Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 212 (2008) 735–752
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

The rank filtration and Robinson's complex

Michele Intermont[a], Brenda Johnson[b,*], Randy McCarthy[c]

[a] *Department of Mathematics, Kalamazoo College, Kalamazoo, MI 49006, United States*
[b] *Department of Mathematics, Union College, Schenectady, NY 12308, United States*
[c] *Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, United States*

Received 28 November 2006; received in revised form 5 July 2007; accepted 13 July 2007
Available online 22 October 2007

Communicated by C.A. Weibel

Abstract

For a functor from the category of finite sets to abelian groups, Robinson constructed a bicomplex in [A. Robinson, Gamma homology, Lie representations and E_∞ multiplications, Invent. Math. 152 (2) (2003) 331–348] which computes the stable derived invariants of the functor as defined by Dold–Puppe in [A. Dold, D. Puppe, Homologie nicht-additiver Funktoren. Anwendungen., Ann. Inst. Fourier (Grenoble) 11 (1961) 201–312]. We identify a subcomplex of Robinson's bicomplex which is analogous to a normalization and also computes these invariants. We show that this new bicomplex arises from a natural filtration of the functor obtained by taking left Kan approximations on subcategories of bounded cardinality.

MSC: 55U15; 55U99; 18G35

For a Γ-module F, the Taylor tower of F is a sequence of functors

$$P_1 F \leftarrow P_2 F \leftarrow \cdots \leftarrow P_n F \leftarrow P_{n+1} F \leftarrow \cdots$$

having formal properties analogous to those of the Taylor series of a real-valued function. The reduced component (the direct summand that vanishes on the basepoint), $D_1 F$, of $P_1 F$ is often referred to as the linearization of F. The purpose of this paper is to compare two filtrations of the linearization of F that arise in two different contexts. The first filtration is the rank filtration of the title. The second filtration is the byproduct of a bicomplex constructed by Alan Robinson [13] for classifying E_∞-structures on ring spectra.

More specifically, the terms in the rank filtration play a role similar to that of Lagrange polynomial approximations of real-valued functions. The Lagrange polynomial approximations to F are a sequence of left Kan extensions

$$L_1 F \to L_2 F \to \cdots \to L_n F \to \cdots$$

* Corresponding author.
E-mail addresses: intermon@kzoo.edu (M. Intermont), johnsonb@union.edu (B. Johnson), randy@math.uiuc.edu (R. McCarthy).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.07.009

defined over certain subcategories of Γ. Applying D_1 to this sequence produces what we call the *rank filtration* of D_1F,

$$D_1L_1F \to D_1L_2F \to \cdots \to D_1L_nF \to \cdots .$$

For a functor F, Robinson's complex, $\Xi(F)$, is a bicomplex of R-modules whose homology agrees with that of D_1F when D_1F is evaluated at $[1] = \{0, 1\}$. It is straightforward to extend Robinson's construction to a bicomplex of functors (which we also call $\Xi(F)$) that agrees with D_1F on all objects of Γ. As a bicomplex, $\Xi(F)$ admits a filtration by rows which is similar to, but not the same as, the rank filtration of D_1F. At the same time, certain details of the construction of $\Xi(F)$ suggest that one can reduce $\Xi(F)$ to a smaller bicomplex, $\widetilde{\Xi}(F)$, built out of pieces of F called the cross effects. In this paper we define the reduced Robinson complex $\widetilde{\Xi}(F)$, prove that it is equivalent to $\Xi(F)$ and show that the standard filtration of $\widetilde{\Xi}(F)$ by rows is equivalent to the rank filtration of D_1F. As part of this process, we also determine the filtration of Robinson's complex $\Xi(F)$ that is equivalent to the rank filtration of D_1F. In a subsequent paper, we will show how the reduced bicomplex $\widetilde{\Xi}(F)$ can be used to produce similar bicomplex models for all terms in the Taylor tower of F.

Using $\Xi^{\le n}(F)$ and $\widetilde{\Xi}^{\le n}(F)$ to denote the nth terms in the filtrations of $\Xi(F)$ and $\widetilde{\Xi}(F)$, the main result is

Theorem 5.1. *For any Γ-module F and $n \ge 1$, there is a natural transformation $\phi : \widetilde{\Xi}(F) \to \Xi(F)$ that induces a natural transformation of filtrations*

$$\begin{array}{ccccccccc} \widetilde{\Xi}^{\le 1}(F) & \to & \cdots & \to & \widetilde{\Xi}^{\le n}(F) & \to & \widetilde{\Xi}^{\le n+1}(F) & \to & \cdots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ \Xi^{\le 1}(F) & \to & \cdots & \to & \Xi^{\le n}(F) & \to & \Xi^{\le n+1}(F) & \to & \cdots \end{array}$$

that is a quasi-isomorphism of functors $\widetilde{\Xi}^{\le n}(F) \xrightarrow{\simeq} \Xi^{\le n}(F)$ at each stage of the filtration.

As a corollary to this, we have

Corollary 5.2. (1) *For a Γ-module F, the natural transformation $\phi : \widetilde{\Xi}(F) \to \Xi(F)$ is a quasi-isomorphism.*
(2) *The filtrations $\{D_1L_nF\}$ and $\{\widetilde{\Xi}^{\le n}(F)\}$ are equivalent.*

The paper is organized as follows. In Section 1 we review properties of the cross effects of functors and some key examples. The cross effects are essential components in the construction of the Taylor tower and the reduced Robinson complex $\widetilde{\Xi}(F)$. They also arise in a natural fashion in the cofibers of the rank filtration, and can be used in the context of Pirashvili's Dold–Kan correspondence [10,11] to simplify calculations of the homology of $\Xi(F)[1]$. Section 2 is used to define the rank filtration of F and establish its basic properties. We calculate the terms in the rank filtration of the functors $\widetilde{R}[\mathrm{Hom}([k], -)]$ that take a basepointed set X to the reduced R-module generated by the set of basepoint-preserving maps from $[k] = \{0, 1, 2 \dots, k\}$ to X, and use these calculations to reformulate the definition of L_nF. In addition, we identify the layers in the rank filtration, proving that

$$\mathrm{cofiber}(L_{n-1}F \to L_nF) \simeq \widetilde{R}[\mathrm{Inj}([n], -)] \otimes_{\Sigma_n} cr_nF[1],$$

where cr_nF is the nth cross effect of F and Inj denotes the collection of injective set maps. In Section 3, we review the construction of Robinson's complex $\Xi(F)$ and its relation to D_1F. We also begin comparing filtrations in earnest. We determine the filtration of $\Xi(F)$ that is equivalent to the rank filtration of D_1F. In Section 4, we define $\widetilde{\Xi}(F)$, and establish that it is a bicomplex. We also review tools developed by Pirashvili [10,11] and Betley and Słomińska [4] for calculating the homology of $\Xi(F)[1]$ and use these results to show that $\Xi(F)[1]$ and $\widetilde{\Xi}(F)[1]$ are quasi-isomorphic for a certain class of functors. We prove that the row filtration of $\widetilde{\Xi}(F)$ agrees with the rank filtration of D_1F in Section 5. The key to proving this is to use the results of Section 2 to show that $\mathrm{cofiber}(L_{n-1}F \to L_nF)$ belongs to the class of functors for which $\Xi(-)[1]$ and $\widetilde{\Xi}(-)[1]$ are shown to be quasi-isomorphic in Section 4. As a consequence, we also establish that $\Xi(F)$ and $\widetilde{\Xi}(F)$ are equivalent in all cases.

Conventions and notation

For $n \ge 0$, $[n]$ denotes the finite basepointed set $\{0, 1, 2, \dots, n\}$ where 0 is the basepoint of $[n]$. We will use $\langle n \rangle$ to denote the set without basepoint, $\langle n \rangle = \{1, 2, \dots, n\}$. The category Γ is the category of finite based sets and basepoint-preserving set maps. Throughout this paper we tend to use the equivalent full subcategory of Γ generated by the objects $[n]$. For a fixed commutative ring R with unit, a left (respectively, right) Γ-module is a covariant (respectively, contravariant) functor from Γ to the category of R-modules. We will also be working with functors to chain complexes of R-modules. By a quasi-isomorphism of such functors we mean a natural transformation that is a quasi-isomorphism when evaluated at any object. We say that the functors F and G are equivalent if there is a sequence of quasi-isomorphisms between them. These quasi-isomorphisms need not go in the same direction. For example, an equivalence between F and G could consist of two quasi-isomorphisms to a third functor H:

$$F \to H \leftarrow G.$$

We will use the symbol $\simeq$ to denote both equivalences and quasi-isomorphisms.

1. Taylor towers and cross effects for Γ-modules

To any Γ-module, one can associate a sequence of functors, called the Taylor tower, whose members have properties analogous to those of Taylor polynomial approximations to real-valued functions. Taylor towers were originally developed for functors of spaces by Tom Goodwillie (see [6]). One way to understand the Taylor tower of a Γ-module is via cross effect functors. We use this section to review the definition of cross effects, their basic properties and some key examples. The cross effect functors are used in subsequent sections to analyze cofibers in the rank filtration, simplify certain homology calculations, and modify Robinson's complex to produce the reduced Robinson complex. For more details about cross effects see Section 1 of [7].

Definition 1.1. Let F be a Γ-module or chain complex (bounded below) of Γ-modules and $n \geq 1$. The nth cross effect of F is the functor $cr_n F : \Gamma^{\times n} \to R-\text{Mod}$ defined inductively for objects $M_1, \dots, M_n$ by natural direct sum decompositions

$$cr_1 F(M_1) \oplus F[0] \cong F(M_1),$$
$$cr_2 F(M_1, M_2) \oplus cr_1 F(M_1) \oplus cr_1 F(M_2) \cong cr_1 F(M_1 \vee M_2),$$

and in general,

$$cr_n F(M_1, \dots, M_n) \oplus cr_{n-1} F(M_1, M_3, \dots, M_n) \oplus cr_{n-1} F(M_2, M_3, \dots, M_n)$$

is isomorphic to

$$cr_{n-1} F(M_1 \vee M_2, M_3, \dots, M_n).$$

Remark 1.2. (1) For a Γ-module F, the cross effect functors satisfy the following properties:

(a) For any $n \geq 0$,
$$F[n] \cong F[0] \oplus \bigoplus_{\{s_1, s_2, \dots, s_t\} \subseteq [n]} cr_t F([1], \dots, [1]).$$
(b) Cross effects are reduced functors in each variable. That is, for any $1 \leq i \leq k$ and any objects $X_1, \dots, X_k$ with $X_i \cong [0]$, $cr_k F(X_1, \dots, X_i, \dots, X_k) \cong 0$.
(c) For any objects $X_1, \dots, X_k$ and $\sigma \in \Sigma_k$, the symmetric group on k letters, there is a natural isomorphism $cr_k F(X_1, \dots, X_k) \cong cr_k F(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(k)})$.

(2) There are many equivalent definitions of cross effects for Γ-modules.

(a) For a Γ-module F and objects $X_1, \dots, X_n$, $cr_n F(X_1, \dots, X_n)$ is quasi-isomorphic to the total complex of the following n-complex of objects. Let $\mathcal{P}(\langle n \rangle)$ denote the power set of $\langle n \rangle = \{1, 2, \dots, n\}$. Let $\mathcal{C}_n(X_1, \dots, X_n)$ be the n-cubical diagram in Γ with $\mathcal{C}_n(X_1, \dots, X_n)(U) = \bigvee_{u \in U} X_u$ for $U \in \mathcal{P}(\langle n \rangle)$ and $\mathcal{C}_n(X_1, \dots, X_n)(\emptyset) = [0]$, with maps the natural inclusions. Then
$$cr_n F(X_1, \dots, X_n) \simeq \text{Tot}(F(\mathcal{C}_n(X_1, \dots, X_n))).$$
For details (in a slightly different formulation) see Remark 1.5 of [7].

(b) The nth cross effect $cr_n F(X_1, \dots, X_n)$ is also quasi-isomorphic to
$$\text{cofiber}(\text{hocolim}_{U \subseteq \langle n \rangle, U \neq \langle n \rangle} F(\mathcal{C}_n(X_1, \dots, X_n)(U)) \to F(X_1 \vee \cdots \vee X_n)),$$
where cofiber denotes the homotopy cofiber, i.e., in this context, the mapping cone.

(c) When $cr_n F$ is evaluated at the same object X in all of its variables, we use $cr_n F(X)$ to denote $cr_n F(X, \ldots, X)$. In this case, the cross effects can be determined by the surjections $r_i : [n] \to [n-1]$, $1 \le i \le n$, with

$$r_i(j) = \begin{cases} j & \text{if } j < i \\ 0 & \text{if } j = i \\ j-1 & \text{if } j > i. \end{cases}$$

These induce natural maps $r_i : \vee_n X \to \vee_{n-1} X$ and we have

$$cr_n F(X) \cong \bigcap_{i=1}^{n} \ker F(r_i).$$

Example 1.3. Let $\mathrm{Hom}([n], [m])$ denote the set of morphisms in Γ from $[n]$ to $[m]$ and $\mathrm{Inj}([n], [m])$ denote the injective morphisms. For a based set $[n]$,

$$\widetilde{R}[n] := R[n]/R[0],$$

the reduced free R-module generated by $[n]$. For calculations in later sections of this work, we need to know the cross effects of the functors $\widetilde{R}[\mathrm{Hom}([n], -)]$ and $\widetilde{R}[\mathrm{Inj}([n], -)]$. (To make $\widetilde{R}[\mathrm{Inj}([n], -)]$ a functor, we set noninjective compositions equal to 0 and $\widetilde{R}[\emptyset] = 0$.) To determine the cross effects of $\widetilde{R}[\mathrm{Hom}([n], -)]$, note that for any n,

$$R[\mathrm{Hom}([n], -)] \cong \otimes^n R[\mathrm{Hom}([1], -)] \cong \otimes^n (R[0] \oplus \widetilde{R}[\mathrm{Hom}([1], -)]).$$

Then, for example,

$$\begin{aligned} R[\mathrm{Hom}([n], X_1 \vee X_2)] &\cong \bigotimes_{i=1}^{n} (R[0] \oplus \widetilde{R}[\mathrm{Hom}([1], X_1)] \oplus \widetilde{R}[\mathrm{Hom}([1], X_2)]) \\ &\cong R[0] \oplus \widetilde{R}[\mathrm{Hom}([n], X_1)] \oplus \widetilde{R}[\mathrm{Hom}([n], X_2)] \oplus \\ &\quad \bigoplus_{\sigma \in \mathrm{surj}(\langle n \rangle, \langle 2 \rangle)} \widetilde{R}[\mathrm{Hom}(\sigma^{-1}\{1\}_+, X_1)] \otimes \widetilde{R}[\mathrm{Hom}(\sigma^{-1}\{2\}_+, X_2)], \end{aligned}$$

where $\mathrm{surj}(\langle n \rangle, \langle 2 \rangle)$ is the set of surjections from $\langle n \rangle$ to $\langle 2 \rangle$ and for a set U, $U_+ = U \cup [0]$. From this it follows that

$$cr_2 \widetilde{R}[\mathrm{Hom}([n], -)](X_1, X_2)$$

is isomorphic to

$$\bigoplus_{\sigma \in \mathrm{surj}(\langle n \rangle, \langle 2 \rangle)} \widetilde{R}[\mathrm{Hom}(\sigma^{-1}\{1\}_+, X_1)] \otimes \widetilde{R}[\mathrm{Hom}(\sigma^{-1}\{2\}_+, X_2)].$$

To describe the higher order cross effects, let $\mathrm{surj}(\langle n \rangle, \langle m \rangle)$ denote the set of surjections from $\langle n \rangle$ to $\langle m \rangle$. Working inductively, one can deduce that

$$cr_m \widetilde{R}[\mathrm{Hom}([n], -)](X_1, \ldots, X_m)$$

is isomorphic to

$$\bigoplus_{\sigma \in \mathrm{surj}(\langle n \rangle, \langle m \rangle)} \widetilde{R}[\mathrm{Hom}(\sigma^{-1}\{1\}_+, X_1)] \otimes \cdots \otimes \widetilde{R}[\mathrm{Hom}(\sigma^{-1}\{m\}_+, X_m)].$$

It follows from this that $cr_m \widetilde{R}[\mathrm{Hom}([n], -)] \cong 0$ for $m > n$. If we restrict our attention to injective morphisms, then similar reasoning leads to the conclusion that

$$cr_m \widetilde{R}[\mathrm{Inj}([n], -)](X_1, \ldots, X_m)$$

is isomorphic to

$$\bigoplus_{\sigma \in \mathrm{surj}(\langle n \rangle, \langle m \rangle)} \widetilde{R}[\mathrm{Inj}(\sigma^{-1}\{1\}_+, X_1)] \otimes \cdots \otimes \widetilde{R}[\mathrm{Inj}(\sigma^{-1}\{m\}_+, X_m)].$$

Note that $cr_m \widetilde{R}[\mathrm{Inj}([n], -)]$ is also 0 for $m > n$. Because we have now restricted to injective maps, the cross effects will also vanish when evaluated on small sets. For example,

$$cr_m \widetilde{R}[\mathrm{Inj}([n], -)]([1], \dots, [1]) \cong \begin{cases} \widetilde{R}[\Sigma_n] & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

The following fact about cross effects of Γ-modules shows how we can reduce the size of the sets on which we evaluate the cross effects by composing the functor with an appropriate coproduct. We use this result in Section 5 to reduce establishing equivalences of certain constructions to proving that they agree when evaluated at [1].

Proposition 1.4. *Let F be a Γ-module and $n \geq 1$. Let $\bigvee_n$ be the functor that takes a finite based set X to the n-fold coproduct $\bigvee_n X$. Then for any $k \geq 1$ and $l_1, \dots, l_k \geq 0$, $cr_k F([nl_1], \dots, [nl_k]) \cong cr_k(F \circ \bigvee_n)([l_1], \dots, [l_k])$.*

Proof. This can be proved by induction on k. The key is to compare the isomorphisms:

$$cr_k F([nl_1], \dots, [nl_k] \vee [nl_{k+1}])$$

is isomorphic to

$$cr_k F([nl_1], \dots, [nl_k]) \oplus cr_k F([nl_1], \dots, [nl_{k+1}]) \oplus cr_{k+1} F([nl_1], \dots, [nl_k], [nl_{k+1}])$$

and

$$cr_k \left(F \circ \bigvee_n \right) ([l_1], \dots, [l_k] \vee [l_{k+1}])$$

is isomorphic to

$$cr_k \left(F \circ \bigvee_n \right) ([l_1], \dots, [l_k]) \oplus cr_k \left(F \circ \bigvee_n \right) ([l_1], \dots, [l_{k+1}]) \oplus cr_{k+1} \left(F \circ \bigvee_n \right) ([l_1], \dots, [l_k], [l_{k+1}]).$$

Assuming that $cr_k F([nl_1], \dots, [nl_k]) \cong cr_k(F \circ \bigvee_n)([l_1], \dots, [l_k])$ gives the isomorphism $cr_{k+1} F([nl_1], \dots, [nl_{k+1}]) \cong cr_{k+1}(F \circ \bigvee_n)([l_1], \dots, [l_{k+1}])$.

Cross effects are used to measure the degree of a functor.

Definition 1.5. A Γ-module F is degree n provided that $cr_{n+1} F \cong 0$. If F is a chain complex of Γ-modules, then F is degree n if and only if $cr_{n+1} F$ is quasi-isomorphic to 0.

Example 1.6. By Example 1.3, both $\widetilde{R}[\mathrm{Hom}([n], -)]$ and $\widetilde{R}[\mathrm{Inj}([n], -)]$ are degree n functors.

For more examples of cross effects and functors of various degrees, see [7, Section 1].

The existence of a Taylor tower for a Γ-module is established in [12], and more generally in [7]. In [7], the terms in the Taylor tower are constructed by using cotriples associated to the cross effect functors.

Theorem 1.7 (*[12, Section 3], [7, Section 2]*). *For a Γ-module F there is a sequence of functors $P_0 F = F[0]$, $P_1 F$, $\dots$, $P_n F, \dots$ and a commuting diagram of natural transformations*

$$\begin{array}{ccccccc} & & & F & & & \\ & p_{n+1} \swarrow & & \downarrow p_n & & \searrow p_{n-1} & \\ \cdots \to & P_{n+1} F & \xrightarrow{q_{n+1}} & P_n F & \xrightarrow{q_n} & P_{n-1} F & \to \cdots \to P_1 F \to P_0 F = F(0) \end{array}$$

in which $P_n F$ is degree n for each n, and $P_n F$ is universal up to quasi-isomorphism among degree n functors with natural transformations from F.

Remark 1.8. The nth *layer* in the Taylor tower,

$$D_n F := \mathrm{fiber}(P_n F \xrightarrow{q_n F} P_{n-1} F)$$

is a *homogeneous* degree n functor in the sense that it is degree n and $P_{n-1}D_nF \cong 0$. As such, it is much better understood and generally easier to determine than P_nF. Here, fiber denotes the homotopy fiber, i.e., the mapping cone shifted down one degree homologically.

2. The rank filtration of a functor

The terms in the Taylor tower of a functor play a role similar to that of the Taylor polynomial approximations to a real-valued function. A real-valued function can also be approximated by a Lagrange polynomial, i.e., a degree n polynomial that agrees with the original function at $n+1$ points. We use this section to describe an analog of the Lagrange construction for Γ-modules, and establish several properties of this construction.

To describe a Lagrange polynomial approximation to a Γ-module F over n "points", we use the full subcategory, $\Gamma_{\leq n}$, of Γ generated by objects X of cardinality less than or equal to $n+1$. Equivalently, we regard $\Gamma_{\leq n}$ as the full subcategory of Γ determined by the objects $[0], [1], \ldots, [n]$. For a right Γ-module F and left Γ-module G, one can define their tensor product over a subcategory, Λ, of Γ as the coend of the bifunctor $F \otimes G : \Lambda^{op} \times \Lambda \to R-\mathrm{mod}$, i.e., as the coequalizer of

$$\bigoplus_{[n]\xrightarrow{f}[m]} F[n] \otimes G[m] \underset{\mathrm{id}\otimes f_*}{\overset{f^*\otimes\mathrm{id}}{\rightrightarrows}} \bigoplus_{[k]} F[k] \otimes G[k],$$

where the sum on the left is over all morphisms f in Λ. We use $F \otimes_\Lambda G$ to denote this tensor product.

We define Lagrangian approximations to functors as follows.

Definition 2.1. Let F be a Γ-module and $n \geq 0$. By L_nF we mean the homotopy left Kan extension of F over $\Gamma_{\leq n}$. That is,

$$L_nF(-) := \widetilde{R}[\mathrm{Hom}(*,-)]\widehat{\otimes}_{\Gamma_{\leq n}} F(*),$$

where $\widehat{\otimes}$ denotes the derived tensor. More specifically, $L_nF(-)$ is the simplicial R-module that in simplicial degree k is

$$\bigoplus_{0\leq t_0,\ldots,t_k\leq n} \widetilde{R}[\mathrm{Hom}([t_0],[t_1])] \otimes \cdots \otimes \widetilde{R}[\mathrm{Hom}([t_{k-1}],[t_k])] \otimes \widetilde{R}[\mathrm{Hom}([t_k],-)] \otimes F(t_0)$$

with face and degeneracy maps defined as follows:

$$d_i(\alpha_1,\ldots,\alpha_k;\beta;x) = \begin{cases} (\alpha_2,\ldots,\alpha_k;\beta;F(\alpha_1)(x)) & \text{if } i=0 \\ (\alpha_1,\ldots,\alpha_{i+1}\circ\alpha_i,\ldots,\alpha_k;\beta;x) & \text{if } 1\leq i\leq k-1 \\ (\alpha_1,\alpha_2,\ldots;\beta\circ\alpha_k;x) & \text{if } i=k \end{cases}$$

$$s_j(\alpha_1,\ldots,\alpha_k;\beta;x) = (\alpha_1,\ldots,\alpha_j,\mathrm{id}_{[t_j]},\alpha_{j+1},\ldots,\alpha_k;\beta;x).$$

We will also use L_nF to denote the (unnormalized) chain complex associated to this simplicial object.

The inclusion of categories $\Gamma_{\leq n} \hookrightarrow \Gamma_{\leq n+1}$ induces a natural transformation of functors $L_nF \to L_{n+1}F$, and more generally, we have a sequence of functors

$$L_0F \to L_1F \to \cdots \to L_{n-1}F \to L_nF \to L_{n+1}F \to \cdots.$$

We let $L_\infty F$ denote the colimit of this sequence, i.e.,

$$L_\infty F = \mathrm{colim}_n L_nF.$$

The functors L_nF satisfy the following properties.

Remark 2.2. (1) There is an augmentation $L_nF(X) \xrightarrow{\epsilon} F(X)$ that takes $(\beta;x)$ to $F(\beta)(x)$. When $X=[k]$ for $k \leq n$, this augmented simplicial object is contractible. In particular, a contracting homotopy is given by $f_m(\alpha_1,\ldots,\alpha_m;\beta;x) = (\alpha_1,\ldots,\alpha_m,\beta;\mathrm{id};x)$. Thus, $L_nF[k] \simeq F[k]$ for $k \leq n$.

(2) The augmentations of (1) induce an equivalence of functors $L_\infty F \xrightarrow{\simeq} F$ in the colimit.

(3) The functor $L_n F$ is a degree n functor. To see this, note that in each simplicial degree, the $n+1$st cross effect of $L_n F$ vanishes by Example 1.3. Since cross effects are defined levelwise for simplicial objects, the claim follows.

The functor $L_n F$ can be defined for any functor $F : \mathcal{C} \to \mathcal{D}$ (where $\mathcal{C}$ is an arbitrary pointed category with finite coproducts and $\mathcal{D}$ is an abelian category). However, in this general setting, one cannot guarantee that $L_n F$ is degree n. Instead $L_n F$ is characterized by the fact that, roughly, it is determined by its values on n objects, and we call a functor with this property a rank n functor. (See [9].) Hence, we call the sequence

$$L_0 F \to L_1 F \to \cdots \to L_{n-1} F \to L_n F \to L_{n+1} F \to \cdots .$$

the *rank filtration* of F. In the present setting, any degree n Γ-module is also determined by its values on n objects as follows.

Proposition 2.3. *Let F be a degree n Γ-module. Then for $m > n$ the natural map*

$$\operatorname*{hocolim}_{U \subseteq \langle m \rangle,\, U \neq \langle m \rangle} F[|U|] \longrightarrow F[m]$$

is a quasi-isomorphism.

Proof. Recall from Remark 1.2.2(b) that

$$cr_m F[1] \simeq \text{cofiber}\left(\operatorname*{hocolim}_{U \subseteq \langle m \rangle,\, U \neq \langle m \rangle} F(\mathcal{C}_m([1], \ldots, [1])(U)) \to F[m] \right).$$

Since F is degree n, this cofiber is acyclic, and as a consequence, the map

$$\operatorname*{hocolim}_{U \subseteq \langle m \rangle,\, U \neq \langle m \rangle} F(\mathcal{C}_m([1], \ldots, [1])(U)) \to F[m]$$

is a quasi-isomorphism. But, by definition, $F(\mathcal{C}_m([1], \ldots, [1])(U)) = F(\vee_{u \in U}[1]) = F[|U|]$ and the result follows.

Corollary 2.4. *If $\mu : F \to G$ is a natural transformation of degree n Γ-modules (or chain complexes of Γ-modules) that induces a quasi-isomorphism $\mu_{[k]} : F[k] \to G[k]$ for all $k \leq n$, then $\mu_X : F(X) \to G(X)$ is a quasi-isomorphism for all objects X of Γ.*

Proof. Let $m > n$. By Proposition 2.3, the fact that F and G are degree n implies that

$$\operatorname*{hocolim}_{\substack{V \subseteq \langle m \rangle \\ V \neq \langle m \rangle}} F[|V|] \xrightarrow{\simeq} F[m]$$

and

$$\operatorname*{hocolim}_{\substack{V \subseteq \langle m \rangle \\ V \neq \langle m \rangle}} G[|V|] \xrightarrow{\simeq} G[m].$$

The result for $m = n+1$ then follows from the facts that μ induces quasi-isomorphisms

$$F[|V|] \to G[|V|]$$

for all $V \subseteq \langle n+1 \rangle$, $V \neq \langle n+1 \rangle$, and the fact that homotopy colimits preserve these. The general result follows inductively by a similar argument.

We wish to identify the cofibers of the rank filtration.

Definition 2.5. For a Γ-module F, we let $R_n F = \text{cofiber}(L_{n-1} F \to L_n F)$.

Proposition 2.6. *Let $C_n F(-) = \widetilde{R}[\text{Inj}([n], -)] \otimes_{\Sigma_n} cr_n F[1]$. There is a natural transformation $\eta_n F : R_n F \to C_n F$ that is an equivalence of functors.*

Proof. By Example 1.3 we know that $\widetilde{R}[\mathrm{Inj}([n], -)]$ and hence $C_n F$ are degree n functors. As the cofiber of degree n and degree $n-1$ functors, $R_n F$ is also degree n. By Corollary 2.4, it suffices to show that the natural transformation from $R_n F$ to $C_n F$ induces quasi-isomorphisms on the objects $[1], \ldots, [n]$. We begin by defining this natural transformation η_n.

Since $\widetilde{R}[\mathrm{Inj}([n], -)]$ is a free Σ_n-module, there is an equivalence

$$(\widetilde{R}[\mathrm{Inj}([n], -)] \otimes cr_n F[1])_{h\Sigma_n} \xrightarrow{\simeq} \widetilde{R}[\mathrm{Inj}([n], -)] \otimes_{\Sigma_n} cr_n F[1],$$

where $h\Sigma_n$ denotes the homotopy orbits with respect to the Σ_n-action. Recall that $(\widetilde{R}[\mathrm{Inj}([n], -)] \otimes cr_n F[1])_{h\Sigma_n}$ is the simplicial object that in degree p is $\widetilde{R}[\mathrm{Inj}([n], -)] \otimes \Sigma_n^p \otimes cr_n F[1]$. We define $\widetilde{\eta}_n F : L_n F \to (\widetilde{R}[\mathrm{Inj}([n], -)] \otimes cr_n F[1])_{h\Sigma_n}$ to be the simplicial map that in simplicial degree p assigns

$$(\alpha_1, \ldots, \alpha_p; \beta; x) \mapsto \begin{cases} (\alpha_1, \ldots, \alpha_p; \beta; c_n(x)) & \text{if } \alpha_1, \ldots, \alpha_n, \beta \text{ are all isomorphisms of } [n] \\ 0 & \text{otherwise.} \end{cases}$$

Here, $c_n : F[n] \to cr_n F[1]$ is the natural projection onto the direct summand. Note that $\widetilde{\eta}_n F$ induces a map on the cofiber $R_n F$ as the diagram

$$\begin{array}{ccc} L_{n-1}F & \xrightarrow{i_n} & L_n F \\ & \searrow & \downarrow{\scriptstyle \widetilde{\eta}_n F} \\ & & C_n F \end{array}$$

commutes (where the diagonal arrow represents the trivial map). The transformation $\eta_n F$ is the composition

$$R_n F \xrightarrow{\widetilde{\eta}_n F} (\widetilde{R}[\mathrm{Inj}([n], -)] \otimes cr_n F[1])_{h\Sigma_n} \xrightarrow{\simeq} \widetilde{R}[\mathrm{Inj}([n], -)] \otimes_{\Sigma_n} cr_n F[1].$$

We claim that $\eta_n F$ induces a quasi-isomorphism $\eta_n : R_n F[k] \to C_n F[k]$ for $k \leq n$. For $k < n$, this is the case because

$$\begin{aligned} R_n F[k] &= \mathrm{cofiber}(L_{n-1}F[k] \to L_n F[k]) \\ &\simeq \mathrm{cofiber}(F[k] \to F[k]) \\ &\simeq 0, \end{aligned}$$

and $\widetilde{R}[\mathrm{Inj}([n], [k])] \cong 0$. To see that we have a quasi-isomorphism for $k = n$, we consider the nth cross effects of the functors. We have

$$\begin{aligned} cr_n R_n F[1] &= \mathrm{cofiber}(cr_n L_{n-1} F[1] \to cr_n L_n F[1]) \\ &\simeq \mathrm{cofiber}(0 \to cr_n L_n F[1]) \\ &\simeq cr_n F[1], \end{aligned}$$

where the first quasi-isomorphism follows from the fact that $L_{n-1}F$ is degree $n-1$. The second quasi-isomorphism follows from the facts that $L_n F$ and F agree on objects $[m]$ for $m \leq n$, and in particular that $cr_n F[1]$ is determined as a direct summand of $F[n]$. Moreover, by Example 1.3,

$$\begin{aligned} cr_n C_n F[1] &\cong cr_n \widetilde{R}[\mathrm{Inj}([n], -)][1] \otimes_{\Sigma_n} cr_n F[1] \\ &\cong \Sigma_n \otimes_{\Sigma_n} cr_n F[1] \\ &\cong cr_n F[1]. \end{aligned}$$

Since $R_n F$ and $C_n F$ also agree at $[1], [2], \ldots, [n-1]$, it follows by using Remark 1.2.2(b) that they agree at $[n]$. That $R_n F$ and $C_n F$ agree everywhere now follows from Corollary 2.4.

We use the previous results to determine $L_n \widetilde{R}[\mathrm{Hom}([k], -)]$ and rewrite the definition of $L_n F$.

Definition 2.7. For $n \geq 0$ and any based finite sets X and Y, $\mathrm{Hom}^{\leq n}(X, Y)$ is the set of all basepoint-preserving maps $\alpha : X \to Y$ such that $|\mathrm{im}(\alpha)| \leq n + 1$.

Lemma 2.8. *For any $n, k \geq 0$, there is a quasi-isomorphism*

$$L_n\widetilde{R}[\mathrm{Hom}([k], -)] \simeq \widetilde{R}[\mathrm{Hom}^{\leq n}([k], -)].$$

Proof. We show this by first computing the homotopy groups of $L_n\widetilde{R}[\mathrm{Hom}([k], -)]$. In degree 0, we have

$$\pi_0 L_n\widetilde{R}[\mathrm{Hom}([k], -)] \cong \bigoplus_{0\leq t\leq n} \widetilde{R}[\mathrm{Hom}([t], -)] \otimes \widetilde{R}[\mathrm{Hom}([k], [t])]/\mathrm{im}(d_0 - d_1).$$

For any m, the augmentation

$$\epsilon : L_n\widetilde{R}[\mathrm{Hom}([k], -)]_0[m] \longrightarrow \widetilde{R}[\mathrm{Hom}^{\leq n}([k], [m])] \subseteq \widetilde{R}[\mathrm{Hom}([k], [m])]$$

that takes $\alpha \otimes \beta$ to $\alpha \circ \beta$ is well-defined on $\pi_0 L_n\widetilde{R}[\mathrm{Hom}([k], -)][m]$. To prove that it is an isomorphism from $\pi_0 L_n\widetilde{R}[\mathrm{Hom}([k], -)][m]$ to $\widetilde{R}[\mathrm{Hom}^{\leq n}([k], [m])]$, we take advantage of the fact that any based set map $\gamma : [t] \to [m]$ (for any t) can be written uniquely as a composition $\gamma = \gamma_2 \circ \gamma_1$ where $\gamma_1 : [t] \to [|\mathrm{im}(\gamma)| - 1]$ is a surjection and $\gamma_2 : [|\mathrm{im}(\gamma)| - 1] \to [m]$ is an order-preserving inclusion. We use this to define $f : \widetilde{R}[\mathrm{Hom}^{\leq n}([k], [m])] \to \pi_0 L_n\widetilde{R}[\mathrm{Hom}([k], -)][m]$ to be the homomorphism that takes $\gamma \in \mathrm{Hom}([k], [m])$ to $\gamma_2 \otimes \gamma_1$. We claim that in $\pi_0 L_n\widetilde{R}[\mathrm{Hom}([k], -)][m]$,

$$\alpha \otimes \beta \simeq (\alpha \circ \beta)_2 \otimes (\alpha \circ \beta)_1 \tag{2.9}$$

for any $\alpha \in \widetilde{R}[\mathrm{Hom}([t], [m])]$ and $\beta \in \widetilde{R}[\mathrm{Hom}([k], [t])]$. From this claim it follows that f and ϵ are inverses of one another and, as a consequence,

$$\pi_0 L_n\widetilde{R}[\mathrm{Hom}([k], -)][m] \cong \widetilde{R}[\mathrm{Hom}^{\leq n}([k], [m])].$$

To show that (2.9) is true, for $\alpha : [t] \to [m]$ and $\beta : [k] \to [t]$, we define $\tau : [|\mathrm{im}\ \beta| - 1] \to [|\mathrm{im}(\alpha \circ \beta)_1| - 1]$ by $\tau(x) = (\alpha \circ \beta)_1(y)$ where y is any element of $\beta_1^{-1}(x)$. (That τ is well-defined follows from the facts that β_1 and $(\alpha \circ \beta)_1$ are surjections and that $(\alpha \circ \beta)_1$ comes from the composition of α with β.) Clearly, $\tau \circ \beta_1 = (\alpha \circ \beta)_1$. Moreover, $(\alpha \circ \beta)_2 \circ \tau = \alpha \circ \beta_2$ since

$$\begin{aligned}(\alpha \circ \beta)_2(\tau(x)) &= (\alpha \circ \beta)_2((\alpha \circ \beta)_1(y)) \\ &= \alpha \circ \beta_2 \circ \beta_1(y) \\ &= \alpha \circ \beta_2(x)\end{aligned}$$

for $y \in \beta_1^{-1}(x)$. Using the fact that the image of $d_0 - d_1$ is generated by elements of the form

$$\sigma \otimes \rho \circ \gamma - \sigma \circ \rho \otimes \gamma,$$

we have

$$\begin{aligned}\alpha \otimes \beta &= \alpha \otimes \beta_2 \circ \beta_1 \\ &\simeq \alpha \circ \beta_2 \otimes \beta_1 \\ &= (\alpha \circ \beta)_2 \circ \tau \otimes \beta_1 \\ &\simeq (\alpha \circ \beta)_2 \otimes \tau \circ \beta_1 \\ &= (\alpha \circ \beta)_2 \otimes (\alpha \circ \beta)_1.\end{aligned}$$

This completes our calculation of the lowest homotopy group.

To show that all other homotopy groups of $L_n\widetilde{R}[\mathrm{Hom}([k], -)]$ are 0, we proceed by induction on n. For $n = 1$, the augmented simplicial object $L_1\widetilde{R}[\mathrm{Hom}([k], -)] \xrightarrow{\epsilon} \widetilde{R}[\mathrm{Hom}^{\leq 1}([k], -)]$ is contractible. To see this, observe that in simplicial degree m, the only nontrivial summand of $L_1\widetilde{R}[\mathrm{Hom}([k], -)]_m$ is the one in which $t_0 = t_1 = \cdots = t_m = 1$. A contraction is obtained by setting $f_{-1} = f : \widetilde{R}[\mathrm{Hom}^{\leq 1}([k], -)] \to L_1\widetilde{R}[\mathrm{Hom}([k], -)]_0$ and $f_n = s_1$ for $n \geq 0$. Hence, the homotopy of $L_1\widetilde{R}[\mathrm{Hom}([k], -)]$ is concentrated in degree 0.

For $n > 1$, suppose that F is any Γ-module for which the homotopy of $L_1 F$ is concentrated in degree 0. Consider $R_n F$. By Proposition 2.6 this cofiber is equivalent to the functor $\widetilde{R}[\mathrm{Inj}_\Gamma([n], -)] \otimes_{\Sigma_n} cr_n F[1]$ which, as a chain complex or simplicial object, is concentrated in degree 0. It follows that the homotopy of $R_n F$ is concentrated in degree 0. By induction, using the cofiber sequence $L_{n-1}F \to L_nF \to R_nF$, we see that the homotopy of L_nF must also be concentrated in degree 0.

As a result, we see that a quasi-isomorphism from $L_n\widetilde{R}[\mathrm{Hom}([k], -)]$ to $\widetilde{R}[\mathrm{Hom}^{\leq n}([k], -)]$ is obtained by using ϵ in degree 0 and the zero homomorphism elsewhere.

Proposition 2.10. *For any Γ-module F, $L_nF(-) \simeq \widetilde{R}[\mathrm{Hom}^{\leq n}(*, -)] \otimes_\Gamma F(*)$.*

Proof. By definition, we know that

$$L_nF(-) = \widetilde{R}[\mathrm{Hom}(\star, -)]\widehat{\otimes}_{\Gamma_{\leq n}} F(\star).$$

However, $F(\star) \simeq \widetilde{R}[\mathrm{Hom}(*, \star)] \otimes_\Gamma F(*)$. It then follows by Lemma 2.8 and associativity of tensors that

$$\begin{aligned} L_nF(-) &= \widetilde{R}[\mathrm{Hom}(\star, -)]\widehat{\otimes}_{\Gamma_{\leq n}} F(\star) \\ &\cong \widetilde{R}[\mathrm{Hom}(\star, -)]\widehat{\otimes}_{\Gamma_{\leq n}} \widetilde{R}[\mathrm{Hom}(*, \star)] \otimes_\Gamma F(*) \\ &= L_n\widetilde{R}[\mathrm{Hom}(*, -)] \otimes_\Gamma F(*) \\ &\simeq \widetilde{R}[\mathrm{Hom}^{\leq n}(*, -)] \otimes_\Gamma F(*). \end{aligned}$$

3. Robinson's complex

The first layer in the Taylor tower of a functor appears in various guises in the literature. For functors of abelian categories, it is equivalent to the stabilization of a functor, in the sense of Dold and Puppe [5,8]. For a Γ-module F, the homology of $D_1F[1]$ is equivalent to the stable homotopy of F, as originally defined for Γ-sets by Segal [14], Bousfield and Friedlander [3], and further developed for Γ-modules by Pirashvili and Richter [10–12].

Pirashvili and Richter recast the concept of stable homotopy in the language of homological algebra. Using the tensor product of Γ-modules as described in the beginning of Section 2, Pirashvili [10,11] and Richter [12] proved that

$$H_*(D_1F[1]) \cong \pi_*^{st}(F) \cong \mathrm{Tor}_*^\Gamma(\widetilde{R}^*, F), \tag{3.1}$$

where π_*^{st} denotes stable homotopy and $\widetilde{R}^*$ is the dual of the functor $\widetilde{R}[-]$ defined in Example 1.3. That is, $\widetilde{R}^*[X] = \mathrm{Hom}_R(\widetilde{R}[X], R)$.

Of greatest interest to us in the present paper is the relationship between $D_1F[1]$ and the bicomplex $\Xi(F)$ constructed by A. Robinson. We review the construction of $\Xi(F)$ in the following pages, but first state some of Robinson's results.

Theorem 3.2 (*[13, 3.5–3.7]*). *For a Γ-module F,*

$$H_*(\Xi(F)) \cong \mathrm{Tor}_*^\Gamma(\widetilde{R}^*, F) \cong H_*(D_1F[1]).$$

Moreover, $\Xi(\widetilde{R}[\mathrm{Hom}_\Gamma(-, *)])$ is a projective resolution of $\widetilde{R}^*$ as a right Γ-module and $\Xi(F) \cong \Xi(\widetilde{R}[\mathrm{Hom}_\Gamma(-, *)]) \otimes_\Gamma F$.

Robinson's complex is a bicomplex of R-modules whose $(n-1)$st row is constructed by using the functor F and the modules Lie_n^* associated to the free Lie algebra on n generators. These modules are defined as follows.

Remark 3.3. Let $\mathcal{L}_n$ be the free Lie algebra over R on the set of generators $\{x_1, \ldots, x_n\}$. The module Lie_n is generated linearly by those monomials in $\mathcal{L}_n$ that contain each of the generators exactly once. The nth symmetric group Σ_n acts on $\mathcal{L}_n$ by permuting the n generators. In $\Xi(F)$, this symmetric group action is twisted by the sign character sgn, so that for $\sigma \in \Sigma_n$ and a monomial $f(x_1, \ldots, x_n) \in \mathcal{L}_n$,

$$\sigma f(x_1, x_2, \ldots, x_n) = \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

We use Lie_n^* to denote the dual of Lie_n, i.e., $\mathrm{Lie}_n^* = \mathrm{Hom}_R(\mathrm{Lie}_n, R)$. Both Lie_n^* and Lie_n are free R-modules of rank $(n-1)!$ and a basis for Lie_n is given by the left-regulated brackets

$$\sigma[x_1, [x_2, [x_3, \ldots, [x_{n-1}, x_n] \cdots]]] \quad \text{for } \sigma \in \Sigma_{n-1}.$$

To define the differentials in $\Xi(F)$ we need the following maps. For more details, see Section 1 of [13].

Definition 3.4. The set of surjections in Γ is generated by the symmetric groups and the collection of surjections $\{c_{ij} : [n] \to [n-1]\}$ defined for $0 \le i < j \le n$ by

$$c_{ij}(t) = \begin{cases} t & \text{if } t < j, \\ i & \text{if } t = j, \\ t-1 & \text{if } t > j. \end{cases}$$

The c_{ij}s induce maps from Σ_n^p to Σ_{n-1}^p as follows.

Definition 3.5. Let $[\sigma_1|\sigma_2|\cdots|\sigma_p] \in \Sigma_n^p$ and $0 \le i < j \le n$. Then $c_{ij}[\sigma_1|\sigma_2|\cdots|\sigma_p] = [\alpha_1|\alpha_2|\cdots|\alpha_p]$ where α_1 is the unique isomorphism that makes the diagram commute:

$$\begin{array}{ccc} [n] & \xrightarrow{\sigma_1} & [n] \\ \Big\downarrow c_{\sigma_1^{-1}(ij)} & & \Big\downarrow c_{ij} \\ [n-1] & \xrightarrow{\alpha_1} & [n-1] \end{array}$$

and, in a similar fashion, $\alpha_2, \ldots, \alpha_p$ are the unique isomorphisms making the diagram below commute:

$$\begin{array}{ccccccccc} [n] & \xrightarrow{\sigma_p} & [n] & \xrightarrow{\sigma_{p-1}} & \cdots & \xrightarrow{\sigma_2} & [n] & \xrightarrow{\sigma_1} & [n] \\ \Big\downarrow c_{(\sigma_1\ldots\sigma_p)^{-1}\{ij\}} & & \Big\downarrow c_{(\sigma_1\ldots\sigma_{p-1})^{-1}\{ij\}} & & & & \Big\downarrow c_{\sigma_1^{-1}\{ij\}} & & \Big\downarrow c_{ij} \\ [n-1] & \xrightarrow{\alpha_p} & [n-1] & \xrightarrow{\alpha_{p-1}} & \cdots & \xrightarrow{\alpha_2} & [n-1] & \xrightarrow{\alpha_1} & [n-1]. \end{array}$$

For each $0 \le i < j \le n$, we also have maps on Lie_{n-1} and Lie_n^*.

Definition 3.6. For $0 \le i < j \le n$, the map $\gamma_{ij}^* : \mathrm{Lie}_{n-1} \to \mathrm{Lie}_n$ is determined as follows. Let $f(x_1, \ldots, x_{n-1})$ be a monomial in Lie_{n-1}. Then

$$(\gamma_{ij}^* f)(x_1, \ldots, x_n) = \begin{cases} (-1)^{j+1}[x_j, f(x_1, \ldots, \widehat{x_j}, \ldots, x_n)] & \text{if } i = 0, \\ (-1)^{j+1} f(x_1, x_2, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_n), & \text{if } i > 0. \end{cases}$$

The maps $\gamma_{ij} : \mathrm{Lie}_n^* \to \mathrm{Lie}_{n-1}^*$, $0 \le i < j \le n$, are the duals of the maps γ_{ij}^*.

This completes the list of ingredients needed to define the bicomplex Ξ.

Definition 3.7 (*[13]*). Let F be a Γ-module. Then $\Xi(F)$ is the bicomplex of R-modules that in bidegree $\{p, q\}$ is

$$\Xi(F)_{p,q} = \mathrm{Lie}_{q+1}^* \otimes R[\Sigma_{q+1}^p] \otimes F[q+1],$$

where tensors are taken over R. The horizontal differential $\partial' : \Xi(F)_{p,q} \to \Xi(F)_{p-1,q}$ is given by

$$\partial'(z \otimes [\sigma_1|\sigma_2|\cdots|\sigma_p] \otimes y) = z\sigma_1 \otimes [\sigma_2|\cdots|\sigma_p] \otimes y + \sum_{i=1}^{p-1}(-1)^i (z \otimes [\sigma_1|\sigma_2|\cdots|\sigma_i\sigma_{i+1}|\cdots|\sigma_p] \otimes y) + (-1)^p (z \otimes [\sigma_1|\sigma_2|\cdots|\sigma_{p-1}] \otimes \sigma_p y).$$

Note that this makes the $(n-1)$st row of $\Xi(F)$ equal to the two-sided bar construction $\mathcal{B}(\mathrm{Lie}_n^*, \Sigma_n, F[n])$. The vertical differential $\partial'' : \Xi(F)_{p,q} \to \Xi(F)_{p,q-1}$ is given by

$$\partial''(z \otimes [\sigma_1|\sigma_2|\cdots|\sigma_p] \otimes y) = (-1)^p \sum_{0 \le i < j \le q+1} \gamma_{ij} z \otimes c_{ij}[\sigma_1|\sigma_2|\cdots|\sigma_p] \otimes c_{(\sigma_1\cdots\sigma_p)^{-1}\{i,j\}} y.$$

For more details, and in particular, to see why $\Xi(F)$ is a bicomplex, the reader is referred to [13, Section 2].

We consider $\Xi(F)$, as defined above, as a functor $\Xi(F)$ (abusing notation) evaluated at the object [1]. When evaluated on a finite based set X,

$$\Xi(F)_{p,q}(X) = \mathrm{Lie}^*_{q+1} \otimes R[\Sigma^p_{q+1}] \otimes F(\vee_{q+1} X).$$

The differentials of $\Xi(F)(X)$ are defined as they are for $\Xi(F)[1]$ – in place of the surjections $c_{ij} : [n] \to [n-1]$, we use the natural surjections $\vee_n X \to \vee_{n-1} X$ that they induce. We show $\Xi(F)$ is equivalent to $D_1 F$ as a functor.

Proposition 3.8. *For a Γ-module F, $\Xi(F)$ is a degree 1 functor. Moreover, $\Xi(F) \simeq D_1 F$.*

Proof. It suffices to prove this in the case that F is a projective generator, i.e., when

$$F = \widetilde{R}[\mathrm{Hom}([n], -)] \tag{3.9}$$

for any based set $[n]$. To show that $\Xi(F)$ is degree 1, we must show that $cr_2\Xi(F)$ is acyclic. However, since Ξ preserves direct sums of functors, this means showing that $\Xi(cr_2 F)$ is acyclic. But, when F is of the form (3.9), its second cross effect is a direct sum of functors of the form $\widetilde{R}[\mathrm{Hom}(U_1, -)] \otimes \widetilde{R}[\mathrm{Hom}(U_2, -)]$, as we saw in Example 1.3. By Proposition 3.4 of [13], applying Ξ to such functors produces an acyclic complex. Hence, $\Xi(F)$ is degree one.

To see that $\Xi(F) \simeq D_1 F$, note that Theorem 3.2 tells us that $\Xi(F)[1] \simeq D_1 F[1]$. The functor F is isomorphic to a direct sum $F[0] \oplus \widetilde{F}(-)$ where $\widetilde{F}[0] \cong 0$. By Lemma 3.3 of [13], $\Xi(F[0])$ is acyclic. Hence, $\Xi(F)[0] \cong \Xi(\widetilde{F})[0] \cong 0$. Since $\Xi(F)[0] \cong 0$, $\Xi(F)$ is a homogeneous degree 1 functor, as is $D_1 F$. From the classification of homogeneous degree n functors of [8, Section 5], we know that both functors are completely determined by their values at [1]. As a result, they must be equivalent.

Corollary 3.10. *As functors, $\Xi(\widetilde{R}[\mathrm{Hom}(*, -)]) \otimes_\Gamma F(*) \simeq \Xi(F)(-) \simeq D_1 F(-)$.*

Proof. To treat $\Xi(\widetilde{R}[\mathrm{Hom}(*, -)]) \otimes_\Gamma F(*)$ as a functor evaluated at the object X we evaluate $\Xi(\widetilde{R}[\mathrm{Hom}(*, -)])$ as a covariant functor at X. That is,

$$\Xi(\widetilde{R}[\mathrm{Hom}(*, -)]) \otimes_\Gamma F(*)(X) = \Xi(\widetilde{R}[\mathrm{Hom}(*, -)])(X) \otimes_\Gamma F(*).$$

Since $\widetilde{R}[\mathrm{Hom}(*, \vee_{q+1} X)] \otimes_\Gamma F(*) \cong F(\vee_{q+1} X)$, it follows that in bidegree (p, q),

$$\Xi(\widetilde{R}[\mathrm{Hom}(*, -)]) \otimes_\Gamma F(*)(X)_{p,q} \cong \Xi(F)(X)_{p,q}.$$

With this, we can proceed as in the proof of Proposition 3.8 to show that $\Xi(\widetilde{R}[\mathrm{Hom}(*, -)]) \otimes_\Gamma F(*)$ is degree 1 and use Theorem 3.2 to conclude that the functors agree everywhere.

Applying D_1 to the rank filtration of F produces the rank filtration of $D_1 F$,

$$D_1 L_1 F \to D_1 L_2 F \to \cdots \to D_1 L_n F \to D_1 L_{n+1} F \to \cdots \to D_1 L_\infty F \tag{3.11}$$

that converges to $D_1 F$ by Remark 2.2. Filtering Robinson's complex by rows produces a filtration of $\Xi(F)$, and hence $D_1 F$, that strongly resembles, but is not quasi-isomorphic to the rank filtration of $D_1 F$. Our last step in this section is to describe the filtration of $\Xi(F)$ that is quasi-isomorphic to (3.11). We filter $\Xi(F)$ by filtering $\mathrm{Hom}(-, *)$ by image size and using the fact from Theorem 3.2 that $\Xi(F) \simeq \Xi(\widetilde{R}[\mathrm{Hom}(-, *)]) \otimes_\Gamma F(*)$.

Definition 3.12. For $n \geq 1$, we set $\Xi^{\leq n}(F)(-) := \Xi(\widetilde{R}[\mathrm{Hom}^{\leq n}(*, -)]) \otimes_\Gamma F(*)$. Clearly, there is a natural inclusion $\Xi^{\leq n}(F) \hookrightarrow \Xi^{\leq n+1}(F)$ and, hence, a filtration

$$\Xi^{\leq 1}(F) \to \cdots \to \Xi^{\leq n}(F) \to \Xi^{\leq n+1}(F) \to \cdots.$$

Using the results of Section 2, we readily show that this filtration of $\Xi(F)$ is equivalent to (3.11).

Proposition 3.13. *For any Γ-module F and any $n \geq 1$, $D_1 L_n F \simeq \Xi^{\leq n}(F)$.*

Proof. By Proposition 3.8, Proposition 2.10, and Corollary 3.10,

$$\begin{aligned} D_1 L_n F &\simeq \Xi(L_n F) \\ &\simeq \Xi(\widetilde{R}[\mathrm{Hom}^{\leq n}(*, -)] \otimes_\Gamma F(*)) \\ &\cong \Xi(\widetilde{R}[\mathrm{Hom}(-, \star)]) \otimes_\Gamma (\widetilde{R}[\mathrm{Hom}^{\leq n}(*, -)] \otimes_\Gamma F(*)) \\ &\simeq \Xi(\widetilde{R}[\mathrm{Hom}^{\leq n}(*, \star)]) \otimes_\Gamma F(*) \\ &= \Xi^{\leq n}(F). \end{aligned}$$

4. The reduced Robinson complex

We use this section to identify a subcomplex of Robinson's complex that we refer to as the reduced Robinson complex. We obtain this subcomplex by using certain cross effects of the functor F. Our interest in the reduced complex is motivated by two factors. Primarily, we wish to use Robinson's complex to create a bicomplex that captures the rank filtration of the functor $D_1 F$ in a nice fashion. As we will see in Section 5, restricting to the reduced complex produces a bicomplex whose filtration by rows is quasi-isomorphic to the rank filtration of $D_1 F$.

The second motivational factor is the fact that cross effects can often be used to simplify homology calculations in the setting of Γ-modules. This becomes apparent in Section 5 and Proposition 4.10 where calculations involving the homology of the reduced Robinson complex are relatively straightforward, but calculations with the unreduced complex are carried out by calling upon deeper results of Pirashvili and of Betley and Słomińska. Pirashvili's work reduces Tor calculations for Γ-modules to Tor calculations in a smaller category, while Betley and Słomińska take advantage of this to calculate Tor groups for a particular class of functors. We begin this section by reviewing Pirashvili's result, before summarizing some of Betley and Słomińska's calculations. We finish by describing the reduced Robinson complex and showing that it agrees with the unreduced complex on a particular class of functors.

Pirashvili compared the category of Γ-modules to the category of Ω-modules. Here Ω is the category whose objects are finite sets (without basepoint) and whose morphisms are surjective set maps. We use $\langle m\rangle$ to represent the set $\{1, 2, \ldots, m\}$ with m elements. A left (respectively, right) Ω-module is a covariant (respectively, contravariant) functor from Ω to R-modules. One can transform a Γ-module into an Ω-module via the functor cr, defined for a Γ-module F by

$$cr F\langle n\rangle = cr_n F[1]. \tag{4.1}$$

That crF is a functor from Ω to R-modules is a consequence of the following lemma, a version of which appears in [10].

Lemma 4.2 (*[10, 2.1]*). *If $\alpha : [n] \to [m]$ is a surjection, then for a Γ-module F and object X, the image of $cr_n F(X)$ under the induced map $F(\alpha) : F(\vee_n X) \to F(\vee_m X)$ is contained in $cr_m F(X)$.*

Proof. Recall from Remark 1.2(c) that

$$cr_n F \cong \bigcap_{i=1}^{n} \ker F(r_i).$$

Using this version of the cross effect, it is enough to show that for any $r_j : [m] \to [m-1]$, $1 \leq j \leq m$, there is an i, $1 \leq i \leq n$, such that $F(\alpha)(\ker F(r_i)) \subseteq \ker(F(r_j))$. This can be done by noting that for any such α and j, there is an i, $1 \leq i \leq n$, and a surjection $\beta : [n-1] \to [m-1]$ such that $r_j \circ \alpha = \beta \circ r_i$.

One can define Tor for Ω-modules as one does for Γ-modules. With this we can state Pirashvili's Dold–Kan correspondence.

Theorem 4.3 (*[10, 3.1, 3.2], [11, pp. 159–160]*). *The functor cr induces an equivalence of categories between the category of left (resp., right) Γ-modules and the category of left (resp., right) Ω-modules. For a right Γ-module F and left Γ-module G,*

$$\mathrm{Tor}^{\Gamma}_*(F, G) \cong \mathrm{Tor}^{\Omega}_*(crF, crG).$$

Betley and Słomińska determined $\mathrm{Tor}_*^{\Omega}(M, N)$ in the case where M and N are the following type of functors.

Definition 4.4. An Ω-module F is *atomic* if $F\langle n\rangle$ is 0 at all but one value of n.

Betley and Słomińska's calculations are in terms of the homology of suspensions of partition complexes. Let $\mathcal{P}(n)$ be the set of partitions of $\langle n\rangle$. The set $\mathcal{P}(n)$ is a poset (and hence a category) where the ordering is given by refinements of partitions. This category has both initial and final objects, the partitions $(\{1, 2, \ldots, n\})$ and $(\{1\}, \{2\}, \ldots, \{n\})$, respectively. Let $\mathcal{K}(n)$ be the full subcategory of $\mathcal{P}(n)$ obtained by removing the final and initial objects. The geometric realization of the nerve of $\mathcal{K}(n)$ is the partition complex K_n. The partition complexes play a fundamental role in the calculus of homotopy functors. In particular, their suspensions characterize the derivatives of the identity functor of spaces [2]. As a space, $K_n \simeq \vee_{(n-1)!} S^{n-2}$, and possesses a Σ_n-action inherited from the action on the set $\langle n\rangle$. For this proof, we are interested in the homology of $S^1 \wedge K_n$ as Σ_n-representations. Arone and Kankaanrinta [1, 2.3] prove that

$$D_n := H_{n-1}(S^1 \wedge K_n) \cong \mathrm{Lie}_n^* \tag{4.5}$$

as Σ_n-modules. Betley and Słomińska prove the following.

Theorem 4.6 (*[4, 2.7]*). *Let M and N be R-modules. Let $M^*(1)$ be the atomic contravariant functor that is equal to M at $\langle 1\rangle$ and 0 elsewhere and, similarly, let $N(n)$ be the atomic covariant functor that is N at $\langle n\rangle$ and 0 elsewhere. Then*

$$\mathrm{Tor}_i^{\Omega}(M^*(1), N(n)) \cong H_{i+1-n}(\Sigma_n, D_n \otimes \mathrm{Hom}_R(M, N)).$$

They obtain more general results for any pair of atomic functors (where the contravariant functor is not necessarily concentrated at the object $\langle 1\rangle$), but we only need the above for the current work. In particular, we use the corollary below.

Corollary 4.7. *If F is a Γ-module such that $cr\,F$ is an atomic functor whose only non-zero value occurs at the object $\langle n\rangle$, then*

$$H_*(\Xi(F))[1] \cong H_{*+1-n}(\Sigma_n, D_n \otimes cr\,F\langle n\rangle).$$

Proof. By Theorems 3.2 and 4.3, we know that

$$H_*(\Xi(F))[1] \cong \mathrm{Tor}_*^{\Gamma}(\widetilde{R}^*[-], F) \cong \mathrm{Tor}_*^{\Omega}(cr\,\widetilde{R}^*[-], cr\,F).$$

But $cr\,\widetilde{R}^*[-]$ is an atomic functor with $cr\,\widetilde{R}^*[-]\langle 1\rangle \cong R$. The result now follows from Theorem 4.6.

The subcomplex, $\widetilde{\widetilde{\Xi}}(F)$, of $\Xi(F)$ that we are interested in using is obtained by replacing $F(\vee_n X)$ with $cr_n F(X)$. That reducing to these cross effects produces a bicomplex is a consequence of Lemma 4.2 and the fact that only surjections of Γ are used in the construction of $\Xi(F)$.

Definition 4.8. Let F be a Γ-module. The *reduced Robinson complex of* F is the bicomplex of functors $\widetilde{\widetilde{\Xi}}(F)$ that for an object X, in bidegree (p, q), is given by

$$\widetilde{\widetilde{\Xi}}(F)_{p,q}(X) = \mathrm{Lie}_{q+1}^* \otimes R[\Sigma_{q+1}^p] \otimes cr_{q+1} F(X).$$

The differential $\widetilde{\partial}' : \widetilde{\widetilde{\Xi}}_{p,q}(F) \to \widetilde{\widetilde{\Xi}}_{p-1,q}(F)$ (respectively $\widetilde{\partial}'' : \widetilde{\widetilde{\Xi}}_{p,q}(F) \to \widetilde{\widetilde{\Xi}}_{p,q-1}(F)$) is the restriction of ∂' (respectively ∂'') to the direct summand $\mathrm{Lie}_{q+1}^* \otimes R[\Sigma_{q+1}^p] \otimes cr_{q+1} F(X)$ of $\Xi(F)_{p,q}(X)$. Lemma 4.2 guarantees that these restrictions are differentials since ∂' and ∂'' are differentials in $\Xi(F)(X)$. That $\widetilde{\widetilde{\Xi}}(F)$ is a bicomplex follows from the fact that $\Xi(F)$ is.

Lemma 4.2 and the fact that the maps c_{ij} used to define ∂'' are surjections also imply that for each p and q, and object X, the natural inclusion maps

$$\phi_{p,q} : \mathrm{Lie}_{q+1}^* \otimes R[\Sigma_{q+1}^p] \otimes cr_{q+1} F(X) \to \mathrm{Lie}_{q+1}^* \otimes R[\Sigma_{q+1}^p] \otimes F(\vee_{q+1} X)$$

form a bicomplex homomorphism.

Proposition 4.9. *There is a natural transformation of bicomplexes of functors* $\phi(F) : \widetilde{\Xi}(F) \to \Xi(F)$ *that in bidegree* (p, q) *is given by the map* $\phi_{p,q}(F) : \widetilde{\Xi}(F)_{p,q} \to \Xi(F)_{p,q}$ *induced by the inclusion* $cr_{q+1}F(-) \hookrightarrow F(\vee_{q+1} -)$.

We show in the next section that $\phi(F)$ is a quasi-isomorphism. Since the homology of $\Xi(F)[1]$ is $\mathrm{Tor}_*^{\Gamma}(\widetilde{R}[-], F)$ and $\widetilde{\Xi}(F)$ is obtained from $\Xi(F)$ via cross effects, one may initially suspect that the fact that $\phi(F)$ is a quasi-isomorphism is simply a restatement of Pirashvili's Dold–Kan correspondence. However, when restricted to Ω, the Γ-modules in $\widetilde{\Xi}(F)$ are not the result of applying cr to the Γ-modules of $\Xi(F)$. In particular, applying cr to $\Xi(F)$ yields a bicomplex that when evaluated at $\langle n\rangle$ in bidegree (p, q) is

$$\mathrm{Lie}^*_{q+1} \otimes R[\Sigma^p_{q+1}] \otimes cr_n(F \circ \vee_{q+1})[1]$$

whereas evaluating $\widetilde{\Xi}(F)$ at $[n]$ in bidegree (p, q) yields

$$\mathrm{Lie}^*_{q+1} \otimes R[\Sigma^p_{q+1}] \otimes cr_{q+1}F[n].$$

In general, $cr_n(F \circ \vee_{q+1})[1]$ is not the same as $cr_{q+1}F[n]$. We will see in the next section that the fact that $\phi(F)$ is a quasi-isomorphism is a consequence, though indirectly, of the Dold–Kan correspondence. We conclude this section by showing how this is done in the case when $cr F$ is atomic.

Proposition 4.10. *Let* F *be a* Γ*-module for which* $cr F$ *is an atomic functor whose only nontrivial value occurs at* $\langle n\rangle$*. Then* $\phi(F)[1] : \widetilde{\Xi}(F)[1] \to \Xi(F)[1]$ *is a quasi-isomorphism.*

Proof. By Remark 1.2.1(a), the fact that $cr F$ is atomic with its nontrivial value at $\langle n\rangle$ means that F is completely determined by its value at $[n]$. As a result, $L_{n-1}F \simeq 0$, $L_nF \simeq F$, and so $R_nF \simeq F$. By Proposition 2.6 it follows that F is naturally equivalent to a functor of the form $\widetilde{R}[\mathrm{Inj}([n], -)] \otimes_{\Sigma_n} A$ where A is the Σ_n-module $cr_nF[1]$. Since both Ξ and $\widetilde{\Xi}$ can be defined levelwise on simplicial functors, and the functor F can be resolved simplicially using functors of the form $\widetilde{R}[\mathrm{Inj}([n], -)] \otimes_{\Sigma_n} Y$ where Y is a free Σ_n-module, we can reduce the proof to showing that $\phi(F)[1]$ is a quasi-isomorphism when $F = \widetilde{R}[\mathrm{Inj}([n], -)]$.

The bicomplex $\widetilde{\Xi}(\widetilde{R}[\mathrm{Inj}([n], -)])[1]$ has a single nontrivial row whose term of bidegree $(p, n-1)$ is

$$\mathrm{Lie}^*_n \otimes R[\Sigma^p_n] \otimes cr_n\widetilde{R}[\mathrm{Inj}([n], -)][1].$$

The pth homology group in this row is

$$\mathrm{Tor}^{\Sigma_n}_p(\mathrm{Lie}^*_n, cr_n\widetilde{R}[\mathrm{Inj}([n], -)][1]).$$

By Example 1.3 we know that $cr_n\widetilde{R}[\mathrm{Inj}([n], -)][1] \cong R[\Sigma_n]$ and so

$$\mathrm{Tor}^{\Sigma_n}_p(\mathrm{Lie}^*_n, cr_n\widetilde{R}[\mathrm{Inj}([n], -)][1]) \cong \mathrm{Tor}^{\Sigma_n}_p(\mathrm{Lie}^*_n, R[\Sigma_n]) \cong \begin{cases} \mathrm{Lie}^*_n & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$H_\star\widetilde{\Xi}(\widetilde{R}[\mathrm{Inj}([n], -)])[1] \cong \begin{cases} \mathrm{Lie}^*_n & \text{if } \star = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

In comparison, we know from (4.5) and Corollary 4.7 that

$$H_\star\Xi(\widetilde{R}[\mathrm{Inj}([n], -)])[1] \cong H_{\star+n-1}(\Sigma_n, \mathrm{Lie}^*_n \otimes R[\Sigma_n]) \cong \begin{cases} \mathrm{Lie}^*_n & \text{if } n = \star - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\Xi(\widetilde{R}[\mathrm{Inj}([n], -)])[1]$ and $\widetilde{\Xi}(\widetilde{R}[\mathrm{Inj}([n], -)])[1]$ have the same homology. To conclude that ϕ induces a quasi-isomorphism between them it suffices to note that ϕ is an isomorphism on the first nontrivial rows of the two complexes and that the homology of each of the higher rows is concentrated in degree 0. To see the latter note that for $k > n$,

$$\widetilde{\Xi}_{p,k-1}\widetilde{R}[\mathrm{Inj}([n], -)][1] \cong 0$$

and

$$H_p(\Xi_{-,k-1}\widetilde{R}[\mathrm{Inj}([n],-)][1]) \cong \mathrm{Tor}_p^{\Sigma_k}(\mathrm{Lie}_k^*, \widetilde{R}[\mathrm{Inj}([n],[k])]).$$

However, as a left Σ_k-module, $\widetilde{R}[\mathrm{Inj}([n],[k])] \cong R[\Sigma_k] \otimes_{\Sigma_{k-n}\times\Sigma_n} R[\Sigma_n]$. By Shapiro's lemma, it follows that

$$\mathrm{Tor}_p^{\Sigma_k}(\mathrm{Lie}_k^*, \widetilde{R}[\mathrm{Inj}([n],[k])]) \cong \mathrm{Tor}_p^{\Sigma_{k-n}\times\Sigma_n}(\mathrm{Lie}_k^*, R[\Sigma_n]).$$

Moreover, $R[\Sigma_n] \cong R[(\Sigma_{k-n} \times \Sigma_n)] \otimes_{\Sigma_{k-n}} R[\Sigma_1]$ as left $\Sigma_{k-n} \times \Sigma_n$-modules. Another application of Shapiro's lemma yields

$$\mathrm{Tor}_p^{\Sigma_{k-n}\times\Sigma_n}(\mathrm{Lie}_k^*, R[\Sigma_n]) \cong \mathrm{Tor}_p^{\Sigma_{k-n}}(\mathrm{Lie}_k^*, R[\Sigma_1]).$$

Since Lie_k^* is the regular representation of Σ_{k-1} for $\Sigma_{k-n} \subseteq \Sigma_k$, it follows that for $p > 0$

$$\mathrm{Tor}_p^{\Sigma_{k-n}}(\mathrm{Lie}_k^*, R[\Sigma_1]) \cong 0.$$

Hence $\phi(F)[1]$ is a quasi-isomorphism when $F = \widetilde{R}[\mathrm{Inj}([n],-)]$ and the result follows.

5. Filtrations

As we mentioned at the end of Section 3, the obvious filtration of the Robinson complex by rows is not equivalent to the rank filtration (3.11) of $D_1 F$. We use this section to show that the filtration of the reduced Robinson complex by rows is equivalent to the rank filtration of $D_1 F$ by proving that it is equivalent to the filtration of Definition 3.12. As a consequence, we also deduce that the natural transformation ϕ of Proposition 4.9 is a quasi-isomorphism.

We use $\widetilde{\Xi}^{\leq n}(F)$ to denote the nth stage of the filtration by rows of $\widetilde{\Xi}(F)$, i.e., the bicomplex with

$$\widetilde{\Xi}^{\leq n}(F)_{p,q} = \begin{cases} \widetilde{\Xi}(F)_{p,q} & \text{if } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.1. *For any Γ-module F and $n \geq 1$, the natural transformation $\phi : \widetilde{\Xi}(F) \to \Xi(F)$ induces a natural transformation of filtrations*

$$\begin{array}{ccccccccc} \widetilde{\Xi}^{\leq 1}(F) & \to & \cdots & \to & \widetilde{\Xi}^{\leq n}(F) & \to & \widetilde{\Xi}^{\leq n+1}(F) & \to & \cdots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ \Xi^{\leq 1}(F) & \to & \cdots & \to & \Xi^{\leq n}(F) & \to & \Xi^{\leq n+1}(F) & \to & \cdots \end{array}$$

that is a quasi-isomorphism of functors $\widetilde{\Xi}^{\leq n}(F) \xrightarrow{\simeq} \Xi^{\leq n}(F)$ at each stage of the filtration.

We use the remainder of the section to prove Theorem 5.1. Before we do so, we note the following immediate consequences of this theorem.

Corollary 5.2. (1) *For a Γ-module F, the natural transformation of Proposition 4.9, $\phi : \widetilde{\Xi}(F) \to \Xi(F)$, is a quasi-isomorphism.*
(2) *The filtrations $\{D_1 L_n F\}$ and $\{\widetilde{\Xi}^{\leq n}(F)\}$ are equivalent.*

Theorem 5.1 is proved by induction on the terms in the filtrations of $\Xi(F)$ and $\widetilde{\Xi}(F)$, via the following lemma.

Lemma 5.3. *Let $n > 1$ and F be a Γ-module. Let $\Xi^n(F) = \mathrm{cofiber}(\Xi^{\leq n-1}(F) \to \Xi^{\leq n}(F))$ and $\widetilde{\Xi}^n(F) = \mathrm{cofiber}(\widetilde{\Xi}^{\leq n-1}(F) \to \widetilde{\Xi}^{\leq n}(F))$. The inclusion $\phi : \widetilde{\Xi}(F) \to \Xi(F)$ induces a quasi-isomorphism on the cofibers of the filtrations when evaluated at* [1]*:*

$$\phi : \widetilde{\Xi}^n(F)[1] \xrightarrow{\simeq} \Xi^n(F)[1].$$

Proof of Theorem 5.1. It is enough to prove the result when the functors are evaluated at [1] since to obtain the result for the functor F evaluated at $[m]$, we replace F by the functor $F \circ \bigvee_m$ and evaluate at [1]. More precisely, for any Γ-module F and $m \geq 0$, $\Xi(F)[m] \cong \Xi(F \circ \bigvee_m)[1]$ by definition and $\widetilde{\Xi}(F)[m] \cong \widetilde{\Xi}(F \circ \bigvee_m)[1]$ by Proposition 1.4. Finally, $\Xi^{\leq k}(F)[m] \cong \Xi^{\leq k}(F \circ \bigvee_m)[1]$ and $\widetilde{\Xi}^{\leq k}(F)[m] \cong \widetilde{\Xi}^{\leq k}(F \circ \bigvee_m)[1]$ for any terms in the filtrations of Ξ and $\widetilde{\Xi}$, and the reduction to [1] is complete.

For induction, we first consider $\widetilde{\Xi}^{\leq 1}(F)[1]$ and $\Xi^{\leq 1}(F)[1]$. The bicomplex $\widetilde{\Xi}^{\leq 1}(F)[1]$ consists of a single nontrivial row whose mth term is $\mathrm{Lie}_1^* \otimes R[\Sigma_1^m] \otimes cr_1 F[1]$. Since $\mathrm{Lie}_1^* \cong R$, the homology of this row is concentrated in degree 0, where it is isomorphic to $cr_1 F[1]$.

To determine the homology of $\Xi^{\leq 1}(F)[1]$, we note that Propositions 3.13, 2.10, and Corollary 3.10 imply that

$$\Xi^{\leq 1}(F) \simeq D_1 L_1 F \simeq \Xi(L_1 F). \tag{5.4}$$

But, $L_1 F$ is degree 1 and so $cr L_1 F$ is an atomic functor with $cr L_1 F\langle 1\rangle \cong cr_1 F[1]$. Hence, by Corollary 4.7 and (5.4), $\Xi^{\leq 1}(F)[1]$ is quasi-isomorphic to $cr_1 F[1]$ and the result holds for $n = 1$.

To finish the proof, note that for $n > 1$, we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\Xi}^{\leq n-1}(F)[1] & \to & \widetilde{\Xi}^{\leq n}(F)[1] & \to & \widetilde{\Xi}^{n}(F)[1] \\
\downarrow & & \downarrow & & \downarrow \\
\Xi^{\leq n-1}(F)[1] & \to & \Xi^{\leq n}(F)[1] & \to & \Xi^{n}(F)[1]
\end{array}$$

in which the rightmost arrow is a quasi-isomorphism by Lemma 5.3 and the two rows are quasi-exact. The result follows by induction.

Proof of Lemma 5.3. The kth row of $\Xi^{\leq n}(F)[1]$ has terms of the form $\mathrm{Lie}_{k+1}^* \otimes R[\Sigma_{k+1}^p] \otimes \widetilde{R}[\mathrm{Hom}^{\leq n}(*, [k+1])] \otimes_\Gamma F(*)$. It follows that the first n nontrivial rows of $\Xi^{\leq n}(F)$ agree with those of $\Xi(F)$. Hence the map $\phi : \widetilde{\Xi}(F) \to \Xi(F)$ extends to a map $\phi : \widetilde{\Xi}^{\leq n}(F) \to \Xi^{\leq n}(F)$ which is the same inclusion on the first n rows as ϕ and 0 elsewhere. It follows that we have a commutative square

$$\begin{array}{ccc}
\widetilde{\Xi}^{\leq n-1}(F) & \hookrightarrow & \Xi^{\leq n-1}(F) \\
\downarrow & & \downarrow \\
\widetilde{\Xi}^{\leq n}(F) & \hookrightarrow & \Xi^{\leq n}(F),
\end{array}$$

and, as a result, ϕ induces a map of the cofibers.

To show the map of cofibers is a quasi-isomorphism, we begin by considering the corresponding cofiber in the rank filtration, $R_n F := \mathrm{cofiber}(L_{n-1}F \to L_n F)$. Recall from Proposition 2.6 that $R_n F$ has the form

$$R_n F \simeq C_n F := \widetilde{R}[\mathrm{Inj}([n], -)] \otimes_{\Sigma_n} cr_n F[1].$$

By Example 1.3,

$$\begin{aligned}
cr_k C_n F[1] &\cong cr_k(\widetilde{R}[\mathrm{Inj}([n], -)][1] \otimes_{\Sigma_n} cr_n F[1]) \\
&\cong cr_k \widetilde{R}[\mathrm{Inj}([n], -)][1] \otimes_{\Sigma_n} cr_n F[1] \\
&\cong \begin{cases} R[\Sigma_n] \otimes_{\Sigma_n} cr_n F[1] & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases} \\
&\cong \begin{cases} cr_n F[1] & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}
\end{aligned} \tag{5.5}$$

Thus, $cr C_n F$ is an atomic functor. By Proposition 4.10, it follows that ϕ induces a quasi-isomorphism $\widetilde{\Xi}(C_n F)[1] \simeq \Xi(C_n F)[1]$. The result then follows provided we show that $\Xi^n(F)[1] \simeq \Xi(C_n F)[1]$ and $\widetilde{\Xi}^n(F)[1] \simeq \widetilde{\Xi}(C_n F)[1]$.

But, by Propositions 3.13 and 2.6 and the fact that D_1 is exact we know that

$$\begin{aligned}
\Xi^n(F) &= \mathrm{cofiber}(\Xi^{\leq n-1}(F) \to \Xi^{\leq n}(F)) \\
&\simeq \mathrm{cofiber}(D_1 L_{n-1} F \to D_1 L_n F) \\
&\simeq D_1 R_n F \\
&\simeq \Xi(C_n F).
\end{aligned}$$

Hence $\Xi^n(F) \simeq \Xi(C_n F)$.

On the other hand, $\widetilde{\Xi}^n(F)[1]$ has a single homologically nontrivial row whose term of bidegree $(p, n-1)$ is

$$\mathrm{Lie}_n^* \otimes \Sigma_n^p \otimes cr_n F[1].$$

By (5.5), $\widetilde{\Xi}(C_n F)$ also has a single homologically nontrivial row that is isomorphic to that of $\widetilde{\Xi}^n(F)$. Thus, $\widetilde{\Xi}^n(F) \simeq \widetilde{\Xi}(C_n F)$ and the result follows.

Acknowledgements

We thank the referee for his/her helpful comments. This work was supported by NSF DMS 03-06429 and the Institut Mittag-Leffler (Djursholm, Sweden).

References

[1] G. Arone, M. Kankaanrinta, The homology of certain subgroups of the symmetric group with coefficients in Lie(n), J. Pure Appl. Algebra 127 (1) (1998) 1–14.
[2] G. Arone, M. Mahowald, The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres, Invent. Math. 135 (3) (1999) 743–788.
[3] A.K. Bousfield, E.M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in: Geometric Applications of Homotopy Theory (Proc. Conf., Evanston, Ill., 1977), II, in: Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130.
[4] S. Betley, J. Słomińska, New approach to the groups $H_*(\Sigma_n, \mathrm{Lie}_n)$ by the homology theory of the category of functors, J. Pure Appl. Algebra 161 (1–2) (2001) 31–43.
[5] A. Dold, D. Puppe, Homologie nicht-additiver Funktoren. Anwendungen., Ann. Inst. Fourier (Grenoble) 11 (1961) 201–312.
[6] T. Goodwillie, Calculus III. Taylor series., Geom. Topol. 7 (2003) 645–711.
[7] B. Johnson, R. McCarthy, Deriving calculus with cotriples, Trans. Amer. Math. Soc. 356 (2) (2004) 757–803.
[8] B. Johnson, R. McCarthy, Linearization, Dold–Puppe stabilization, and Mac Lane's Q-construction, Trans. Amer. Math. Soc. 350 (4) (1998) 1555–1593.
[9] B. Johnson, R. McCarthy, A classification of degree n functors I, Cah. Topol. Géom. Différ. Catég. 44 (1) (2003) 2–38.
[10] T. Pirashvili, Dold–Kan type theorem for Γ-groups, Math. Ann. 318 (2) (2000) 277–298.
[11] T. Pirashvili, Hodge decomposition for higher order Hochschild homology, Ann. Sci. École Norm. Sup. (4) 33 (2) (2000) 151–179.
[12] B. Richter, Taylor towers for Γ-modules, Ann. Inst. Fourier (Grenoble) 51 (4) (2001) 995–1023.
[13] A. Robinson, Gamma homology, Lie representations and E_∞ multiplications, Invent. Math. 152 (2) (2003) 331–348.
[14] G. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.