') = T'$. Although proof of this portion of the Lemma involving $\mathrm{GA}_T(d, j)$ for T, T' eventually zero appears already in [I2, Section 4B], we include the argument with further details here for completeness. For now we assume that T, T' are eventually zero: that $c_T = c_{T'} = 0$. We will also now assume that our ideals $I \subset M^j$, by intersecting with M^j if necessary. We first choose the Hilbert function $T(1)$ of $R/(I(1))$, differing from T' in the lowest degree possible, and then the corresponding ideal $I(1)$.

Let $t > j$ be the smallest integer, such that there is a permissible sequence $T(1)$ satisfying the condition (2.28) of Lemma 2.15 for T, and such that $T(1)_t \neq T'_t$ and satisfying both

$$T' \geqslant T(1) \geqslant T: \quad \text{that is } \forall i \geqslant j \ T'_i \geqslant T(1)_i \geqslant T_i, \quad \text{and}$$
$$T(1)_i = T'_i \quad \forall i \mid j \leqslant i < t. \tag{2.73}$$

Let $E' = \Delta T'$ be the difference sequence, and let a be the largest nonnegative integer such that

$$e'_{t+1} = e'_{t+2} = \cdots = e'_{t+a}. \tag{2.74}$$

Claim B. The sequence $T(1)$, defined by

$$T(1)_i = \begin{cases} T'_i & \text{unless } t \leqslant i \leqslant t+a-1, \\ T'_i - 1 & \text{for } t \leqslant i \leqslant t+a-1, \end{cases} \tag{2.75}$$

is a permissible sequence satisfying the condition (2.28) of Lemma 2.15. Furthermore, let $T'' \leqslant T'$ (termwise) be a permissible sequence for which $\exists k, t < k \leqslant t+a$ with $T''_k \neq T'_k$. Then $T'' \leqslant T(1)$.

Proof of Claim B. Because e'_i is non-increasing as $i \geqslant j$ increases, the integer t identifies the largest difference $e'_t \neq e_t$, and we have $e'_i = e_i$ for i satisfying $i \leqslant t-1$. Since $T'_t > T_t$, we have $e'_t = T'_t - T'_{t-1} > T_t - T_{t-1} = e_t$ so we have $e'_t > e_t$. Evidently $e(T(1))_i = e'_i$ unless $i = t$ or $t+a$. We have

$$e\big(T(1)\big)_t = e'_t + 1 \leqslant e_t \leqslant e_{t-1} = e'_{t-1} = e\big(T(1)\big)_{t-1}$$

and

$$e\big(T(1)\big)_{t+a} = e'_{t+a} - 1 \geqslant e'_{t+a+1} = e\big(T(1)\big)_{t+a+1}.$$

Since both T and T' are permissible, the above inequalities show that $T(1)$ also is a permissible sequence—one satisfying the condition (2.28) of Lemma 2.15 for T.

Suppose by way of contradiction that T'' is likewise a permissible sequence satisfying $T'' \leqslant T'$ termwise, but T'' does not satisfy $T'' \leqslant T(1)$, and let u be the smallest integer, $t \leqslant u \leqslant t+a$ such that $T''_u = T'_u$. If $t < u < t+a$ the difference $e''_u < e'_u = e'_{u+1} \leqslant e''_{u+1}$, contradicting the assumption that T'' is permissible for $\mathrm{GA}(d, j)$. This completes the proof of the Claim B.

We now choose an ideal $I(1) \supset I'$ with $H(R/I(1)) = T(1)$, beginning with the *higher* degrees. Clearly $I(1)_i = I'_i$ unless $t \leqslant i \leqslant t + a - 1$, so we need only choose $I(1)_t, \ldots, I(1)_{t+a-1}$. Suppose that u satisfies $t + 1 \leqslant u \leqslant t + a$ and $I(1)_{u+1}, \ldots, I(1)_{t+a}$ have been chosen so that

$$R_{-1} \cdot I(1)_{v+1} \supset I'_v,\ I(1)_v \supset I'_v, \quad \text{and} \quad \operatorname{cod} I(1)_v = T(1)_v \quad \text{for } v > u.$$

Now $R_{-1} \cdot I(1)_{u+1} \supset R_{-1} \cdot I'_{u+1} \supset I'_u$, the first inclusion is by assumption, and the second since I' is an ideal. We need to choose a vector space $I(1)_u$ between $R_{-1} \cdot I(1)_{u+1}$ and I'_u, having codimension $T(1)_u$ in R_u. This is possible if and only if $\operatorname{cod}(R_{-1} \cdot I(1)_{u+1}) \leqslant T(1)_u = T'_u - 1$. We have

$$\begin{aligned}
\dim I(1)_{u+1} - \dim R_{-1} \cdot I(1)_{u+1} &= \tau\big(I(1)_{u+1}\big) = \dim R_1 \cdot I_{u+1} - \dim I_{u+1} \\
&\leqslant \dim I(1)_{u+2} - \dim I(1)_{u+1} \quad \text{by (1.7)} \\
&\leqslant 1 + e_{u+2}\big(T(1)\big) \\
&\leqslant 1 + e_{u+1}\big(T(1)\big), \quad \text{since } T(1) \text{ is permissible.}
\end{aligned}$$

Thus

$$\begin{aligned}
u + 1 - \dim R_{-1} \cdot I(1)_{u+1} &\leqslant u + 2 - \dim I(1)_{u+1} + e_{u+1}\big(T(1)\big), \\
\operatorname{cod} R_{-1} \cdot I(1)_{u+1} &\leqslant T(1)_{u+1} + e_{u+1}\big(T(1)\big) = T(1)_u
\end{aligned} \tag{2.76}$$

by our choice of $T(1)$. Therefore, we may choose $I(1)_u$ such that $I'_u \supset I(1)_u \subset R_{-1} \cdot I(1)_{u+1}$, satisfying $\operatorname{cod} I(1)_u = \operatorname{cod} I'_u - 1$. Continuing this process, we may choose an ideal $I(1) \supset I(0) = I'$ of Hilbert function $H(R/I(1)) = T(1)$, as claimed. Continuing in this manner, we eventually construct $I(s)$ of Hilbert function $H(R/I(s)) = T(s) = T$, and satisfying $I(s) \supset I'$, as claimed. This completes the proof of the lemma for the pair (T, T') when $c_T = c_{T'} = 0$.

When $c_T \neq 0$, by Corollary 2.14 any ideal I with $H(R/I) = T$ must have a common factor $f = \mathrm{GCD}(I)$ of degree c_T. We have $T \leqslant T' \Rightarrow c(T) \leqslant c(T')$. Suppose the pair of ideals I, I' satisfies $I \supset I', H(R/I) = T, H(R/I') = T'$, then $f = \mathrm{GCD}(I)$ divides any common factor $f' = \mathrm{GCD}(I')$ of I'. Given I', we now refine the choice of I by choosing in advance a degree $c(T)$ factor f of $\mathrm{GCD}(I')$ to be the common factor of I. Now it will suffice to choose $J = I : f$ of Hilbert function $T : c(T')$ containing $I' : f$, of Hilbert function $T' : c(T')$, and then set $I = fJ$. Thus we have reduced to showing the lemma when T is eventually zero, but $c_{T'} > 0$.

Suppose now that $c_T = 0, c' = c_{T'} \neq 0$, and define s' by $T'_{s'-1} > T'_s = c_{T'} > 0$. (When no such integer s' exists, then $I' = (f')$ and choosing $I \supset (f')$ poses no difficulty.) Let f' be the degree c' common factor of I'. When e'_i of (2.74) satisfies $e'_i > 0$ we choose $T(1)$ as in the case $c_T = c_{T'} = 0$, however to construct $I(1)$, we first construct $I(1) : f'$ of Hilbert function $T(1) : c'$ such that $I(1) : f' \supset I' : f'$, as above, then we let $I(1) = f' \cdot (I(1) : f')$. When $i = s' + 1$ and $e'_i = 0$ in (2.74), then $a = +\infty$ in (2.74). We choose $I(1) \cap M^{s+1} = (f'_1) \cap M^{s+1}$ with f'_1 a degree $c' - 1$ divisor of f'. Continuing in this way, we obtain finally an ideal $I \supset I'$ of Hilbert function $H(R/I) = T$. This completes the proof of the statements involving T, T' of the lemma in all cases.

We now turn to the case of a pair H, H' of acceptable Hilbert functions. When H is eventually zero, one uses the above methods to first construct $I + M^{j+1}$ and then construct $I \cap M^j$, which together determine the ideal I (since $I_j = I'_j$ is given). When H is eventually c, then one chooses f of degree c dividing the common factor f' of I' of degree $c(T') \geqslant c$. Then one chooses $I : f$ of Hilbert function $T : c$, as above from $I' : f$ of Hilbert function $T' : c$, then sets $I = f \cdot (I : f)$. Since $H = H(N, T)$ is acceptable (Definition 2.7) if and only if N, T have the same τ and are both permissible (satisfy (2.27) or (2.28), respectively), this completes the proof of the lemma. $\square$

Example 2.31. We illustrate the process of choosing $N(1)$ in the proof above. Suppose that the two sequences N', N are $N' = (1, 2, \ldots, 13, 11, 9, 7, 4, 0)$ with $N'_{16} = 4$, and $N = (1, 2, \ldots, 13, 12, 11, 8, 4, 0)$. We choose $N(1)$: here $t = 15$, and one chooses $N(1)_{15,16} = (8, 4)$. However, if this were the only change, the intermediate sequence $(1, \ldots, 13, 11, 9, 8, 4, 0)$ would violate the condition on first differences, as it has first differences $(\ldots 2, 1, 4, 4)$, which has a decrease from 2 to 1. Instead, we must choose $N(1) = (1, \ldots, 13, 12, 10, 8, 4)$, which is also next to N' in the partial order among the subset of sequences possible for level algebras LA(13, 16) and having $N(1)_{15} > 7$. Then $N(2) = N$. Note that $N(0) = (1, \ldots, 13, 12, 10, 7, 4, 0)$ is next to N' in the partial order, but we have chosen to step to $N(1)$, which is the closest to N' among those between N' and N and differing from N' in the highest possible degree. Note that in the proof of Lemma 2.30, the occurring Hilbert functions $N(i), T(i)$ must be permissible for a level algebra, graded ideal, respectively of a vector space of forms. But the intermediate ideals $I(1), \ldots$ that we construct are not themselves level ideals, nor ideals generated by I_j, respectively.

Recall from Definition 1.14 that we denote by $\mathcal{P} = \mathcal{P}(d, j)$ the partial order on the set $\mathcal{H}(d, j)$ of acceptable Hilbert functions. The acceptable Hilbert functions are described in Definition 2.7, and further in Lemma 2.8. Recall that we showed in Theorem 2.19 that these $H \in \mathcal{H}(d, j)$ are exactly the sequences occurring as Hilbert functions of ancestor algebras.

Theorem 2.32. *Let d, j be positive integers satisfying $d \leqslant j$, assume that the field k is algebraically closed, and suppose that H is an acceptable O-sequence* (*Definition* 2.7).

(A) Frontier property. *The Zariski closure $\overline{\mathrm{Grass}_H(d, j)}$ satisfies*

$$\overline{\mathrm{Grass}_H(d, j)} = \bigcup_{H' \geqslant_{\mathcal{P}} H} \mathrm{Grass}_{H'}(d, j). \tag{2.77}$$

The analogous equalities hold for $\overline{\mathrm{LA}_N(d, j)}$ and for $\overline{\mathrm{GA}_T(d, j)}$.

(B) $G(H)$ *is a* desingularization *of $\overline{\mathrm{Grass}_H(d, j)}$. There is a surjective morphism $\pi : G(H) \to \overline{\mathrm{Grass}(H)}$ from the nonsingular variety $G(H)$, given by $I \to I_j$. The inclusion $\iota : \mathrm{Grass}_H(d, j) \subset G(H), \iota : V \to \overline{V}$ is a dense open immersion. For*

$H' \in \mathcal{H}(d, j)$, $H' \geqslant_{\mathcal{P}} H$, the fibre of π over $V' \in \overline{\mathrm{Grass}_H(d, j)} \cap \mathrm{Grass}_{H'}(d, j)$ parametrizes the family of graded ideals

$$\{I \mid H(R/I) = H \text{ and } I_j = V'\}. \tag{2.78}$$

The schemes $\overline{\mathrm{LA}_N(d, j)}$ and $\overline{\mathrm{GA}_T(d, j)}$ have desingularizations $G(N)$ and $G(T)$, respectively, with analogous properties.

Proof. By Theorem 1.10(i), (iii) $G(H)$ is nonsingular and has as open dense subset the subfamily of ideals with minimum number of generators; by Proposition 2.11(v), this subfamily is $\iota(\mathrm{Grass}_H(d, j))$ (see also Theorem 2.17(A)). By definition of π the fibre of π is the family specified in (2.78). That π is surjective we will show next, thus completing the proof of (B).

We now show (2.77). Suppose that $H' \geqslant H \in \mathcal{H}(d, j)$: so H, H' satisfy the condition of Proposition 2.6 and each occurs as the Hilbert function of an ancestor ideal, and let $V' \in \mathrm{Grass}_{H'}(d, j)$. By Lemma 2.30 there is an ideal I of Hilbert function H satisfying $I_j = V'$. Since $G(H)$ is irreducible with open dense subscheme $\mathrm{Grass}_H(d, j)$ we have that there is a family $I(t), t \in \mathfrak{Z}$ of ideals parametrized by a curve $\mathfrak{Z} \subset G(H)$ such that for $t \neq t_0$, $I(t) \in \iota(\mathrm{Grass}_H(d, j))$, with $I = \lim_{t \to t_0} I(t)$; it follows that $V' = \lim_{t \to t_0} V(t) = I(t)_j$ is in the closure of $\mathrm{Grass}_H(d, j)$. This shows that the closure $\overline{\mathrm{Grass}_H(d, j)}$ includes the union of lower strata in (2.77). By Theorem 1.15 the closure $\overline{\mathrm{Grass}_H(d, j)}$ is a subset of $\bigcup_{H' \geqslant_{\mathcal{P}} H} \mathrm{Grass}_{H'}(d, j)$. This completes the proof of (2.77) and (A), as well as (B) for $\overline{\mathrm{Grass}_H(d, j)}$. An analogous argument proves the results in (A) concerning the closures $\overline{\mathrm{LA}_N(d, j)}$ and $\overline{\mathrm{GA}_T(d, j)}$. This completes the proof. □

Corollary 2.33. *The scheme $\mathrm{Grass}_\tau(d, R_j)$ is irreducible and $\mathrm{Grass}_{H_\tau}(d, j)$ (see (2.33)) is a dense open subscheme. The Zariski closure of $\mathrm{Grass}_\tau(d, j)$ satisfies $\overline{\mathrm{Grass}_\tau(d, j)} = \bigcup_{\tau' \leqslant \tau} \mathrm{Grass}_{\tau'}(d, j)$.*

Proof. We fix (d, j, τ). Evidently, by Lemma 2.3(ii) and Eq. (2.66), the Hilbert function $N(H_\tau)$ is maximum, among the Hilbert functions $N(H)$ for H satisfying $\tau(H) \leqslant \tau$. Similarly (2.53) and (2.68) show that $T(H_\tau)$ has the minimum values among such H. Thus, Theorem 2.32 implies the corollary. □

Definition 2.34. We denote by $\mathcal{P}A(d, j)$ the partially ordered set of pairs of partitions (P, Q) such that P partitions d, Q partitions an integer no greater than $j + 1 - d$, and the largest part p_1 of P and the largest part q_1 of Q satisfy $p_1 = q_1 + 1$. We let $(P, Q) \leqslant (P', Q')$ if both $P \leqslant P'$ and $Q \leqslant Q'$ in the respective majorization partial orders.

Theorem 2.35. *There is an isomorphism of partially ordered sets $\mathcal{H}(d, j)$ under the partial order $\mathcal{P}(d, j)$ and the partially ordered set $\mathcal{P}A(d, j)$, under the product of the majorization partial orders (see Definition 2.34) given by $H \to (P, Q)$, $P = P(H) = A(H)^*$, $Q = Q(H) = B(H)^*$ (see Definitions 2.9 and 2.21). This is the same order as is induced by specialization (closure) of the strata $\mathrm{Grass}(H)$.*

Table 2.1
Hilbert functions H for $(d, j) = (4, 5)$

Stratum	τ	A	B	$P = A^*$	$Q = B^*$	c	cod	H
$H(0)$	3	$(2, 1, 1)$	$(1, 1)$	$(3, 1)$	(2)	0	0	$(1, 2, 3, 4, 4, 2, 0, \underline{0})$
$H(1)$	2	$(2, 2)$	(2)	$(2, 2)$	$(1, 1)$	0	1	$(1, 2, 3, 4, 3, 2, 1, \underline{0})$
$H(2)$	2	$(2, 2)$	(1)	$(2, 2)$	(1)	1	3	$(1, 2, 3, 4, 3, 2, \underline{1})$
$H(3)$	1	(4)	–	$(1, 1, 1, 1)$	–	2	6	$(1, \underline{2})$

Proof. This is immediate from (2.77), Theorem 2.19(iii), and Lemma 2.28. □

Example 2.36. We consider the partial order on all sequences H for $(d, j) = (4, 5)$ (see Table 2.1). Thus, A partitions the dimension $d = 4$ into $\tau \leqslant 3$ parts, and B partitions the integer $\operatorname{cod}(V) - c = 2 - c$ into $\tau - 1$ parts. Grass(4, R_5) has dimension 8; the open cell is given by the pair $A = (2, 1, 1)$, $B = (1, 1)$. When $\tau = 2$ there are two sequences, and for $\tau = 1$ a single sequence. They are here linearly ordered by $\geqslant_{\mathcal{P}(4,5)}$, so by Theorem 1.16 the closure of each stratum listed in Table 2.1 is the union of the stratum itself with the strata below it. Note that the A, P and Q columns of partitions in Table 2.1 are simply ordered in the majorization partial order, but the B column is not. The order on $\mathcal{H}(d, j)$ is equivalent to the product of majorization orders on the pairs (P, Q).

Remark 2.37. Possibly relevant to the frontier property, given Theorem 2.32(A) and Theorem 2.35, C. Greene and D.J. Kleitman have studied the longest simple chains in the lattice of partitions of an integer [GreK].

Relevant to the desingularization of Theorem 2.32(B), a basis for the homology of $G(H)$ is given in [IY], in terms of the classes $\pi_*(E(J))$ determined by the monomial ideals J of Hilbert function $H(R/J) = H$: here $E(J)$ is the affine cell parametrizing graded ideals having initial ideal J, and it the set $\{E(J)\}$ form a cell decomposition of $G(H)$. A natural cobasis of a monomial ideal of colength n, $H(R/J) = H$ is a vector space $E^c(J)$ of monomials whose graph is the Ferrers graph of a partition $P(E^c)$ of n with diagonal lengths H. The dimension of the cell $E(J)$ is the number of difference one hooks (arm-leg = 1) in the partition $P(E^c)$ When $|H| = \sum H_i = n$ a basis for the degree-i homology corresponds one-to-one with the partitions of n having the given diagonal lengths H; and having the given number i of hooks of difference one. In a few cases the homology ring structure of $G(H)$ is known, but in general the homology ring structure is not known (see [IY]).

3. Waring problem, related vector spaces

In Section 3.1 we apply the previous results to a refinement of the simultaneous Waring problem for a vector space of forms. In Section 3.2 we first return to polynomial rings R of arbitrary dimension r, to develop the notion of a space $W \subset R_i$ related to a vector space $V \subset R_j$ if W is obtained by a chain whose elements are each a homogeneous component of the ancestor ideal of the predecessor space. When $r = 2$ we bound the number of classes $\overline{W}$ related to $\overline{V}$ in terms of the τ invariant $\tau(V)$. Finally, we state some open problems.

3.1. The simultaneous Waring problem for degree-j binary forms

We let $r=2$ and denote by $\mathcal{R}=k[X,Y]$ the *dual* polynomial ring to R. We suppose that $\operatorname{char} k=0$ or $\operatorname{char} k=p>j$ throughout this section. The *simultaneous Waring problem* is to find the minimum number $\mu(c,j)$ of linear forms, needed to write each element of a general dimension-c vector space $\mathcal{W}\subset\mathcal{R}_j$ as a sum of jth powers of the linear forms; here the choice of the linear forms depends on $\mathcal{W}$. Our refinement is to fix also the differential τ invariant of $\mathcal{W}$.

The case $c=1$ of a single binary form F is quite classical: it is related to the secant varieties of rational normal curves, and is resumed along with this connection in [IK, Section 1.3]. Note that in this section $c=\dim\mathcal{W}$ satisfies $c=\operatorname{cod}(V)=j+1-\dim V$ where $V=(\operatorname{Ann}\mathcal{W})_j$ (see (3.3)). Letting $\mu(W)$ denote the minimal length of a simultaneous (generalized) additive decomposition of W, our results rest on the identity $\mu(W)=\mu(L(V))$, the order of the level ideal $L(V)$ determined by V (Lemma 3.2), valid for $r=2$ only. For $u\leqslant c$ we let $c_a=c(c-1)\cdots(c+1-a)$.

Definition 3.1. The ring $R=k[x,y]$ acts on $\mathcal{R}$ by differentiation

$$x^ay^b\circ X^cY^d=\begin{cases}(c_a\cdot d_b)X^{c-a}Y^{d-b} & \text{if } c\leqslant a \text{ and } b\leqslant d,\\ 0 & \text{otherwise.}\end{cases}\tag{3.1}$$

Let $V\subset R_j$ be a vector subspace. We denote by $V^\perp\subset\mathcal{R}_j$ the subspace

$$V^\perp=\{F\in\mathcal{R}_j\mid v\circ F=0\ \forall v\in V\}.\tag{3.2}$$

Given $\mathcal{W}\subset\mathcal{R}_j$ we denote by $\operatorname{Ann}(\mathcal{W})\subset R$ the ideal

$$\operatorname{Ann}\mathcal{W}=\{f\in R\mid f\circ w=0\ \forall w\in\mathcal{W}\}.\tag{3.3}$$

Let $V=(\operatorname{Ann}(\mathcal{W}))_j\subset R_j$. We define the *differential τ-invariant* $\tau_\delta(\mathcal{W})$ as

$$\tau_\delta(W)=\tau(V)=\dim R_1\cdot V-\dim V.\tag{3.4}$$

We need also the following notions of additive decomposition: let $F\in\mathcal{W}$ then $F=\sum_{i=1}^s\alpha_iL_i^j$ is an additive decomposition of length μ of F, assuming that the $\{L_i\}$ are pairwise linearly independent. The form $F\in\mathcal{R}_j$ has a *generalized additive decomposition* (GAD) of length μ and weights $\beta_1,\dots,\beta_t$ into powers of the linear forms $L_1,\dots,L_t\in\mathcal{R}_1$ if

$$F=\sum_{i=1}^tG_iL_i^{j+1-\beta_i}\quad\text{where }\deg G_i=\beta_i-1\text{ and }\sum\beta_i=\mu.\tag{3.5}$$

The vector space $\mathcal{W}\subset\mathcal{R}_j$ has a *simultaneous decomposition* of length μ if there is a single ordered set $L=(L_1,\dots,L_t)$ of linear forms $L_i\in R_1$ (which may depend on W) and weights $\beta=(\beta_1,\dots,\beta_t)$ such that each $F\in W$ has a GAD of length μ and weights

β into the forms L. We denote by $\mu(\mathcal{W})$ the shortest length of a simultaneous additive decomposition of W.

We define $\mu(c, j), \mu(\tau, c, j)$, respectively, as the common value of $\mu(W)$ for $\mathcal{W}$ in a suitable open dense subset of $\operatorname{Grass}(c, \mathcal{R}_j)$, or of $\operatorname{Grass}_{\tau_\delta}(c, R_j)$ (where $\tau_\delta(\mathcal{W}) = \tau$), respectively.

Note that we defined $\tau_\delta(\mathcal{W})$ for $\mathcal{W} \subset \mathcal{R}_j$ using the annihilating degree-j space $V = (\operatorname{Ann}(\mathcal{W}))_j$. Here is a direct definition. Let $R_1 \circ \mathcal{W} \subset \mathcal{R}_{j-1}$ be $R_1 \circ \mathcal{W} = \{\ell \circ w, \ell \in R_1, w \in \mathcal{W}\}$. Letting $N = (n_0, n_1, \ldots) = H(R/\operatorname{Ann}(\mathcal{W}))$, we have from $(\operatorname{Ann}(\mathcal{W})_{j-1})^\perp = R_1 \circ \mathcal{W}$ and (2.4)

$$\tau_\delta(\mathcal{W}) = 1 + e_j(N) = 1 + n_{j-1} - n_j = 1 + \dim R_1 \circ \mathcal{W} - \dim \mathcal{W}. \tag{3.6}$$

For $L_i = a_i X + b_i Y \in \mathcal{R}_1$ we let $\ell_i = b_i x - a_i y \in \mathcal{R}_1$: then $\ell_i \circ L_i = 0$. We have the following well-known result. Recall that $\mu(L(V))$ is the *order* of the level ideal $L(V)$.

Lemma 3.2. *Let $V \subset R_j$ and set $\mathcal{W} = V^\perp$. The level ideal $L(V)$ satisfies*

$$L(V) = \operatorname{Ann}(\mathcal{W}), \quad \mathcal{W} = V^\perp. \tag{3.7}$$

Let $F \in \mathcal{R}_j$. Then F has a GAD of length μ as in (3.5) *if and only if*

$$\exists f \in \operatorname{Ann}(F) \textit{ such that } \quad \deg f = \mu \quad \textit{and} \quad f = \prod \ell_i^{\beta_i}, \quad \ell_i \in R_1. \tag{3.8}$$

Let $\mathcal{W} \subset \mathcal{R}_j$ and $\dim \mathcal{W} = c$. Then $\mu(\mathcal{W}) = \mu(L(V))$ for $V = (\operatorname{Ann}(\mathcal{W})_j$. Also $1 \leqslant \tau_\delta$ and

$$\tau_\delta(\mathcal{W}) \leqslant \min\{c + 1, j + 1 - c\}, \tag{3.9}$$

with equality in (3.9) *for a generic choice of $\mathcal{W} \subset R_j$ of dimension c.*

Proof. The identity (3.7) is a basic property of inverse systems—see in general [Mac1, Section 60ff], [EmI1,G] or for a modern proof, [IK, Lemma 2.17]. Eq. (3.8) is [IK, Lemma 1.33]; that $\mu(W) = \mu(L(V))$ is a straightforward consequence. The last statement is a consequence of the upper bound on $\tau(V)$, $V = (\operatorname{Ann} W)_j$ from Lemma 2.2, rewritten in terms of c, j, since $\tau_\delta(\mathcal{W}) = \tau(V)$. □

We let $c = j + 1 - d$ and define $\mu(\tau, d, j) = j + 1 - \lceil d/\tau \rceil$. When $\mu \leqslant \mu(\tau, d, j)$, we define the Hilbert function sequence $N(\mu, \tau, d, j)$ by

$$N(\mu, \tau, d, j)_i = \begin{cases} \min\{i + 1, \mu, c + (\tau - 1)(j - i)\} & \text{for } i \leqslant j, \\ 0 & \text{for } i > j. \end{cases} \tag{3.10}$$

We define $N(\tau, d, j) = N(H_\tau(d, j))$ with $H_\tau(d, j)$ from Eq. (2.33): thus we have $N(\tau, d, j)_i = \min\{i + 1, c + (\tau - 1)(j - i)\}$ for $i \leqslant j$. We define $a, \kappa \in \mathbb{N}$ by $\mu - c = a(\tau - 1) + \kappa$ with $0 \leqslant \kappa = \operatorname{rem}(\tau - 1, \mu - c) < \tau - 1$.

Lemma 3.3. *$N(\tau, d, j)$ is the maximum level algebra Hilbert function for a d-dimensional vector space $V \subset R_j$ with $\tau(V) = \tau$; it has order $\mu(\tau, d, j)$ and partition $P(\tau, d, j) = (\tau^{\lfloor d/\tau \rfloor}, \mathrm{rem}(\tau, j))$ from (2.32). $N(\mu, \tau, d, j)$ is the maximum level algebra Hilbert function that is both bounded above by μ and possible for a vector space $V \subset R_j$ with $\tau(V) = \tau$. It has order μ and partitions P, A of d*

$$P = P(\mu, \tau, d, j) = \left(\tau^a, \kappa + 1, 1^{j-\mu-a}\right), \tag{3.11}$$

$$A = A(\mu, \tau, d, j) = P^* = \left(j + 1 - \mu, \lceil(\mu - c)/(\tau - 1)\rceil^{(\kappa-1)^+}, a^{\tau-\kappa}\right). \tag{3.12}$$

The dimension of $\mathrm{LA}_N(d, j)$, $N = N(\tau, d, j)$ is $\tau(j + 2 - \tau) - d$.

Proof. The order $\mu = \mu(\tau, d, j)$ of $N(\tau, d, j)$ satisfies

$$\mu = \max\left\{i \mid N(\tau, d, j)_{i-1} \geqslant i\right\} = \max\left\{i \mid c + \left(j - (i - 1)\right)(\tau - 1) \geqslant i\right\},$$

which leads to $\mu = \mu(\tau, d, j)$. The calculation of $P(\mu, \tau, d, j)$, $A(\mu, \tau, d, j)$ is routine, and the dimension formula for $\mathrm{LA}_N(d, j)$, is (2.41). □

One part (ii) of the following theorem may be classical; it was shown by J. Emsalem and the author in an unpublished preprint, and also in [Ca,CaCh].

Theorem 3.4. *We will suppose that $\mathcal{W} \subset \mathcal{R}_j$, $\mathcal{R} = k[X, Y]$, $\dim \mathcal{W} = c$, and $d = j + 1 - c$.*

(i) *Each dimension c subspace $\mathcal{W} \subset \mathcal{R}_j$ with $\tau_\delta(\mathcal{W}) = \tau$ satisfies $c \leqslant \mu(W) \leqslant \mu(\tau, d, j)$, with equality $\mu(W) = \mu(\tau, d, j)$ for a generic choice of such $\mathcal{W}$.*
(ii) *For general $\mathcal{W}$ the value of $\mu(\mathcal{W})$ is $\lfloor c(j + 2)/(c + 1)\rfloor$ if $c < j/2$, and j otherwise.*
(iii) *Let $c \leqslant \mu \leqslant \mu(\tau, d, j)$. When k is algebraically closed, the subfamily $\mathrm{GAD}_\mu(\tau, c, j)$ of $\mathrm{Grass}_{\tau_\delta}(c, \mathcal{R}_j)$ parametrizing $\mathcal{W}$ satisfying $\tau_\delta(\mathcal{W}) = \tau$ and $\mu(\mathcal{W}) \leqslant \mu$ is isomorphic under $\mathcal{W} \to (\mathrm{Ann}\,\mathcal{W})_j$ to $\overline{\mathrm{LA}_N(d, j)}$, where $N = N(\mu, \tau, d, j)$. The codimension of $\mathrm{LA}_N(d, j)$ in $\mathrm{Grass}_\tau(d, j)$ satisfies, for $1 \leqslant \mu < \mu(\tau, d, j)$*

$$\mathrm{cod}_{\tau_\delta} \mathrm{GAD}_\mu(\tau, c, j) = \ell(A) = (j - \mu)\tau - (d + 1). \tag{3.13}$$

Proof. By Lemmas 3.2 and 3.3 each of the statements (i), (ii), and the first part of (iii) translates into one about the order of $N(\tau, d, j)$, or the dimension of $N(\mu, \tau, d, j)$. Corollary 2.33 implies that for an open dense set of $V \in \mathrm{Grass}_\tau(d, j)$, the Hilbert function of $\mathrm{LA}(V)$ is $N(\tau, d, j)$, derived from $H(\tau, d, j)$ of (2.33). Thus, the order $\mu(\tau, d, j)$ of $N(\tau, d, j)$, is the generic value for $\mu(W)$, W, $\tau_\delta(W) = \tau$. This gives (i), and (ii) follows from substituting $\tau = c + 1$ or $j + 1 - c$ from (3.9) into the formula of (i). The codimension of $\mathrm{LA}_N(d, j)$ in $\mathrm{Grass}_\tau(d, j)$ of (iii) is by (2.55) the invariant $\ell(A)$ of (2.54) for the partition $A = A(\mu, \tau, d, j)$ from (3.12); however a routine calculation using $\dim N(\tau, d, j)$ from Lemma 3.3 and (2.35)—assuming $e_\mu = 0$ for $N = N(\mu, \tau, d, j)$—gives (3.13) for $\mu < \mu(\tau, d, j)$ (when $\mu = \mu(\tau, d, j)$ the assumption $e_\mu = 0$ for (3.13) may not hold). Theorem 2.32 completes the proof of (iii). □

Remark 3.5. Theorem 3.4 states that vector spaces $\mathcal{W}$ with higher τ in general require a larger number of linear forms $L_1, \dots, L_\mu$ so that

$$\mathcal{W} \subset \langle L_1^j, \dots, L_\mu^j \rangle. \tag{3.14}$$

Thus, letting $V = (\mathrm{Ann}(W))_j$ when $\tau(V) = 1$ so $V = f_c R_{j-c}$, we have $\mu(\mathcal{W}) = c$. When $c \geqslant j/2$ and $\tau(V) = j + 1 - c$, the maximum value, then $\mu(\mathcal{W}) = j$ in general. Note that, given (μ, τ, d, j) satisfying $c \leqslant \mu \leqslant \mu(\tau, d, j)$, the proof of Theorem 1.10 in [I2] shows that one can choose a vector space $V \in \mathrm{LA}_N(d, j)$, $N = N(\mu, \tau, d, j)$ such that there is a form $f \in L(V)_\mu$ with distinct roots, thus one may suppose that a general $\mathcal{W} \in \mathrm{GAD}_\mu(\tau, c, j)$ satisfies (3.14).

3.2. Vector spaces related to V; open problems

In Section 3.2 the dimension r of R is arbitrary unless otherwise specified. We say that $W \subset R_i$ is related to $V \subset R_j$ if there is a sequence $(i_1, \dots, i_k) \in \mathbb{Z}^k$ such that

$$W = R_{i_k} \cdot R_{i_{k-1}} \cdots R_{i_1} V = R_{i_k} \cdot \big(R_{i_{k-1}} \cdot (\cdots R_{i_1} V) \cdots\big). \tag{3.15}$$

We give some basic identities, valid for $R = k[x_1, \dots, x_r]$.

Lemma 3.6. *We have for arbitrary vector spaces $V \subset R_j$,*

$$R_s R_t V = R_{s+t} V \quad \text{if } s, t \leqslant 0 \text{ or } s, t \geqslant 0; \tag{3.16}$$

$$R_s R_t V \subset R_{s+t} V \quad \text{if } s \geqslant 0 \text{ or } t \leqslant 0; \tag{3.17}$$

$$R_s R_t V \supset R_{s+t} V \quad \text{if } s \leqslant 0 \text{ or } t \geqslant 0. \tag{3.18}$$

Also,

$$\begin{aligned} R_s R_t R_u V = R_{s+t+u} V \quad & \text{if } s, t, v \text{ have the same sign,} \\ & \text{or if } \operatorname{sign} s = \operatorname{sign} u \text{ and } |t| \leqslant |s|, |u|, \end{aligned} \tag{3.19}$$

$$R_s R_t R_u V \subset R_{s+t+u} V \quad \text{if } s, s+t \geqslant 0 \text{ or } u, t+u \leqslant 0, \tag{3.20}$$

$$R_s R_t R_u V \supset R_{s+t+u} V \quad \text{if } s, s+t \leqslant 0 \text{ or } u, t+u \geqslant 0. \tag{3.21}$$

The proofs are immediate from the definitions. The following lemma gives a normal form for relations, that need not be unique.

Lemma 3.7. *Let W be related to V. Then there is an expression $W = R_{i_k} \cdot R_{i_{k-1}} \cdots R_{i_1} V$ satisfying*

(i) *The sequence $i_1, \dots, i_k$ is alternating in sign.*
(ii) *$\exists t, 1 \leqslant t \leqslant k$ such that $|i_1| < \cdots < |i_t|$, and if $k > t$, $|i_t| \geqslant |I_{t+1}| \geqslant \cdots \geqslant |i_s|$.*

Proof. First, using (3.16) to collect $R_a \cdot R_b$ for which sign $a =$ sign b, we may assume the expression is alternating in sign and is no longer than the original expression. Then using (3.19) we collect adjacent triples $R_a \cdot R_b \cdot R_c$ in the expression for W, for which $|b| \leqslant |a|, |c|$. Since collecting terms shortens the length of the relation, after a finite number of steps of collecting such triples and assuring that the signs alternate, we will arrive at an expression where the indices alternate in sign, and for which each adjacent triple $R_a \cdot R_b \cdot R_c$ we have $|b| > |a|, |c|$. This is possible only if the indices satisfy the condition (ii). □

One might ask whether W related to V and V related to W imply equality $\overline{V} = \overline{W}$. We will shortly show that this holds when $r = 2$ (Corollary 3.10). The following counterexample when $r = 3$ is due to David Berman [Be].

Example 3.8 (*D. Berman: loops in the natural partial order*). Let $V = \langle x^2y^3, y^2z^3, x^3z^2\rangle \subset R_5$, $R = k[x, y, z]$, and let $W = R_2V$. Then $V = R_{-2}W$ but $R_{-1}W$ contains $x^2y^2z^2$, which is not in R_1V, hence $\overline{V} \neq \overline{W}$.

We now restrict to $r = 2$.

Proposition 3.9. *Suppose that $r = 2$ and $V \subset R_j$ satisfies $\tau(V) = \tau$. Then there are at most $2^\tau - 1$ nonzero equivalence classes $\overline{W}$ of vector spaces related to V. Any nonzero W related to V has an expression of length $k \leqslant \tau(V) - \tau(W) + 1$.*

Proof. When $\tau(V) = 1$, Lemma 2.2 implies that the vector space V satisfies $V = f \cdot R_{j-d}$, and $\overline{V} = (f)$. Evidently, any nonzero W related to V must satisfy $\overline{W} = (f)$. Let $n > 1$ and assume inductively that the statement is true for all j, for vector spaces V satisfying $\tau(V) \leqslant n-1$. Let $V \subset R_j$ satisfies $\tau(V) = n$, and let u, v be the minimum positive integers such that $\overline{R_{-u}V}$ and $\overline{R_vV}$ are each not equivalent to V. Since both $\tau(R_{-u}V) \leqslant n - 1$ and $\tau(R_v(V)) \leqslant n - 1$, the induction step would follow from the following claim, as we would then have that the number of classes $\overline{W}$ related to V would satisfy

$$\begin{aligned}\#\{\overline{W} \text{ related to } V\} &= \#\{\overline{W} \text{ related to } R_{-u}V\} + \#\{\overline{W} \text{ related to } R_vV\} + \text{ one for } \overline{V}\\ &\leqslant 2\bigl(2^{n-1} - 1\bigr) + 1 = 2^n - 1.\end{aligned}$$

Claim. Let $W \neq 0$ be related to V, and assume $\overline{W} \neq \overline{V}$. Then W is related to $R_{-u}V$ or to R_vV, where u, v are defined above.

Proof of claim. We first observe that

$$\overline{R_wV} = \overline{V} \quad \Rightarrow \quad R_aR_wV = R_{a+w}V \quad \text{for } a \in \mathbb{Z}. \tag{3.22}$$

When sign $a =$ sign w, this is just (3.16); when sign $a \neq$ sign w and $|a| \geqslant |w|$ then

$$\begin{aligned} R_a \cdot R_w &= R_{a+w} R_{-w} \cdot R_w V \quad \text{by (3.16) as sign } a+w = \text{sign } -w \\ &= R_{a+w} V \quad \text{since } \overline{V} = \overline{R_w V}. \end{aligned}$$

Suppose now that W is related to V. Unless $\overline{V} = \overline{W}$, by (3.22) we may assume that in the expression $W = R_{i_k} \cdot R_{i_{k-1}} \cdots R_{i_1} V$ for W we have $i_1 \leqslant -u$ or $i_1 \geqslant v$. Then by (3.16) $R_{i_1} V = R_{i_1+u} \cdot R_{-u} V$ in the first case, or $R_{i_1} V = R_{i_1-v} R_v V$ in the second case. This completes the proof of the claim, and of the first statement of the proposition.

The claim and above proof shows that we need only allow at most one factor of the form R_{i_t} in the expression for W for each reduction by one in τ, and one more for the last step, giving us $k \leqslant \tau(V) - \tau(W) + 1$ as claimed. □

Corollary 3.10. *Let $r = 2$, and suppose that $V \subset R_j$ and $W \subset R_w$ satisfy W is related to V in the sense of* (3.15), *and also V is related to W. Then $\overline{V} = \overline{W}$.*

Proof. By repeated application of Proposition 2.3(i), we have $\tau(W) \leqslant \tau(V)$, and vice-versa, hence $\tau(W) = \tau(V)$. Then there is an expression $W = R_a V$ by the second part of Proposition 3.9. Proposition 2.3(iii) now implies that $\overline{V} = \overline{W}$. □

Open problems

A. The dimension and closure results of Theorems 2.17, 2.24, and 2.32 have a naturality that suggest they might extend to strata not only by the Hilbert function and partial Hilbert functions (analogous to [I2, Section 4B]), but also to more refined strata closer to the complete Hilbert function where the dimension of each vector space W related to V is specified (see Section 3.2 and [Be]). For example, suppose that $D(u, v)(V) = \dim R_u R_v V$ is specified for all u, v: what is the dimension and closure of the stratum of $\mathrm{Grass}(d, R_j)$ determined by $D = \{D(u, v)\}$?

B. The desingularization morphism $G(H) \to \overline{\mathrm{Grass}_H(d, j)}$ is a semi-small resolution. What can be said about the singularities of $\overline{\mathrm{Grass}_H(d, j)}$? What is the class of $\overline{\mathrm{Grass}_{H'}(d, j)}$ in the homology ring $H_*(G(H))$? Is $\overline{\mathrm{Grass}_H(d, j)}$ Cohen–Macaulay? A. King and C. Walter have shown that the homomorphism $i_*: H_*(G(H)) \hookrightarrow \prod_{\mu \leqslant i \leqslant s} H_*(\mathrm{Grass}(i + 1 - H_i, R_i))$ is an inclusion [KW].

C. In Corollary 2.18 we showed that $\mathrm{Grass}_H(d, j) = \mathrm{LA}_N(d, j) \cap \mathrm{GA}_T(d, j)$, is a proper intersection in $\mathrm{Grass}_\tau(d, j)$. Thus, the only condition tying $\mathrm{LA}_N(d, j)$ and $\mathrm{GA}_T(d, j)$, with $N = N_H$ and $T = T_H$ is that $\tau(N) = \tau(T)$. Do these subvarieties intersect transversely?

D. Is there a relation between the cohomology rings $H^*(\overline{\mathrm{LA}_N(d, j)})$ and $H^*(\overline{\mathrm{GA}_T(d, j)}$, when the related partitions A, B correspond? Or a relation between $H^*(\overline{\mathrm{LA}_N(d, j)})$ and $H^*(\overline{\mathrm{LA}'_N(d, j)})$ when the partition A' determining N' has one more part than the partition A determining N?

E. There is a well-known geometric interpretation of the Hilbert function stratum $\mathrm{GA}_T(d, j)$. The vector space V determines a rational curve $X \subset \mathbb{P}^{d-1}$; the restriction $\mathcal{T}$ to X of the tangent bundle to $\mathbb{P}^{d-1}$ decomposes into a direct sum of the line bundles $\mathcal{T} \cong \bigoplus \mathcal{O}(-j - d_i)$ where D is the partition we defined in Definition 2.21 [GhISa]. Also, the partition C corresponds to the generator degrees of the ancestor ideal $\overline{V}$, and these are related to the minimum dimension rational scroll containing the rational curve determined by (a basis of) V [I5]. Is there a natural geometric interpretation of the pair C, D, that could generalize to other curves in $\mathbb{P}^{d-1}$?

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References

[Be] D. Berman, Simplicity of a vector space of forms: finiteness of the number of complete Hilbert functions, J. Algebra 45 (1977) 52–57.

[BiGe] A. Bigatti, A. Geramita, Level algebras, lex segments, and minimal Hilbert functions, Comm. Algebra 31 (3) (2003) 1427–1451.

[Bj] M. Boij, Betti numbers of compressed level algebras, J. Pure Appl. Algebra 134 (1) (1999) 11–16.

[BrPV] A. Bruguières, Fibrés de Harder–Narasimham et stratification de Shatz, in: J. Le Potier, J.-L. Verdier (Eds.), Module des Fibrés Stables sur les Courbes Algébriques, in: Progress in Math., vol. 54, Birkhäuser, Boston, 1985, pp. 81–204.

[BrH] W. Bruns, J. Herzog, Cohen–Macaulay Rings, in: Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1993; revised paperback edition, 1998.

[BuEi] D. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for codimension three, Amer. J. Math. 99 (1977) 447–485.

[Ca] E. Carlini, Varieties of simultaneous sums of powers for binary forms, math.AG/0202050.

[CaCh] E. Carlini, J. Chipalkatti, On Waring's problem for several algebraic forms, math.AG/0112110.

[ChGe] J. Chipalkatti, A. Geramita, On parameter spaces for Artin level algebras, math.AG/0204017, Michigan Math. J. 51 (2003) 187–207.

[ChoI] Y. Cho, A. Iarrobino, Hilbert functions of level algebras, J. Algebra 241 (2001) 745–758.

[Di] S.J. Diesel, Some irreducibility and dimension theorems for families of height 3 Gorenstein algebras, Pacific J. Math. 172 (1996) 365–397.

[DF] C. Dionosi, C. Fontanari, Grassmann defectivity à la Terracini, math.AG/0112149.

[EmI1] J. Emsalem, A. Iarrobino, Inverse system of a symbolic power I, J. Algebra 174 (1995) 1080–1090.

[FL] R. Fröberg, D. Laksov, Compressed algebras, in: S. Greco, R. Strano (Eds.), Conf. on Complete Intersections in Acireale, in: Lecture Notes in Math., vol. 1092, Springer-Verlag, Berlin, 1984, pp. 121–151.

[G] A.V. Geramita, Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals, in: The Curves Seminar at Queen's, Vol. X (Kingston, ON, 1995), in: Queen's Papers in Pure and Appl. Math., vol. 102, Queen's Univ., Kingston, ON, 1996, pp. 2–114.

[GHS1] A.V. Geramita, T. Harima, Y.S. Shin, Some special configurations of points in $\mathbb{P}^n$, J. Algebra, in press.

[GHMS1] A.V. Geramita, T. Harima, J. Migliore, Y.S. Shin, The Hilbert function of a level algebra, preprint #346, Univ. Notre Dame, 2003.

[GhISa] F. Ghione, A. Iarrobino, G. Sacchiero, Restricted tangent bundles of rational curves in $\mathbb{P}^n$, preprint.

[Go1] G. Gotzmann, Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes, Math. Z. 158 (1) (1978) 61–70.

[GreK] C. Greene, D.J. Kleitman, Longest chains in the lattice of integer partitions ordered by majorization, European J. Combin. 7 (1986) 1–10.

[Gro] A. Grothendieck, Techniques de construction et théorèmes d'existence en géometrie algébrique, in: Sem. Bourbaki, vol. 221, 1961; in: Fondements de la Géometrie Algébrique, Sem. Bourbaki, Secretariat Math., Paris, 1962, pp. 1957–1962.

[HN] G. Harder, M. Narasimham, On the cohomology groups of moduli spaces, Math. Ann. 212 (1975) 215–248.

[I1] A. Iarrobino, Vector spaces of forms I, Ancestor ideals of a vector space of forms, preprint, 1975.

[I2] A. Iarrobino, Punctual Hilbert schemes, in: Mem. Amer. Math. Soc., vol. 10 (188), Amer. Math. Soc., Providence, RI, 1977.

[I3] A. Iarrobino, Deforming complete intersection Artin algebras. Appendix: Hilbert functions of $\mathbb{C}[x, y]/I$, in: Proc. Sympos. Pure Math., vol. 40 I, 1983, pp. 593–608.

[I4] A. Iarrobino, Compressed algebras: Artin algebras having given socle degrees and maximal length, in: Trans. Amer. Math. Soc., vol. 285 (1), 1984, pp. 337–378.

[I5] A. Iarrobino, Rational curves on scrolls and the restricted tangent bundle: the ancestor ideal of a vector space of forms in $k[x, y]$, preprint.

[I6] A. Iarrobino, Betti strata of height two ideals, preprint.

[IK] A. Iarrobino, V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, in: Springer Lecture Notes in Math., vol. 1721, Springer, Heidelberg, 1999.

[IKl] A. Iarrobino, S. Kleiman, The Gotzmann theorems and the Hilbert scheme, in: A. Iarrobino, V. Kanev (Eds.), Power Sums, Gorenstein Algebras, and Determinantal Loci, Springer Lecture Notes in Math., vol. 1721, Springer, Heidelberg, 1999, pp. 289–312, Appendix C.

[IY] A. Iarrobino, J. Yaméogo, The family G_{T} of graded Artinian quotients of $k[x, y]$ of given Hilbert function, Comm. Algebra 31 (2003) 3863–3916.

[KW] A. King, C. Walter, On Chow rings of fine moduli spaces, J. Reine Angew. Math. 461 (1995) 179–187.

[Klp] J.O. Kleppe, The smoothness and the dimension of PGOR(H) and of other strata of the punctual Hilbert scheme, J. Algebra 200 (1998) 606–628.

[Mac1] F.H.S. Macaulay, The Algebra of Modular Systems, Cambridge Univ. Press, Cambridge, 1916, reprinted with a foreword by P. Roberts, Cambridge Univ. Press.

[Mac2] F.H.S. Macaulay, Some properties of enumeration in the theory of modular systems, Proc. London Math. Soc. 26 (1927) 531–555.

[Mall] D. Mall, Connectedness of Hilbert function strata and other connectedness results, J. Pure Appl. Algebra 150 (2) (2000) 175–205.

[Par] K. Pardue, Deformations of graded modules and connected loci on the Hilbert scheme, in: Queen's Papers Pure Appl. Math., vol. 105, 1997, pp. 131–149.

[Ra] L. Ramella, La stratification du schéma de Hilbert des courbes rationelles de $\mathbb{P}^n$ par le fibré tangent restreint, C. R. Acad. Sci. Paris Sér. I Math. 311 (3) (1990) 181–184.

[St1] R. Stanley, Hilbert functions of graded algebras, Adv. Math. 28 (1978) 57–83.

[St2] R. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth, Belmont, 1986.

[Ve] J.-L. Verdier, Two-dimensional σ-models and harmonic maps from S^2 to S^n, in: Group Theoretical Methods in Physics, Istanbul, 1982, in: Lecture Notes Phys., vol. 180, Springer, Berlin, 1983, pp. 136–141.