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# $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and nonperiodic perturbations ☆

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**Abstract**

We consider the $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and nonperiodic perturbations being stationary at infinity, where the perturbations are done not only for the coefficients but also for the exponents. Using concentration–compactness principle, under appropriate assumptions, we prove the existence of ground state solutions vanishing at infinity for the equations.
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*Keywords:* $p(x)$-Laplacian equation; Variable exponent Sobolev space; Concentration–compactness; Ground state solution

## 1. Introduction and statement of main results

In this paper, we consider the $p(x)$-Laplacian equation in $\mathbb{R}^N$ with periodic data of form

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + V(x)|u|^{p(x)-2}u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}\big(\mathbb{R}^N\big), \end{cases} \tag{1.1}$$

and its nonperiodic perturbation being stationary at infinity of form

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)+\theta(x)-2}\nabla u\big) + a(x)V(x)|u|^{p(x)+\theta(x)-2}u = b(x)f(x,u)|u|^{-\tau(x)} \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)+\theta(x)}\big(\mathbb{R}^N\big). \end{cases} \tag{1.2}$$

In this paper, $\mathbb{R}$ denotes the space of all real numbers, $\mathbb{R}^+ = [0, +\infty)$.

Let $\{e_1, e_2, \ldots, e_N\}$ be the standard basis of $\mathbb{R}^N$. Let $T_i > 0$, $i = 1, 2, \ldots, N$. Denote $T = (T_1, T_2, \ldots, T_N)$. A function $p : \mathbb{R}^N \to \mathbb{R}$ is called $T$-periodic if

$$p(x + T_i e_i) = p(x), \quad \forall x \in \mathbb{R}^N,\ i = 1, 2, \ldots, N.$$

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For problem (1.1) we introduce the following assumptions.

($p_1$) The function $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and

$$1 < p_- := \inf_{\mathbb{R}^N} p(x) \leqslant \sup_{\mathbb{R}^N} p(x) := p_+ < N.$$

($p_2$) $p$ is $T$-periodic.
($V_1$) $V \in C^0(\mathbb{R}^N, \mathbb{R}^+)$, $0 < V_- \leqslant V_+ < \infty$.
($V_2$) $V$ is $T$-periodic.
($f_1$) $f \in C^0(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and

$$\bigl|f(x,t)\bigr| \leqslant C_1\bigl(|t|^{p(x)-1} + |t|^{q(x)-1}\bigr), \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

where $C_1$ is a positive constant, $q \in C^0(\mathbb{R}^N, \mathbb{R})$ and $p \leqslant q \ll p^*$, $p^*$ is defined by

$$p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{for } x \in \mathbb{R}^N,$$

the notation "$q \ll p^*$" means that $\inf\{p^*(x) - q(x)\colon x \in \mathbb{R}^N\} > 0$.
($f_2$) There is a positive constant $\beta > p_+$ such that

$$0 < \beta F(x,t) \leqslant t f(x,t), \quad \forall x \in \mathbb{R}^N,\ t \neq 0,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.
($f_3$) $f(x,t) = o(|t|^{p_+-1})$ as $t \to 0$, uniformly in $x$.
($f_4$) $f(\cdot,t)$ is $T$-periodic for every $t \in \mathbb{R}$.
($f_5$) For each $x \in \mathbb{R}^N$, $\frac{f(x,t)}{|t|^{p_+-1}}$ is an increasing function of $t$ on $\mathbb{R} \setminus \{0\}$.

For problem (1.2) we introduce the following assumptions.

(a) $a \in C^0(\mathbb{R}^N, \mathbb{R}^+)$, $0 < a_- \leqslant a_+ < \infty$ and $\lim_{|x|\to\infty} a(x) = a_+$.
(b) $b \in C^0(\mathbb{R}^N, \mathbb{R}^+)$, $0 < b_- \leqslant b_+ < \infty$ and $\lim_{|x|\to\infty} b(x) = b_-$.
($\theta$) $\theta : \mathbb{R}^N \to \mathbb{R}^+$ is Lipschitz, there exists $R_* > 0$ such that $\theta(x) = 0$ for $|x| \geqslant R_*$, and $(p+\theta)_+ < N$.
($\tau$) $\tau : \mathbb{R}^N \to \mathbb{R}^+$ is Lipschitz, and there exists $R_* > 0$ such that $\tau(x) = 0$ for $|x| \geqslant R_*$.
($\theta,\tau$) $(p+\theta)_+ < \beta - \tau_+$, where $\beta$ is as in ($f_2$).
$(f_3)_*$ $f(x,t)|t|^{-\tau(x)} = o(|t|^{(p+\theta)_+-1})$ as $t \to 0$, uniformly in $x$.
$(f_5)_*$ For each $x \in \mathbb{R}^N$, $\frac{f(x,t)}{|t|^{\beta-1}}$ is an increasing function of $t$ on $\mathbb{R} \setminus \{0\}$, where $\beta$ is as in ($f_2$).

A typical example of $f$ satisfying ($f_1$)–($f_3$), ($f_5$) and $(f_5)_*$ is $f(x,t) = |t|^{q(x)-2}t$, where $q \in C^0(\mathbb{R}^N, \mathbb{R})$, $q_- > p_+$ and $q \ll p^*$. ($V_1$) means that the left-hand side of Eq. (1.1) is positive definite. ($f_1$) means that $f$ satisfies the subcritical growth condition.

The main results of this paper are the following theorems.

**Theorem 1.1.** *Suppose the assumptions* ($p_1$), ($p_2$), ($V_1$), ($V_2$), ($f_1$)–($f_4$) *hold. Then*

(1) *problem* (1.1) *has a nontrivial solution*;
(2) *problem* (1.1) *has a positive solution and a negative solution.*

**Theorem 1.2.** *Suppose that in addition to the assumptions of Theorem* 1.1, ($f_5$) *holds. Then*

(1) *problem* (1.1) *has a ground state solution $u_*$, that is, $u_*$ is a nontrivial solution of* (1.1) *and*

$$J(u_*) = \inf\bigl\{J(u)\colon J'(u)u = 0,\ u \neq 0\bigr\},$$

*where $J(u)$ is the energy functional associated with problem* (1.1) (*for the definition of $J$ see Section* 2);

(2) *problem* (1.1) *has a positive solution* $v$ *and a negative solution* $w$ *such that*

$$J(v) = \inf\{J(u):\ J'(u)u = 0,\ u > 0\},$$
$$J(w) = \inf\{J(u):\ J'(u)u = 0,\ u < 0\}.$$

**Theorem 1.3.** *Suppose that in addition to the assumptions of Theorem* 1.2*, the assumptions* (a)*,* (b)*,* $(\theta)$*,* $(\tau)$*,* $(\theta,\tau)$*,* $(f_3)_*$ *and* $(f_5)_*$ *hold. Then*

(1) *problem* (1.2) *has a ground state solution*;
(2) *problem* (1.2) *has a positive solution* $v$ *and a negative solution* $w$ *such that*

$$J_*(v) = \inf\{J_*(u):\ J_*'(u)u = 0,\ u > 0\},$$
$$J_*(w) = \inf\{J_*(u):\ J_*'(u)u = 0,\ u < 0\},$$

*where* $J_*(u)$ *is the energy functional associated with problem* (1.2).

The problems studied in this paper involve the variable exponent $p(x)$. The variable exponent problems are interesting for some applications (see [24,35]). The study of various mathematical problems with variable exponent has been received considerable attention in recent years. We refer to the survey papers [8,13,36] for the advances and references in this area. The $p(x)$-Laplacian is a generalization of the $p$-Laplacian, and it possesses more complicated nonlinearities than the $p$-Laplacian.

It is well known that a main difficulty in studying the elliptic equations in $\mathbb{R}^N$ is the lack of compactness. To overcome this difficulty, many methods can be used. One type of methods is that under some additional conditions there holds the required compact imbedding theorem, for example, the weighting method and the symmetry method (see e.g. [34,37,39]). In [17] the equations of type (1.1) with weighted function $f(x,u)$ were studied. In [17, Remark 3.3] it was point out that the similar method is also applicable to the case that $V(x) \to +\infty$ as $|x| \to \infty$, and the case of radial symmetry. In [23] a compact imbedding theorem with symmetry of Strauss–Lions type for the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ was obtained and in [21] the nodal solutions of $p(x)$-Laplacian equations possessing radial symmetry were considered. Another type of methods is concentration–compactness principle, discovered by P.L. Lions [28,29]. By this principle, under suitable conditions, a noncompact minimizing or $(\mathrm{PS})_c$ sequence can be changed into a new sequence possessing some compactness. For such purpose the following methods are often used:

($1^0$) Translations. It is applicable to homogeneous equations, i.e., equations not clearly including $x$. In this case the corresponding energy functionals are invariant under translations. For applying this method to the $p$-Laplacian equations, we refer to [27–29,37–39] and references therein.
($2^0$) Periodicity. It is applicable to the equations possessing periodicity. In this case the corresponding energy functionals are invariant under period-translations. For applying this method to the $p$-Laplacian equations, the Schrödinger equations and the biharmonic equations, we refer to [2–6,9,10,26,30–33,40] and references therein. Pankov [31] and Pankov and Pflüger [32] have used the method of periodic approximations.
($3^0$) In comparison with a limiting equation. The idea of this method is to compare the original equation with its limiting equation at infinity, especially to compare the corresponding critical values for these two equations, where the existence of the ground state solutions for the limiting equation is known. For applying this method to the $p$-Laplacian equations, the Schrödinger equations and the biharmonic equations, we refer to [3–6,11,12, 14,28–30,34,37–39,41] and references therein. Usually the limiting equations are homogeneous, but in [3–6,30] the limiting equations are periodic. Alves and Souto [7] have studied the $p(x)$-Laplacian equations such that the variable exponent $p(x)$ is constant outside a ball, and thus in [7] the limiting equation is homogeneous.

For $p$-Laplacian equations, the constant exponent $p$, as a function on $\mathbb{R}^N$, is periodic and is also invariant under translations. For $p(x)$-Laplacian equations, of course, we cannot require that the variable exponent $p(x)$ is invariant under all translations. In this paper we study $p(x)$-Laplacian equations (1.1) and (1.2) by using methods ($2^0$) and ($3^0$). Equation (1.1) is periodic and Eq. (1.2) is a nonperiodic perturbation of (1.1). Note that all the coefficients and the exponents in (1.1) are perturbed, and the limiting equation of (1.2) is not homogeneous but periodic. The perturbation of the exponents is a distinguishing characteristic of variable exponent problems. In this paper the idea of Pankov [30] and the idea of Alves et al. [3–7] are used together. Theorem 1.3 is a generalization of the corresponding results of [7] and [30].

This paper is organized as follows. In Section 2, we present some necessary preliminaries. In Section 3, we prove Theorems 1.1 and 1.2. The proof of Theorem 1.1 is based on Proposition 2.3, a Lions type lemma for the variable exponent space $W^{1,p(x)}(\mathbb{R}^N)$ obtained by Fan, Zhao and Zhao [23]. The proof of Theorem 1.2 is done according to the idea of the concentration–compactness but not directly applying the first concentration–compactness principle of Lions. Such a proof seems to be simpler because the "dichotomy" case mentioned in the first concentration–compactness principle is evaded. In Section 4 we give the proof of Theorem 1.3.

## 2. Preliminaries

Let $\Omega$ be an open subset of $\mathbb{R}^N$, $p \in L^\infty(\Omega)$ and $p_-(\Omega) = \operatorname{ess\,inf}_{x\in\Omega} p(x) \geqslant 1$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)}\,dx < \infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{ \lambda > 0\colon \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leqslant 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega)\colon\ |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

When $V$ satisfies $(\mathrm{V}_1)$, it is easy to see that $\|u\|_{W_V^{1,p(x)}(\Omega)}$, defined by

$$\|u\|_{W_V^{1,p(x)}(\Omega)} = \inf\left\{ \lambda > 0\colon \int_\Omega \left( \left|\frac{\nabla u}{\lambda}\right|^{p(x)} + V(x)\left|\frac{u}{\lambda}\right|^{p(x)} \right) dx \leqslant 1 \right\},$$

is an equivalent norm in $W^{1,p(x)}(\Omega)$.

For the basic properties of spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ we refer to [13,15,18,20,25,36]. In the following we list some facts which will be used later. In this paper, for $x \in \mathbb{R}^N$ and $R > 0$, $B(x,R) := \{y \in \mathbb{R}^N\colon\ |y-x| < R\}$ and $B_R = B(0,R)$. The symbols $u_n \to u_0$ and $u_n \rightharpoonup u_0$ denote the strong convergence and weak convergence of a sequence $\{u_n\}$ in a Banach space, respectively.

**Proposition 2.1.** *(See [15,20,25].) The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable Banach spaces, and they are reflexive when $p_-(\Omega) > 1$.*

**Proposition 2.2.** *(See [18].) Suppose that $p$ satisfies $(\mathrm{p}_1)$, $q \in C^0(\mathbb{R}^N, \mathbb{R})$ and $p \leqslant q \ll p^*$. Then there is a continuous embedding $W^{1,p(x)}(\mathbb{R}^N) \to L^{q(x)}(\mathbb{R}^N)$. If $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with cone property, then the embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$ is compact.*

**Proposition 2.3.** *(See [23].) Suppose that $p$ satisfies $(\mathrm{p}_1)$. If $\{u_n\}$ is a bounded sequence in $W^{1,p(x)}(\mathbb{R}^N)$ and for some $R > 0$,*

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_n|^{q(x)}\,dx \to 0 \quad \text{as } n \to \infty, \tag{2.1}$$

*for some* $R > 0$ *and some* $q \in C^0(\mathbb{R}^N, \mathbb{R})$ *satisfying* $p \leqslant q \ll p^*$, *then* $u_n \to 0$ *in* $L^{r(x)}(\mathbb{R}^N)$ *for any* $r$ *satisfying* $p \ll r \ll p^*$.

Let $\Omega$ be an open subset of $\mathbb{R}^N$. Define for $u \in W^{1,p(x)}(\Omega)$,

$$J_\Omega(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx - \int_\Omega F(x,u)\,dx,$$

$$I_\Omega(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx,$$

$$\Psi_\Omega(u) = \int_\Omega F(x,u)\,dx. \tag{2.2}$$

When $\Omega = \mathbb{R}^N$, $J_{\mathbb{R}^N}$, $I_{\mathbb{R}^N}$ and $\Psi_{\mathbb{R}^N}$ are written simply by $J$, $I$ and $\Psi$, respectively. $J$ is the energy functional associated with problem (1.1).

**Proposition 2.4.** *(See [17].) Suppose* $(\mathrm{p}_1)$, $(\mathrm{V}_1)$ *and* $(\mathrm{f}_1)$ *hold. Then the following assertions are true.*

(1) $J_\Omega \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ *and for every* $u, v \in W^{1,p(x)}(\Omega)$,

$$J'_\Omega(u)v = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right)dx - \int_\Omega f(x,u)v\,dx. \tag{2.3}$$

(2) *The mapping* $I'_\Omega : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ *is a strictly monotone, bounded homeomorphism, and is of* $(S_+)$ *type, namely*

$$u_n \rightharpoonup u \quad \text{and} \quad \varlimsup_{n\to\infty} I'_\Omega(u_n)(u_n - u) \leqslant 0 \quad \text{imply } u_n \to u.$$

$u \in W^{1,p(x)}(\mathbb{R}^N)$ is called a weak solution of problem (1.1) if $u$ is a critical point of $J$, that is, for every $v \in W^{1,p(x)}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right)dx - \int_{\mathbb{R}^N} f(x,u)v\,dx = 0.$$

**Proposition 2.5.** *Suppose* $(\mathrm{p}_1)$, $(\mathrm{V}_1)$ *and* $(\mathrm{f}_1)$ *hold. If* $u$ *is a weak solution of problem* (1.1)*, then* $u \in C^{1,\alpha}(\mathbb{R}^N)$, $u(x) \to 0$ *and* $|\nabla u(x)| \to 0$ *as* $|x| \to \infty$.

**Proof.** Let $u \in W^{1,p(x)}(\mathbb{R}^N)$ be a weak solution of problem (1.1). By the regularity result on local boundedness of the weak solutions (see [19]), we know that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and for every bounded open subsets $\Omega \subset \overline{\Omega} \subset \Omega' \subset \mathbb{R}^N$, $|u|_{L^\infty(\Omega)}$ depends only on $N, p_-, p_+, q_-, q_+, C_1$, $\operatorname{dist}(\Omega, \partial\Omega')$ and $\int_{\Omega'}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx$. Given any $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx \leqslant \varepsilon.$$

For every $x_0 \in \mathbb{R}^N$ with $|x_0| \geqslant R_\varepsilon + 2$, we have that $|u|_{L^\infty(B(x_0,1))} \leqslant C(\varepsilon)$, where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. From this we can see that $u(x) \to 0$ as $|x| \to \infty$. Similarly, by the $C^{1,\alpha}$ regularity of the bounded weak solutions (see [1,16]), we can see that $u \in C^{1,\alpha}(\mathbb{R}^N)$ and $|\nabla u(x)| \to 0$ as $|x| \to \infty$. $\square$

**Remark 2.1.**

(1) Let assumptions $(f_1)$ and $(f_2)$ hold. $(f_1)$ implies that

$$\left|F(x,t)\right| \leqslant C_1\left(|t|^{p(x)} + |t|^{q(x)}\right), \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{2.4}$$

$(f_2)$ implies that

$$F(x,t) \geqslant C_2|t|^{\beta}, \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{2.5}$$

So there holds that $C_2|t|^{\beta} \leqslant C_1(|t|^{p(x)} + |t|^{q(x)})$, which implies that $q(x) \geqslant \beta$ for all $x \in \mathbb{R}^N$ and hence

$$p_+ < \beta \leqslant q_- < p_-^*. \tag{2.6}$$

(2) Let $(f_1)$, $(f_2)$ and $(f_3)$ hold. It follows from $(f_1)$, $(f_3)$ and (2.6) that, given any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

$$\left|f(x,t)\right| \leqslant \varepsilon|t|^{p_+-1} + C(\varepsilon)|t|^{q(x)-1}, \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{2.7}$$

and consequently

$$\left|F(x,t)\right| \leqslant \varepsilon|t|^{p_+} + C(\varepsilon)|t|^{q(x)}, \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{2.8}$$

(3) It is clear that $(f_3)_*$ implies $(f_3)$, and $(f_5)_*$ implies $(f_5)$. Note that when $(f_2)$ with $\beta_1 > p_+$ and $(f_5)_*$ with $\beta_2 > p_+$ hold, $(f_2)$ and $(f_5)_*$ with $\beta := \min\{\beta_1, \beta_2\}$ hold.

**Remark 2.2.** It is easy to see that, when $\theta$ satisfies $(\theta)$, there is a continuous embedding $W^{1,p(x)+\theta(x)}(\mathbb{R}^N) \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$.

## 3. Solutions of problem (1.1)

In this section, we consider problem (1.1) and prove Theorems 1.1 and 1.2.

Let $J = J_{\mathbb{R}^N}$, $I = I_{\mathbb{R}^N}$ and $\Psi = \Psi_{\mathbb{R}^N}$ be as in Section 2. We write $\|u\| = \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}$ and $\|u\|_V = \|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)}$.

**Lemma 3.1.** *Let* $(p_1)$, $(V_1)$ *and* $(f_1)$ *hold. Suppose that* $\{u_n\}$ *is a sequence in* $W^{1,p(x)}(\mathbb{R}^N)$ *such that* $u_n \rightharpoonup u$ *in* $W^{1,p(x)}(\mathbb{R}^N)$*,* $u_n(x) \to u(x)$ *a.e.* $x \in \mathbb{R}^N$ *and* $J'(u_n) \to 0$ *in* $(W^{1,p(x)}(\mathbb{R}^N))^*$ *as* $n \to \infty$*. Then the following assertions are true.*

(1) $u_n \to u$ *in* $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$.
(2) $J'(u_n) \rightharpoonup J'(u)$ *in* $(W^{1,p(x)}(\mathbb{R}^N))^*$ *and consequently* $J'(u) = 0$. *So* $u$ *is a solution of* (1.1).

**Proof.** (1) It follows from $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$ that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$ for every bounded open ball $\Omega \subset \mathbb{R}^N$, and consequently

$$u_n \to u \quad \text{in } L^{q(x)}(\Omega) \text{ for } q \in C^0\left(\mathbb{R}^N, \mathbb{R}\right) \text{ satisfying } p \leqslant q \ll p^*. \tag{3.1}$$

Now let $R > 0$ be given. We will prove that $u_n \to u$ in $W^{1,p(x)}(B(0,R))$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi(x) = 1$ if $|x| \leqslant R$, $\varphi(x) = 0$ if $|x| \geqslant R+2$, $\varphi(x) \in [0,1]$ and $|\nabla\varphi(x)| \leqslant 1$ for all $x \in \mathbb{R}^N$. Put

$$\begin{aligned} Q_n(x) := {}& \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u) \\ &+ V(x)\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u). \end{aligned}$$

We have

$$0 \leqslant \int_{B_R} Q_n(x)\,dx \leqslant \int_{\mathbb{R}^N} \varphi(x) Q_n(x)\,dx$$
$$= \int_{\mathbb{R}^N} \big(|\nabla u_n|^{p(x)-2}\nabla u_n \varphi(\nabla u_n - \nabla u) + V(x)|u_n|^{p(x)-2} u_n \varphi(u_n - u)\big)\,dx + o(1)$$
$$= J'(u_n)(\varphi u_n - \varphi u) - \int_{B_{R+2}\setminus B_R} (u_n - u)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi\,dx + \int_{B_{R+2}} f(x,u_n)\varphi(u_n - u)\,dx + o(1).$$

From $J'(u_n) \to 0$ and (3.1) it follows that $\int_{B_R} Q_n(x)\,dx \to 0$, that is $(I'_{B_R}(u_n) - I'_{B_R}(u))(u_n - u) \to 0$. Thus $I'_{B_R}(u_n)(u_n - u) \to 0$ and $u_n \rightharpoonup u$ in $W^{1,p(x)}(B_R)$. Since $I'_{B_R}$ is of $(S_+)$ type, $u_n \to u$ in $W^{1,p(x)}(B_R)$. Assertion (1) is proved.

(2) Denote $W_c^{1,p(x)}(\mathbb{R}^N) = \{v \in W^{1,p(x)}(\mathbb{R}^N)\colon \operatorname{supp} v \text{ is compact}\}$. Then $W_c^{1,p(x)}(\mathbb{R}^N)$ is dense in $W^{1,p(x)}(\mathbb{R}^N)$. For each $v \in W_c^{1,p(x)}(\mathbb{R}^N)$, taking $R > 0$ sufficiently large such that $\operatorname{supp} v \subset B_R$ and noting that $u_n \to u$ in $W^{1,p(x)}(B_R)$, we have that

$$J'(u_n)v = J'_{B_R}(u_n)v \to J'_{B_R}(u)v = J'(u)v.$$

This shows $J'(u_n) \rightharpoonup J'(u)$ in $(W^{1,p(x)}(\mathbb{R}^N))^*$. Since $J'(u_n) \to 0$, we have $J'(u) = 0$. Assertion (2) is proved. □

**Lemma 3.2.** *Let* ($p_1$), ($V_1$), ($f_1$) *and* ($f_2$) *hold. If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* $J$*, that is,* $J(u_n) \to c$ *and* $J'(u_n) \to 0$ *as* $n \to \infty$*, then* $\{\|u_n\|\}$ *is bounded.*

**Proof.** We may assume that $\|u_n\|_V \geqslant 1$ for all $n$. For $n$ sufficiently large, we have

$$c + 1 + \|u\|_V \geqslant J(u_n) - \frac{1}{\beta} J'(u_n)u_n \geqslant \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\|u\|_V^{p_-},$$

which implies that $\{\|u_n\|\}$ is bounded. □

**Lemma 3.3.** *Let* ($p_1$), ($V_1$), ($f_1$), ($f_2$) *and* ($f_3$) *hold. Then* $J$ *satisfies the Mountain Pass Geometry, that is,*

(1) *there exist positive numbers* $\rho$ *and* $\alpha$ *such that* $J(u) \geqslant \alpha$ *for* $\|u\| = \rho$*;*
(2) *there exists* $v \in W^{1,p(x)}(\mathbb{R}^N)$ *such that* $\|v\| > \rho$ *and* $J(v) < 0$*.*

**Proof.** (1) By (2.6), (2.8) and Proposition 2.2, we have that, for any $\varepsilon > 0$, when $\|u\|_V \leqslant 1$,

$$J(u) \geqslant \|u\|_V^{p_+} - \int_{\mathbb{R}^N} \big(\varepsilon|u|^{p_+} + C(\varepsilon)|u|^{q(x)}\big)\,dx$$
$$\geqslant \|u\|_V^{p_+} - \varepsilon C_3\|u\|_V^{p_+} - C(\varepsilon)C_4\|u\|_V^{q_-}.$$

Taking $\varepsilon = \frac{1}{2C_3}$ in the above inequality and noting that $q_- > p_+$, we can see that assertion (1) holds.

(2) Assertion (2) follows easily from (2.5). □

**Lemma 3.4.** *Let* ($p_1$), ($V_1$), ($f_1$)–($f_3$) *hold. Suppose that* $\{u_n\}$ *is a sequence in* $W^{1,p(x)}(\mathbb{R}^N)$ *such that* $\{\|u_n\|\}$ *is bounded and* $J'(u_n) \to 0$ *in* $(W^{1,p(x)}(\mathbb{R}^N))^*$ *as* $n \to \infty$*. Then, passing to a subsequence still labeled by* $n$*, either*

($1^0$) $u_n \to 0$ *in* $W^{1,p(x)}(\mathbb{R}^N)$ *as* $n \to \infty$*,*

*or*

($2^0$) *there exist a sequence* $\{y_n\} \subset \mathbb{R}^N$ *and positive numbers* $R$ *and* $\delta$ *such that*

$$\int_{B(y_n,R)} \left|u_n(x)\right|^{p(x)}\,dx \geqslant \delta \quad \textit{for all } n.$$

**Proof.** Assume that ($2^0$) does not hold. Then, passing to a subsequence still labeled by $n$, for some $R > 0$,

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_n|^{p(x)}\,dx \to 0 \quad \text{as } n \to \infty.$$

By Proposition 2.3, $u_n \to 0$ in $L^{q(x)}(\mathbb{R}^N)$. Now $J'(u_n)u_n = I'(u_n)u_n - \Psi'(u_n)u_n \to 0$. By (2.7) we know that, for any $\varepsilon > 0$,

$$\left|\Psi'(u_n)u_n\right| = \left|\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx\right| \leqslant \left|\int_{\mathbb{R}^N} \left(\varepsilon|u_n|^{p_+} + C(\varepsilon)|u_n|^{q(x)}\right)dx\right|. \tag{3.2}$$

The boundedness of $\{\|u_n\|\}$ implies the boundedness of $\{|u_n|_{L^{p_+}(\mathbb{R}^N)}\}$. Noting that $u_n \to 0$ in $L^{q(x)}(\mathbb{R}^N)$, from (3.2) and the arbitrariness of $\varepsilon > 0$ it follows that $\Psi'(u_n)u_n \to 0$. Thus

$$I'(u_n)u_n = \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)dx \to 0,$$

which implies $u_n \to 0$ in $W^{1,p(x)}(\mathbb{R}^N)$. □

**Proof of Theorem 1.1.** (1) Define

$$\begin{aligned} &\Gamma = \left\{\gamma \in C^0\left([0,1], W^{1,p(x)}\left(\mathbb{R}^N\right)\right):\ \gamma(0) = 0 \text{ and } J\left(\gamma(1)\right) < 0\right\},\\ &c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J\left(\gamma(t)\right). \end{aligned} \tag{3.3}$$

By Lemma 3.3 and Mountain Pass theorem (see e.g. [38,39]), there exists a sequence $\{u_n\} \subset W^{1,p(x)}(\mathbb{R}^N)$ such that $J(u_n) \to c \geqslant \alpha > 0$ and $J'(u_n) \to 0$. By Lemma 3.2, $\{\|u_n\|\}$ is bounded. Applying Lemma 3.4 to $\{u_n\}$ and noting that the case ($1^0$) does not hold because $c > 0$ and $J(0) = 0$, we know that, for a subsequence of $\{u_n\}$, denoted still by $\{u_n\}$, the case ($2^0$) holds, that is, there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and positive numbers $R$ and $\delta$ such that

$$\int_{B(y_n,R)} \left|u_n(x)\right|^{p(x)}\,dx \geqslant \delta \quad \text{for all } n.$$

Put

$$D = \left\{x = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N:\ -\frac{T_i}{2} \leqslant \xi_i < \frac{T_i}{2},\ i = 1, 2, \ldots, N\right\}.$$

Denote by $L$ the diameter of $D$. For each $y_n$, there are integers $k_1^{(n)}, k_2^{(n)}, \ldots, k_N^{(n)}$ such that $z_n = y_n - (k_1^{(n)}T_1, k_2^{(n)}T_2, \ldots, k_N^{(n)}T_N) \in D$.

Denote $h_n = (k_1^{(n)}T_1, k_2^{(n)}T_2, \ldots, k_N^{(n)}T_N)$ and put $v_n(x) = u_n^{h_n}(x) = u_n(x + h_n)$. Since $p$, $V$ and $f(\cdot,t)$ are $T$-periodic, we have that

$$\|v_n\| = \|u_n\|, \qquad J(v_n) = J(u_n) \to c, \qquad J'(v_n) \to 0,$$

and

$$\int_{B(z_n,R)} \left|v_n(x)\right|^{p(x)}\,dx = \int_{B(y_n,R)} \left|u_n(x)\right|^{p(x)}\,dx \geqslant \delta \quad \text{for all } n.$$

Noting that $B(z_n, R) \subset B(0, R + L)$, we have

$$\int_{B(0,R+L)} \left|v_n(x)\right|^{p(x)}\,dx \geqslant \int_{B(z_n,R)} \left|v_n(x)\right|^{p(x)}\,dx \geqslant \delta \quad \text{for all } n. \tag{3.4}$$

We may assume, taking a subsequence if necessary, that $v_n \rightharpoonup v$ in $W^{1,p(x)}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$. By Lemma 3.1, $v_n \to v$ in $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ and $J'(v) = 0$. From (3.4) it follows that

$$\int_{B(0,R+L)} \left|v(x)\right|^{p(x)}\,dx = \lim_{n\to\infty} \int_{B(0,R+L)} \left|v_n(x)\right|^{p(x)}\,dx \geqslant \delta > 0,$$

which shows $v \neq 0$. So $v$ is a nontrivial solution of (1.1).

(2) Define $f^+ : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$f^+(x,t) = \begin{cases} f(x,t) & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Then, similar to (1), we can prove that the problem

$$\begin{cases} -\operatorname{div}\bigl(|\nabla u|^{p(x)-2}\nabla u\bigr) + V(x)|u|^{p(x)-2}u = f^+(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}\bigl(\mathbb{R}^N\bigr) \end{cases} \tag{3.5}$$

has a nontrivial solution $v$. It is easy to see that, $v$, as a solution of (3.5), is nonnegative, and hence $v$ is a solution of (1.1). By the strong maximum principle of [22], $v(x) > 0$ for all $x \in \mathbb{R}^N$ and so $v$ is a positive solution of (1.1). Similarly, problem (1.1) has a negative solution. Theorem 1.1 is proved. □

**Lemma 3.5.** *Let* $(\mathrm{p}_1)$, $(\mathrm{V}_1)$ *and* $(\mathrm{f}_1)$ *hold, and let* $\{u_n\}$ *and* $u$ *be as in Lemma* 3.1. *Then, given any* $\varepsilon > 0$, *there exists* $R_\varepsilon > 0$ *such that for each* $R \geqslant R_\varepsilon$ *and* $n$ *sufficiently large,* $|J'_{\mathbb{R}^N\setminus B_R}(u_n)u_n| \leqslant \varepsilon$, *i.e.*

$$\left| \int_{\mathbb{R}^N\setminus B_R} \bigl(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} - f(x,u_n)u_n\bigr)\,dx \right| \leqslant \varepsilon.$$

**Proof.** Let $\varepsilon > 0$ be given. Since $u \in W^{1,p(x)}(\mathbb{R}^N)$, there exists $R_\varepsilon > 2$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_\varepsilon - 2}} \bigl(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} + f(x,u)u + |\nabla u|^{p(x)-1}|u|\bigr)\,dx \leqslant \frac{\varepsilon}{2}. \tag{3.6}$$

Let $R \geqslant R_\varepsilon$ and $\psi \in C^\infty(\mathbb{R}^N)$ be such that $\psi(x) = 1$ for $|x| \geqslant R_\varepsilon$, $\psi(x) = 0$ for $|x| \leqslant R_\varepsilon - 2$, $\psi(x) \in [0,1]$ and $|\nabla\psi(x)| \leqslant 1$ for all $x \in \mathbb{R}^N$. Then $\{\psi u_n\}$ is a bounded sequence in $W^{1,p(x)}(\mathbb{R}^N)$. Hence $J'(u_n)(\psi u_n) \to 0$, that is

$$\int_{\mathbb{R}^N} \bigl(|\nabla u_n|^{p(x)-2}\nabla u_n(\psi\nabla u_n + u_n\nabla\psi) + \psi V(x)|u_n|^{p(x)} - f(x,u_n)\psi u_n\bigr)\,dx \to 0. \tag{3.7}$$

From (3.7) we have that

$$\begin{aligned} &\bigl|J'_{\mathbb{R}^N\setminus B_R}(u_n)u_n\bigr| \\ &\quad = \left| \int_{\mathbb{R}^N\setminus B_R} \bigl(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} - f(x,u_n)u_n\bigr)\,dx \right| \\ &\quad \leqslant \left| \int_{B_R\setminus B_{R_\varepsilon-2}} \bigl(u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi + \psi\bigl(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} - f(x,u_n)u_n\bigr)\bigr)\,dx \right| + o(1) \\ &\quad \leqslant \left| \int_{B_R\setminus B_{R_\varepsilon-2}} \bigl(|\nabla u_n|^{p(x)-1}|u_n| + |\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} + f(x,u_n)u_n\bigr)\,dx \right| + o(1). \end{aligned} \tag{3.8}$$

From Lemma 3.1 we know that $u_n \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$. Thus the limit of the right side of (3.8) equals the left side of (3.6) and consequently the assertion of Lemma 3.5 is true. □

**Proposition 3.1.** *Let* ($p_1$), ($V_1$), ($f_1$) *and* ($f_2$) *hold. Suppose that* $\{u_n\}$ *and* $u$ *are as in Lemma* 3.1 *and* $J(u_n) \to c > 0$. *Then* $0 \leqslant J(u) \leqslant c$, *and* $J(u) > 0$ *provided* $u \neq 0$.

**Proof.** Note that for any critical point $u$ of $J$,

$$J(u) \geqslant \frac{1}{p_+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx - \frac{1}{\beta}\int_{\mathbb{R}^N} f(x,u)u\,dx$$
$$= \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx \geqslant 0,$$

and $J(u) > 0$ provided $u \neq 0$. It remains to prove that $J(u) \leqslant c$. Given any $\varepsilon > 0$, let $R_\varepsilon > 0$ be as in Lemma 3.5, then for each $R \geqslant R_\varepsilon$ and for $n$ sufficiently large,

$$J_{\mathbb{R}^N\setminus B_R}(u_n) \geqslant \frac{1}{p_+}\int_{\mathbb{R}^N\setminus B_R}\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)dx - \frac{1}{\beta}\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx$$
$$= \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx + \frac{1}{\beta}J'_{\mathbb{R}^N\setminus B_R}(u_n)u_n$$
$$\geqslant -\frac{\varepsilon}{\beta} \geqslant -\varepsilon,$$

and consequently

$$J_{B_R}(u_n) = J(u_n) - J_{\mathbb{R}^N\setminus B_R}(u_n) \leqslant c + 2\varepsilon.$$

Since $u_n \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, $J_{B_R}(u) = \lim_{n\to\infty} J_{B_R}(u_n) \leqslant c + 2\varepsilon$. Furthermore, $J(u) = \lim_{R\to\infty} J_{B_R}(u) \leqslant c + 2\varepsilon$ and consequently, by the arbitrariness of $\varepsilon > 0$, $J(u) \leqslant c$. □

Define

$$\mathbf{N} = \left\{u \in W^{1,p(x)}\left(\mathbb{R}^N\right) \setminus \{0\}\colon\ J'(u)u = 0\right\}.$$

**Lemma 3.6.** *Let* ($p_1$), ($V_1$), ($f_1$)–($f_3$) *and* ($f_5$) *hold. Then for any* $u \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\}$ *there exists a unique* $t_*(u) > 0$ *such that* $t_*(u)u \in \mathbf{N}$. *The maximum of* $J(tu)$ *for* $t \geqslant 0$ *is achieved at* $t_* = t_*(u)$. *The function* $u \mapsto t_*(u)$ *is continuous on* $W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\}$ *and the mapping* $u \mapsto t_*(u)u$ *defines a homeomorphism of the unit sphere of* $W^{1,p(x)}(\mathbb{R}^N)$ *with* $\mathbf{N}$.

**Proof.** Let $u \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\}$ be fixed and define the function $g(t) := J(tu)$ on $[0,\infty)$. It is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Therefore $\max_{t\in[0,\infty)} g(t)$ is achieved at some $t_* = t_*(u) > 0$. Thus $g'(t_*) = J'(t_*u)u = 0$. Put $v = t_*u$. Then $J'(v)v = 0$ and so $v \in \mathbf{N}$. Define the function $h(t) = J(tv)$ on $[0,\infty)$. We already know that $h(1) = \max_{t\in[0,\infty)} h(t)$ and $h'(1) = J'(v)v = 0$, that is

$$\int_{\mathbb{R}^N}\left(|\nabla v|^{p(x)} + V(x)|v|^{p(x)}\right)dx = \int_{\mathbb{R}^N} f(x,v)v\,dx.$$

When $t > 1$,

$$h'(t) = J'(tv)v = \int_{\mathbb{R}^N} t^{p(x)-1}\left(|\nabla v|^{p(x)} + V(x)|v|^{p(x)}\right)dx - \int_{\mathbb{R}^N} f(x,tv)v\,dx$$

$$\leqslant t^{p_+-1}\left(\int\limits_{\mathbb{R}^N}\big(|\nabla v|^{p(x)}+V(x)|v|^{p(x)}\big)\,dx-\int\limits_{\mathbb{R}^N}\frac{1}{t^{p_+-1}}f(x,tv)v\,dx\right),$$

and consequently $h'(t)<0$ because from $(f_5)$ it follows that $\int_{\mathbb{R}^N}\frac{1}{t^{p_+-1}}f(x,tv)v\,dx>\int_{\mathbb{R}^N}f(x,v)v\,dx$. Analogously, when $t\in(0,1)$ there holds $h'(t)>0$. This shows that the positive number $t_*$ satisfying $g'(t_*)=J'(t_*u)u=0$ is unique. The proof of the remainder statements is standard (see e.g. [39]) and is omitted here. □

**Definition 3.1.** A nontrivial solution $u$ of (1.1) is called a ground state if $J(u)=\inf_{v\in\mathbf{N}}J(v)$.

**Lemma 3.7.** *Let* $(p_1)$, $(V_1)$, $(f_1)$–$(f_3)$ *and* $(f_5)$ *hold. Let $c$ be the value defined by* (3.3). *Then* $c=\inf_{u\in\mathbf{N}}J(u)$.

**Proof.** Put $c_1=\inf_{u\in\mathbf{N}}J(u)$. By Lemma 3.6, $\mathbf{N}$ separates $W^{1,p(x)}(\mathbb{R}^N)$ into two components. It is easy to verify that every $\gamma\in\Gamma$ has to cross $\mathbf{N}$. Thus $c\geqslant c_1$. Given any $u\in\mathbf{N}$, there exists $s>0$ such that $J(su)<0$. Define $\gamma(t)=tsu$ for $t\in[0,1]$. Then $\gamma\in\Gamma$ and $J(u)=\max_{t\in[0,1]}J(\gamma(t))\geqslant c$. This shows $c_1\geqslant c$ and hence $c=c_1$. □

**Lemma 3.8.** *Under the assumptions of Lemma* 3.7, *if $v\in N$ and $J(v)=\inf_{u\in\mathbf{N}}J(u)$, then $J'(v)=0$ and so $v$ is a ground state solution of problem* (1.1).

The proof of Lemma 3.8 is similar to the proof of Theorem 4.3 of [39] and is omitted here.

**Proof of Theorem 1.2.** (1) By Theorem 1.1 and Proposition 3.1, problem (1.1) has a nontrivial solution $v$ such that $0<J(v)\leqslant c$. Since $v\in\mathbf{N}$, by Lemma 3.7, $J(v)\geqslant c$ and hence $J(v)=c$. So $v$ is a ground state.

(2) Define $\tilde f^+:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ by

$$\tilde f^+(x,t)=\begin{cases}f(x,t) & \text{if } t\geqslant 0,\\ -f(x,-t) & \text{if } t<0,\end{cases}$$

and consider the problem

$$\begin{cases}-\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big)+V(x)|u|^{p(x)-2}u=\tilde f^+(x,u) & \text{in } \mathbb{R}^N,\\ u\in W^{1,p(x)}\big(\mathbb{R}^N\big).\end{cases}\tag{3.9}$$

Denote by $\widetilde J^+(u)$ the energy functional associated with problem (3.9). Then, applying the above assertion (1) to problem (3.9), (3.9) has a ground state solution $u_0$. Put $v(x)=|u_0(x)|$ for $x\in\mathbb{R}^N$. It is easy to see that $\widetilde J^+(v)=\widetilde J^+(u_0)$ and $(\widetilde J^+)'(v)v=(\widetilde J^+)'(u_0)u_0=0$. By Lemma 3.8, $v$ is a ground state solution of (3.9). Since $v$ is nonnegative, $v$ is also a solution of (1.1). By the strong maximum principle of [22], $v$ is a positive solution of (1.1). Since $v$ is a ground state solution of (3.9), we have that $J(v)=\widetilde J^+(v)=\inf\{J(u):\ J'(u)u=0,\ u>0\}$. Similarly, we can prove that (1.1) has a negative solution $w$ satisfying required condition. □

The following proposition shows that the ground state solution of (1.1) is the strong limit of the corresponding $(\mathrm{PS})_c$ sequence in the norm topology.

**Proposition 3.2.** *Let* $(p_1)$, $(V_1)$, $(f_1)$–$(f_3)$ *and* $(f_5)$ *hold. Suppose that $\{u_n\}$ is a sequence in $W^{1,p(x)}(\mathbb{R}^N)$ such that $u_n\rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$, $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^N$, $J'(u_n)\to 0$ in $(W^{1,p(x)}(\mathbb{R}^N))^*$ and $J(u_n)\to c$ as $n\to\infty$, where $c$ is as in* (3.3). *If $J(u)=c$, then*

(1) $\lim_{R\to\infty}\int_{\mathbb{R}^N\setminus B_R}(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)})\,dx=0$ *uniformly in $n$,*

(2) $u_n\to u$ *in* $W^{1,p(x)}(\mathbb{R}^N)$.

**Proof.** (1) Let $\varepsilon>0$ be given. By Lemma 3.6, there exists $R_\varepsilon>0$ such that for each $R\geqslant R_\varepsilon$ and $n$ sufficiently large, $|J'_{\mathbb{R}^N\setminus B_R}(u_n)u_n|\leqslant\varepsilon$. We may assume that $R_\varepsilon$ is large enough such that $|J_{\mathbb{R}^N\setminus B_R}(u)|\leqslant\varepsilon$ for $R\geqslant R_\varepsilon$. Let $R\geqslant R_\varepsilon$. Then

$$J_{B_R}(u) = J(u) - J_{\mathbb{R}^N\setminus B_R}(u) \geqslant c - \varepsilon.$$

By Lemma 3.1, $u_n \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ and consequently $J_{B_R}(u_n) \to J_{B_R}(u)$. Thus for $n$ sufficiently large, $J_{B_R}(u_n) \geqslant c - 2\varepsilon$, $J_{\mathbb{R}^N\setminus B_R}(u_n) \leqslant 3\varepsilon$ and consequently

$$\begin{aligned}
3\varepsilon &\geqslant J_{\mathbb{R}^N\setminus B_R}(u_n) \\
&\geqslant \frac{1}{p_+}\int_{\mathbb{R}^N\setminus B_R}\big(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\big)\,dx - \frac{1}{\beta}\int_{\mathbb{R}^N\setminus B_R} f(x,u_n)u_n\,dx \\
&= \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N\setminus B_R}\big(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\big)\,dx + \frac{1}{\beta}J'_{\mathbb{R}^N\setminus B_R}(u_n)u_n \\
&\geqslant \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N\setminus B_R}\big(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\big)\,dx - \varepsilon,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N\setminus B_R}\big(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\big)\,dx \leqslant \left(\frac{1}{p_+} - \frac{1}{\beta}\right)^{-1} 4\varepsilon.$$

Assertion (1) is proved.

(2) Define $\rho_\Omega(u) = \int_\Omega(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx$ and write $\rho_{\mathbb{R}^N} = \rho$. Since $\{\|u_n\|\}$ is bounded, we may assume that $\rho(u_n) \to d$. Since $u_n \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, we may assume that $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \mathbb{R}^N$. Given any $\varepsilon > 0$, by assertion (1), there exists $R_\varepsilon > 0$ such that $\rho_{\mathbb{R}^N\setminus B_R}(u_n) \leqslant \varepsilon$ for $R \geqslant R_\varepsilon$. Then for $n$ sufficiently large, $\rho_{B_R}(u_n) \geqslant d - 2\varepsilon$. Noting that $\rho_{B_R}(u_n) \to \rho_{B_R}(u)$ because $u_n \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, we have that $\rho_{B_R}(u) \geqslant d - 2\varepsilon$ for all $R \geqslant R_\varepsilon$, furthermore $\rho(u) \geqslant d - 2\varepsilon$ and so $\rho(u) \geqslant d$. It follows from $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$ that $\rho(u) \leqslant \lim_{n\to\infty}\rho(u_n) = d$. Thus $\rho(u) = d = \lim_{n\to\infty}\rho(u_n)$. Noting that

$$|\nabla u_n - \nabla u|^{p(x)} + V(x)|u_n - u|^{p(x)} \leqslant 2^{p_+}\big(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} + |\nabla u|^{p(x)} + V(x)|u|^{p(x)}\big),$$

by the Vitali convergence theorem, we can obtain that

$$\int_{\mathbb{R}^N}\big(\big(|\nabla u_n - \nabla u|\big)^{p(x)} + V(x)\big(|u_n - u|\big)^{p(x)}\big)\,dx \to 0 \quad \text{as } n \to \infty,$$

which implies that $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$. $\square$

## 4. Solutions of problem (1.2)

In this section we consider problem (1.2) and prove Theorem 1.3.

Let the assumptions of Theorem 1.3 hold. Denote $p_*(x) = p(x) + \theta(x)$, $f_*(x,t) = f(x,t)|t|^{-\tau(x)}$ and $F_*(x,t) = \int_0^t f_*(x,s)\,ds$. Define for $u \in W^{1,p_*(x)}(\mathbb{R}^N)$,

$$\begin{aligned}
J_*(u) &= \int_{\mathbb{R}^N}\frac{1}{p_*(x)}\big(|\nabla u|^{p_*(x)} + a(x)V(x)|u|^{p_*(x)}\big)\,dx - \int_{\mathbb{R}^N} b(x)F_*(x,u)\,dx, \\
\Gamma_* &= \big\{\gamma \in C^0\big([0,1], W^{1,p_*(x)}\big(\mathbb{R}^N\big)\big)\colon\ \gamma(0) = 0 \text{ and } J_*\big(\gamma(1)\big) < 0\big\}, \\
c_* &= \inf_{\gamma\in\Gamma_*}\max_{t\in[0,1]} J_*\big(\gamma(t)\big), \\
\mathbf{N}_* &= \big\{u \in W^{1,p_*(x)}\big(\mathbb{R}^N\big)\setminus\{0\}\colon\ J'_*(u)u = 0\big\},
\end{aligned}$$

and for $u \in W^{1,p(x)}(\mathbb{R}^N)$,

$$J_\infty(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + a_+ V(x)|u|^{p(x)}\big)\,dx - \int_{\mathbb{R}^N} b_- F(x,u)\,dx,$$

$$\Gamma_\infty = \big\{\gamma \in C^0\big([0,1], W^{1,p(x)}\big(\mathbb{R}^N\big)\big)\colon\ \gamma(0) = 0 \text{ and } J_\infty\big(\gamma(1)\big) < 0\big\},$$

$$c_\infty = \inf_{\gamma\in\Gamma_\infty}\max_{t\in[0,1]} J_\infty\big(\gamma(t)\big),$$

$$\mathbf{N}_\infty = \big\{u \in W^{1,p(x)}\big(\mathbb{R}^N\big)\setminus\{0\}\colon\ J'_\infty(u)u = 0\big\}.$$

Denote by $\|u\|_*$ the norm of $u$ in $W^{1,p_*(x)}(\mathbb{R}^N)$.

**Lemma 4.1.** *Under the assumptions of Theorem* 1.3*, the following conditions are satisfied.*

$(\mathrm{p}_*)_1$ *The function $p_* : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and*

$$1 < (p_*)_- \leqslant (p_*)_+ < N.$$

$(\mathrm{f}_*)_1$ *$f_* \in C^0(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and*

$$\big|f_*(x,t)\big| \leqslant C\big(|t|^{p_*(x)-1} + |t|^{q_*(x)-1}\big), \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

*where $C$ is a positive constant, $q_*(x) = q(x) + \theta(x)$, $q_* \in C^0(\mathbb{R}^N, \mathbb{R})$ and $p_* \leqslant q_* \ll (p_*)^*$.*

$(\mathrm{f}_*)_2$ *$\beta - \tau_+ > (p_*)_+$ and*

$$0 < (\beta - \tau_+)F_*(x,t) \leqslant t f_*(x,t), \quad \forall x \in \mathbb{R}^N,\ t \neq 0.$$

$(\mathrm{f}_*)_3$ *$f_*(x,t) = o(|t|^{(p_*)_+-1})$ as $t \to 0$, uniformly in $x$.*

$(\mathrm{f}_*)_5$ *For each $x \in \mathbb{R}^N$, $\frac{f_*(x,t)}{|t|^{(p_*)_+-1}}$ is an increasing function of $t$ on $\mathbb{R}\setminus\{0\}$.*

**Proof.** $(\mathrm{p}_*)_1$ follows immediately from $(\mathrm{p}_1)$ and $(\theta)$. $(\mathrm{f}_*)_3$ is just $(\mathrm{f}_3)_*$. $(\mathrm{f}_3)_*$ implies the continuity of $f_*(x,t)$ at $t = 0$. Put $q_*(x) = q(x) + \theta(x)$. From $(\mathrm{f}_1)$ and $(\mathrm{f}_3)_*$ it follows that when $|t| \leqslant 1$, $|f_*(x,t)| \leqslant C_2|t|^{(p_*)_+-1} \leqslant C_2|t|^{p_*(x)-1}$, and when $|t| > 1$, $|f_*(x,t)| \leqslant |f(x,t)| \leqslant C_3|t|^{q(x)-1} \leqslant C_3|t|^{q_*(x)-1}$. Obviously $p_* \leqslant q_*$. From $q \ll p^*$ we can obtain $q_* \ll (p_*)^*$. Thus $(\mathrm{f}_*)_1$ holds. Noting that

$$\frac{f_*(x,t)}{|t|^{(p_*)_+-1}} = \frac{f(x,t)|t|^{-\tau(x)}|t|^{\beta-(p_*)_+}}{|t|^{\beta-1}} = \frac{f(x,t)}{|t|^{\beta-1}}\cdot|t|^{\beta-(p_*)_+-\tau(x)},$$

from $(\theta,\tau)$ and $(\mathrm{f}_5)_*$ it follows that $(\mathrm{f}_*)_5$ holds. It only remains to prove $(\mathrm{f}_*)_2$. By $(\theta,\tau)$, $(p+\theta)_+ < \beta - \tau_+$. By $(\mathrm{f}_2)$, the definition of $F_*$ and the integration by parts, we have that, for $x \in \mathbb{R}^N$ and $t \neq 0$,

$$\begin{aligned}
0 < F_*(x,t) &= \int_0^t f_*(x,s)\,ds = \int_0^t f(x,s)|s|^{-\tau(x)}\,ds \\
&= \big(F(x,s)|s|^{-\tau(x)}\big)\Big|_0^t + \tau(x)\int_0^t F(x,s)|s|^{-\tau(x)-2}s\,ds \\
&\leqslant \frac{1}{\beta} f(x,t)t|t|^{-\tau(x)} + \frac{\tau(x)}{\beta}\int_0^t f(x,s)|s|^{-\tau(x)}\,ds \\
&= \frac{1}{\beta} f_*(x,t)t + \frac{\tau(x)}{\beta}F_*(x,t),
\end{aligned}$$

and consequently

$$0 < \big(\beta - \tau(x)\big)F_*(x,t) \leqslant t f_*(x,t).$$

Thus $(\mathrm{f}_*)_2$ holds. The proof is complete. □

Now let us to prove assertion (1) of Theorem 1.3. Noting that all the lemmata and propositions obtained in Section 3 hold without the periodicity assumptions, by Lemma 4.1, we know that $J_*$ satisfies Mountain Pass Geometry and there exists a bounded sequence $\{u_n\} \subset W^{1,p_*(x)}(\mathbb{R}^N)$ such that $J_*(u_n) \to c_* > 0$ and $J_*'(u_n) \to 0$. We may assume that $u_n \rightharpoonup u_0$ in $W^{1,p_*(x)}(\mathbb{R}^N)$ and $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^N$. By Lemma 3.1, $u_n \to u_0$ in $W^{1,p_*(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ and $J_*'(u_n) \rightharpoonup J_*'(u_0) = 0$. Thus $u_0$ is a solution of (1.2).

If $u_0 \neq 0$, then $u_0$ is a nontrivial solution of (1.2) and by Proposition 3.1 and Lemmas 3.6 and 3.7, $J_*(u_0) = c_* = \inf\{J_*(v)\colon v \in \mathbf{N}_*\}$. Thus $u_0$ is a ground state solution of (1.2). In this case assertion (1) of Theorem 1.3 already holds. In addition, by Proposition 3.2, $u_n \to u_0$ in $W^{1,p_*(x)}(\mathbb{R}^N)$.

Below let us consider the case that $u_0 = 0$. For this case we first give the following lemma.

**Lemma 4.2.** *Under the above assumptions, if $u_0 = 0$, then*

(1) $J_\infty'(u_n)u_n \to 0$ *and* $J_\infty(u_n) \to c_*$ *as* $n \to \infty$,
(2) $c_* \geqslant c_\infty$.

**Proof.** (1) By Lemma 3.1, $u_n \to 0$ in $W^{1,p_*(x)}_{\mathrm{loc}}(\mathbb{R}^N)$. For any $R \geqslant R_*$ we have

$$J_\infty'(u_n)u_n - J_*'(u_n)u_n = \int\limits_{\mathbb{R}^N \setminus B_R} \big(\big(a_+ - a(x)\big)V(x)|u_n|^{p(x)} + \big(b(x) - b\big)f(x,u_n)u_n\big)\,dx + (J_\infty)_{B_R}'(u_n)u_n - (J_*)_{B_R}'(u_n)u_n.$$

From assumptions (a) and (b), the boundedness of $\{\|u_n\|_*\}$ and $u_n \to 0$ in $W^{1,p_*(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, we can obtain that $J_\infty'(u_n)u_n - J_*'(u_n)u_n \to 0$ and consequently $J_\infty'(u_n)u_n \to 0$ because $J_*'(u_n)u_n \to 0$. Using similar arguments we can prove that $J_\infty(u_n) - J_*(u_n) \to 0$ and consequently $J_\infty(u_n) \to c_*$.

(2) Since $J_\infty'(u_n)u_n \to 0$, we have

$$\int\limits_{\mathbb{R}^N} \big(|\nabla u_n|^{p(x)} + a_+V(x)|u_n|^{p(x)}\big)\,dx = \int\limits_{\mathbb{R}^N} b_- f(x,u_n)u_n\,dx + o(1). \tag{4.1}$$

From $J_\infty(u_n) \to c_* > 0$ it follows that $\{\|u_n\|\}$ has a positive bound from below and consequently there exists $\delta > 0$ such that

$$\int\limits_{\mathbb{R}^N} b_- f(x,u_n)u_n\,dx \geqslant \delta \quad \text{for } n \text{ large enough.} \tag{4.2}$$

For each $u_n$ there exists a unique positive number $t_n$ such that $t_nu_n \in \mathbf{N}_\infty$, which implies that $J_\infty'(t_nu_n)u_n = 0$, that is

$$\int\limits_{\mathbb{R}^N} t_n^{p(x)-1}\big(|\nabla u_n|^{p(x)} + a_+V(x)|u_n|^{p(x)}\big)\,dx = \int\limits_{\mathbb{R}^N} b_- f(x,t_nu_n)u_n\,dx. \tag{4.3}$$

By the mean value theorem for integrals, for each $n$ there exists a number $\bar{p}_n \in [p_-, p_+]$ such that

$$\int\limits_{\mathbb{R}^N} t_n^{p(x)-1}\big(|\nabla u_n|^{p(x)} + a_+V(x)|u_n|^{p(x)}\big)\,dx = t_n^{\bar{p}_n-1}\int\limits_{\mathbb{R}^N} \big(|\nabla u_n|^{p(x)} + a_+V(x)|u_n|^{p(x)}\big)\,dx,$$

and so

$$\int\limits_{\mathbb{R}^N} \big(|\nabla u_n|^{p(x)} + a_+V(x)|u_n|^{p(x)}\big)\,dx = \int\limits_{\mathbb{R}^N} \frac{b_- f(x,t_nu_n)u_n}{t_n^{\bar{p}_n-1}}\,dx. \tag{4.4}$$

(4.4) and (4.1) imply that

$$\int\limits_{\mathbb{R}^N} \frac{b_- f(x,t_nu_n)u_n}{t_n^{\bar{p}_n-1}}\,dx = \int\limits_{\mathbb{R}^N} b_- f(x,u_n)u_n\,dx + o(1). \tag{4.5}$$

When $t_n \geqslant 1$, noting that

$$\frac{b_- f(x, t_n u_n) u_n}{t_n^{\bar{p}_n - 1}} \geqslant \frac{b_- f(x, t_n u_n) u_n}{t_n^{p_+ - 1}} = \frac{b_- f(x, t_n u_n) u_n}{t_n^{\beta - 1}} t_n^{\beta - p_+}$$

and using assumption $(\mathrm{f}_5)_*$ we have that

$$\frac{b_- f(x, t_n u_n) u_n}{t_n^{\bar{p}_n - 1}} \geqslant t_n^{\beta - p_+} b_- f(x, u_n) u_n,$$

and consequently

$$\left(t_n^{\beta - p_+} - 1\right) \int_{\mathbb{R}^N} b_- f(x, u_n) u_n \, dx \leqslant o(1). \tag{4.6}$$

Analogously we can obtain that, when $t_n < 1$,

$$\left(1 - t_n^{\beta - p_+}\right) \int_{\mathbb{R}^N} b_- f(x, u_n) u_n \, dx \leqslant o(1). \tag{4.7}$$

From (4.6), (4.7) and (4.2) it follows that $t_n \to 1$. Because the mapping $J'_\infty$ is bounded, the functional $J_\infty$ is uniformly continuous on every bounded set and so

$$J_\infty(t_n u_n) - J_\infty(u_n) \to 0 \quad \text{as } n \to \infty.$$

Thus we have that

$$\begin{aligned} c_* &= J_*(u_n) + o(1) = J_\infty(u_n) + o(1) \\ &= J_\infty(t_n u_n) + o(1) \geqslant c_\infty + o(1) \end{aligned}$$

and hence $c_* \geqslant c_\infty$. The lemma is proved. □

Now let us continue with the proof of Theorem 1.3. Let $u_0 = 0$. Then by Lemma 4.2, there holds

$$c_* \geqslant c_\infty. \tag{4.8}$$

Applying Theorem 1.2 to $J_\infty$, we know that $J_\infty$ has a nontrivial critical point $w \in W^{1,p(x)}(\mathbb{R}^N)$ such that $J_\infty(w) = c_\infty$. By Proposition 2.5, $w(x) \to 0$ and $|\nabla w(x)| \to 0$ as $|x| \to \infty$. Define for each $k = 1, 2, \ldots,$

$$w_k(x) = w(x - kT_N e_N), \quad \forall x \in \mathbb{R}^N.$$

Then for each $k$, $J'_\infty(w_k) = 0$ and $J_\infty(w_k) = c_\infty$. Take $k_0$ big enough such that $|w_{k_0}(x)| < 1$ and $|\nabla w_{k_0}(x)| < 1$ for $|x| \leqslant R_*$. Applying Lemma 3.6 to $\mathbf{N}_*$, there exists a unique positive number $t_*$ such that $t_* w_{k_0} \in \mathbf{N}_*$, that is $J'_*(t_* w_{k_0}) w_{k_0} = 0$. Noting that

$$\begin{aligned} J'_*(w_{k_0}) w_{k_0} &= \int_{\mathbb{R}^N \setminus B_{R_*}} \left(|\nabla w_{k_0}|^{p(x)} + a(x) V(x) |w_{k_0}|^{p(x)} - b(x) f(x, w_{k_0}) w_{k_0}\right) dx \\ &\quad + \int_{B_{R_*}} \left(|\nabla w_{k_0}|^{p_*(x)} + a(x) V(x) |w_{k_0}|^{p_*(x)} - b(x) f(x, w_{k_0}) |w_{k_0}|^{-\tau(x)} w_{k_0}\right) dx \\ &\leqslant \int_{\mathbb{R}^N} \left(|\nabla w_{k_0}|^{p(x)} + a_+ V(x) |w_{k_0}|^{p(x)} - b_- f(x, w_{k_0}) w_{k_0}\right) dx \\ &= J'_\infty(w_{k_0}) w_{k_0} = 0, \end{aligned}$$

we know that $t_* \leqslant 1$. We claim that, in the case when $u_0 = 0$, there holds $t_* = 1$. Indeed, if $t_* < 1$, then, noting that when $|x| \leqslant R_*$, $|t_* w_{k_0}(x)| < 1$ and $|\nabla (t_* w_{k_0})(x)| < 1$, and when $|t| \leqslant 1$, $F_*(x, t) \geqslant F(x, t)$, we have that

$$
\begin{aligned}
c_* &\leqslant J_*(t_* w_{k_0}) \\
&= \int\limits_{\mathbb{R}^N \setminus B_{R_*}} \frac{1}{p(x)} \left( \left| \nabla (t_* w_{k_0}) \right|^{p(x)} + a(x) V(x) |t_* w_{k_0}|^{p(x)} \right) dx - \int\limits_{\mathbb{R}^N \setminus B_{R_*}} b(x) F(x, t_* w_{k_0}) \, dx \\
&\quad + \int\limits_{B_{R_*}} \left( \frac{1}{p_*(x)} \left( \left| \nabla (t_* w_{k_0}) \right|^{p_*(x)} + a(x) V(x) |t_* w_{k_0}|^{p_*(x)} \right) - b(x) F_*(x, t_* w_{k_0}) \right) dx \\
&\leqslant J_\infty(t_* w_{k_0}) < J_\infty(w_{k_0}) = c_\infty,
\end{aligned}
$$

which contradicts with (4.8). Hence $t_* = 1$, which implies $w_{k_0} \in \mathbf{N}_*$. In this case,

$$
c_* \leqslant J_*(w_{k_0}) \leqslant J_\infty(w_{k_0}) = c_\infty.
$$

From this and (4.8) it follows that $c_* = c_\infty$ and $J_*(w_{k_0}) = c_*$. By Lemma 3.8, $J_*'(w_{k_0}) = 0$ and $w_{k_0}$ is a ground state solution of (1.2). Assertion (1) of Theorem 1.3 is proved.

The proof of (2) of Theorem 1.3 is similar to the proof of (2) of Theorem 1.2 and hence is omitted here.

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