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Pushout of quasi-finite and flat group schemes over a Dedekind ring

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ABSTRACT

Let G, G_1 and G_2 be quasi-finite and flat group schemes over a complete discrete valuation ring R, $\varphi_1 : G \to G_1$ any morphism of R-group schemes and $\varphi_2 : G \to G_2$ a model map. We construct the pushout P of G_1 and G_2 over G in the category of R-affine group schemes. In particular when φ_1 is a model map too we show that P is still a model of the generic fibre of G. We also provide a short proof for the existence of cokernels and quotients of finite and flat group schemes over any Dedekind ring.

Contents

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1. Introduction

1.1. Aim and scope

We are interested in the construction of the pushout (whose definition will be recalled in Section 2) in the category of affine group schemes over a given ring as described hereafter. It is known that in the category of abstract groups the pushout of two groups over a third one always exists but it is not finite even when the three groups are all finite (unless one takes very particular cases). However for group schemes over a Dedekind ring R something new happens when we consider some special important cases: so let G, G_1 and G_2 be R-affine group schemes and consider the diagram

$$\begin{array}{ccc} & G & \\ {}^{\varphi_1}\swarrow & & \searrow^{\varphi_2} \\ G_1 & & G_2 \end{array} \tag{1}$$

where $\varphi_i : G \to G_i$ $(i = 1, 2)$ are R-group scheme morphisms. We first prove the following

Theorem 1.1. *(Cf. Theorem 3.2.) Assume R is a complete discrete valuation ring and G, G_1, G_2 are finite and flat over R. Then if φ_1 is a model map (i.e. generically an isomorphism) the pushout of (1) in the category of affine R-group schemes exists. Moreover it is finite and flat and its generic fibre is isomorphic to $G_{2,K}$, the generic fibre of G_2.*

This immediately implies that when G, G_1 and G_2 are all models of a same K-group scheme G_K (K being the fraction field of R) then the pushout of (1) exists and is still a model of G_K thus proving the existence of a lower bound for models of finite group schemes. This was already known in the commutative case (cf. [9], Proposition 2.2.2). The same will be true for the quasi-finite case under the assumption that $G_{2,K}$ admits a finite and flat R-model:

Theorem 1.2. *(Cf. Theorem 3.5.) Assume R is a complete discrete valuation ring and G, G_1, G_2 are quasi-finite and flat over R. If φ_1 is a model map and $G_{2,K}$ admits a finite and flat model then the pushout of (1) in the category of affine R-group schemes exists. Moreover it is quasi-finite and flat and its generic fibre is isomorphic to $G_{2,K}$.*

Using the fact that $G_{2,K}$ always admits, when it is étale, a finite and flat model up to a finite extension of scalars we finally prove the following

Corollary 1.3. *(Cf. Corollary 3.9.) Assume R is a complete discrete valuation ring and G, G_1, G_2 are quasi-finite and flat over R. Then if φ_1 is a model map and $G_{2,K}$ is étale then the pushout of (1) in the category of affine R-group schemes exists. Again it is quasi-finite and flat and its generic fibre is isomorphic to $G_{2,K}$.*

All the proofs rest on the computation of the pushout in the category of R-Hopf algebras. With the same techniques we briefly study in Section 3.3 the existence of cokernels in the category of affine R-group schemes where R is any Dedekind ring. This will lead to a new and short proof of the following:

Corollary 1.4. *(Cf. Corollary 3.13.) Let R be a Dedekind ring, G and H two finite and flat R-group schemes with H a closed and normal R-subgroup scheme of G. Then the quotient G/H exists in the category of R-affine group schemes.*

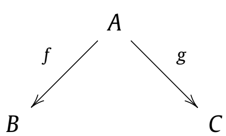
This holds over any base scheme and is in fact a consequence of a much bigger theorem (cf. [4], Théorème 7.1).

1.2. Notations and conventions

Every ring A will be supposed to be associative and unitary, i.e. provided with a unity element denoted by 1_A, or simply 1 if no confusion can arise. However, unless stated otherwise, a ring will not be supposed to be commutative. Every Dedekind ring, instead, will always be supposed to be commutative. For an R-algebra A the morphisms $u_A : R \to A$ and $m_A : A \otimes_R A \to A$ will always denote the unity and the multiplication morphisms (respectively). If moreover A has an R-coalgebra structure then $\Delta_A : A \to A \otimes_R A$, $\varepsilon_A : A \to R$ will denote the comultiplication and the counity respectively. Furthermore if A has an R-Hopf algebra structure then $S_A : A \to A$ will denote the coinverse. All the coalgebra structures will be supposed to be coassociative. Morphisms of R-algebras (resp. R-coalgebras, R-Hopf algebras) are R-module morphisms preserving R-algebra (resp. R-coalgebra, R-Hopf algebra) structure. We denote by $R\text{-}\mathcal{H}opf$ the category of associative and coassociative R-Hopf algebras while $R\text{-}\mathcal{H}opf_{ff}$ will denote the category of associative and coassociative R-Hopf algebras which are finite and flat as R-modules. When $R \to T$ is a morphism of commutative algebras, M is an R-module, X is an R-scheme, $f : M \to N$ is an R-module morphism and $\varphi : X \to Y$ a morphism of R-schemes then we denote by M_T, X_T, $f_T : M_T \to N_T$ and $\varphi_T : X_T \to Y_T$ respectively the T-module $M \otimes_R T$, the T-scheme $X \times_{Spec(R)} Spec(T)$, the T-module morphism induced by f and the T-morphism of schemes induced by φ. When R is a Dedekind ring and K its field of fractions then an R-morphism of schemes $\varphi : X \to Y$ is called a model map if generically it is an isomorphism, i.e. $\varphi_K : X_K \to Y_K$ is an isomorphism.

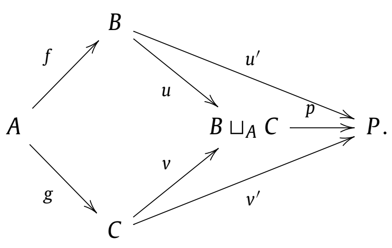
2. Pushout of Hopf algebras

In this section we first study the pushout of algebras over a commutative ring R then we discuss the existence of the pushout in the category of $R\text{-}\mathcal{H}opf_{ff}$ when R is a complete discrete valuation ring. Let us first recall that in a category $\mathcal{C}$ the pushout (see for instance [8], III, §3) of a diagram



(2)

(where clearly A, B, C are objects of $\mathcal{C}$ and f, g morphisms in the same category) is an object of $\mathcal{C}$ that we denote $B \sqcup_A C$ provided with two morphisms $u : B \to B \sqcup_A C$, $v : C \to B \sqcup_A C$ such that $uf = vg$ and satisfying the following universal property: for any object P of $\mathcal{C}$ and any two morphisms $u' : B \to P$, $v' : C \to P$ in $\mathcal{C}$ such that $u'f = v'g$ then there exists a unique morphism $p : B \sqcup_A C \to P$ making the following diagram commute:



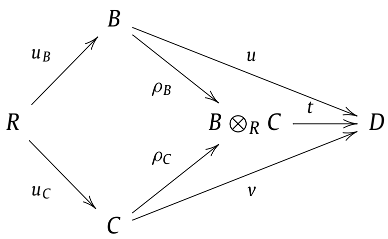
When A is an initial object (provided it exists) of $\mathcal{C}$ then $B \sqcup_A C$ is the coproduct[2] of B and C in $\mathcal{C}$. When $\mathcal{C}$ is the category of commutative R-algebras then the pushout is given by the tensor product

[2] The coproduct can be defined, however, without assuming the existence of an initial object.

$B \otimes_A C$. This is not true anymore if $\mathcal{C}$ is the category of R-algebras (cf. Example 2.9 or create easier examples). However we can always find a pushout even when $\mathcal{C}$ is the category of R-algebras and it will be denoted by $B *_A C$. Before introducing, however, the pushout for non(necessarily)-commutative R-algebras we recall the behavior of the tensor product over R. We put ourselves in the following situation:

Notation 2.1. By R we will denote a commutative ring while A, B and C will be R-algebras and $f : A \to B$, $g : A \to C$ two R-algebra morphisms. We also denote by $\rho_B : B \to B \otimes_R C$ and $\rho_C : C \to B \otimes_R C$ the morphisms sending respectively $b \mapsto b \otimes 1_C$ and $c \mapsto 1_B \otimes c$.

Proposition 2.2. *Let D be any R-algebra and $u : B \to D$, $v : C \to D$ two R-algebra morphisms such that $u \circ u_B = v \circ u_C$ and such that $u(b)v(c) = v(c)u(b)$ for all $b \in B$, $c \in C$. Then there exists a unique R-algebra morphism $t : B \otimes_R C \to D$ making the following diagram commute*:



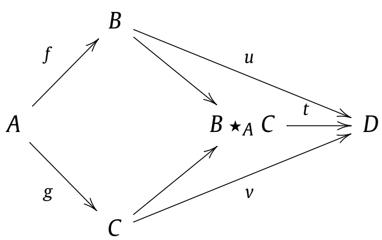
Proof. Cf. for instance [3], I, §3, Proposition 3.2. □

Unfortunately $B \otimes_A C$ behaves badly in general and one can observe that even $A \otimes_A A \simeq A$, as an R-algebra, is not a natural quotient of $A \otimes_R A$. So, instead, let us consider the following construction:

Definition 2.3. We denote by $B \star_A C$, and we call it the star product of B and C over R, the quotient of $B \otimes_R C$ by the two-sided ideal generated by A, i.e. the ideal of $B \otimes_R C$ generated by the set $\{\rho_B f(a) - \rho_C g(a)\}_{a \in A}$.

It is an easy consequence the following universal property of the star product:

Proposition 2.4. *Let D be any R-algebra and $u : B \to D$, $v : C \to D$ two R-algebra morphisms such that $uf = vg$ and such that $u(b)v(c) = v(c)u(b)$ for all $b \in B$, $c \in C$. Then there exists a unique R-algebra morphism $t : B \star_A C \to D$ making the following diagram commute*:



Proof. It is sufficient to take the R-algebra morphism $B \otimes_R C \to D$ and observe that it passes to the quotient. □

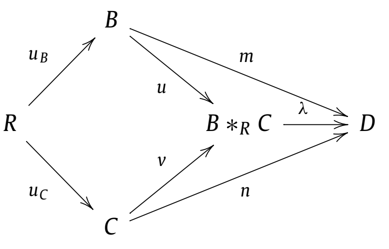
The star product will only be used in Example 2.8 and Section 3.3, so finally let us recall the construction of the pushout of R-algebras: we follow essentially [2] 1.7 and 5.1 with very few modifications in the exposition in order to obtain an easier to handle description. We describe A, B and C giving their presentation as R-algebras thus getting $R\langle X_0; S_0\rangle$, $R\langle X_1; S_1\rangle$ and $R\langle X_2; S_2\rangle$ respectively, where X_i is a generating set with relations S_i $(i = 0, 1, 2)$. We recall that $R\langle X; S\rangle$ is to be intended as the R-algebra whose elements are all R-linear combinations of words on the set X quotiented by the two-sided ideal generated by the relations in S. Observe that for $y, z \in X$ we are not assuming $zy = yz$; if it is the case the information will appear in S. However for any $x \in X$ and any $r \in R$ we do assume $xr = rx$. For example the commutative R-algebra $R[x, y]/f(x, y)$ can be presented as $R\langle x, y; f(x, y) = 0, xy = yx\rangle$. First we observe that the coproduct of B and C (i.e. the pushout of B and C over the initial object R) is given by the R-algebra $B *_R C := R\langle X_1 \cup X_2; S_1 \cup S_2\rangle$ where the union is of course disjoint. Let us denote by $u : B \to B *_R C$ and $v : C \to B *_R C$ the canonical inclusions. Then the pushout of B and C over A is given by the R-algebra

$$B *_A C := R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3\rangle \tag{3}$$

where S_3 consists on the relations given by $uf(x) = vg(x)$ for every $x \in X_0$. Now we relate the pushout just described to the tensor product:

Lemma 2.5. *Assume that $B = R\langle X_1; S_1\rangle$ and $C = R\langle X_2; S_2\rangle$. Then $B \otimes_R C$ can be presented as $R\langle X_1 \cup X_2; S_1 \cup S_2, \{zy = yz\}_{z\in X_1, y\in X_2}\rangle$ thus becoming a quotient of $R\langle X_1 \cup X_2; S_1 \cup S_2\rangle = B *_R C$.*

Proof. Let D be an R-algebra provided with R-algebra morphisms $m : B \to D$ and $n : C \to D$ such that $m \circ u_B = n \circ u_C$, and assume moreover that $m(b)n(c) = n(c)m(b)$ for all $b \in B, c \in C$. Let us denote by $u : B \to B *_R C$ and $v : C \to B *_R C$ the canonical morphisms and by $\lambda : B *_R C \to D$ the universal morphism making the following diagram commute:



By assumption $\lambda u(z)\lambda v(y) = \lambda v(y)\lambda u(z)$ so $u(z)v(y) - v(y)u(z) \in ker(\lambda)$ hence λ factors through $R\langle X_1 \cup X_2; S_1 \cup S_2, \{zy = yz\}_{z\in X_1, y\in X_2}\rangle$ providing it with the universal property stated in Proposition 2.2 and this is enough to conclude. □

Let $q : R \to T$ be an R-commutative algebra. When $f = f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ we denote by $q_*(f)$ the polynomial in $T[x_1, \ldots, x_n]$ whose coefficients are the image in T by q of the coefficients of f, i.e. the image of f through the morphism $q_* : R[x_1, \ldots, x_n] \to T[x_1, \ldots, x_n] = R[x_1, \ldots, x_n] \otimes_R T$. Now take $R\langle X, S\rangle$: by an abuse of notation we denote by $q_*(S)$ the set of relations $\{q_*(s_i) = 0\}$ on the set X. In Lemma 2.6 we observe that the pushout is stable under base change.

Lemma 2.6. *Let $q : R \to T$ be an R-commutative algebra and $R\langle X; S\rangle$ any R-algebra, then*

1. $R\langle X; S\rangle \otimes_R T \simeq T\langle X; q_*(S)\rangle$,
2. $(B *_A C) \otimes_R T \simeq (B \otimes_R T) *_{(A\otimes_R T)} (C \otimes_R T)$.

Proof. As a commutative R-algebra, T is isomorphic to $R[\{y_i\}]/(\{f_r\})$ where $\{y_i\}$ is a set of generators and $\{f_r\}$ a set of polynomials in the variables $\{y_i\}$ with coefficients in R. So by Lemma 2.5 $R\langle X; S\rangle \otimes_R T$ is isomorphic to $R\langle X \cup \{y_i\}; S \cup \{f_r = 0\} \cup \{y_i y_j = y_j y_i\} \cup \{xy_i = y_i x\}_{x\in X}\rangle$ which is isomorphic to $R\langle X \cup \{y_i\}; q_*(S) \cup \{f_r = 0\} \cup \{y_i y_j = y_j y_i\} \cup \{xy_i = y_i x\}_{x\in X}\rangle$ and the latter is isomorphic to $T\langle X; q_*(S)\rangle$ since T commutes with X and this proves 1. Let us describe A, B and C as $R\langle X_0; S_0\rangle$, $R\langle X_1; S_1\rangle$ and $R\langle X_2; S_2\rangle$ respectively. As a consequence of point 1 we have $A \otimes_R T \simeq T\langle X_0; q_*(S_0)\rangle$, $B \otimes_R T \simeq T\langle X_1; q_*(S_1)\rangle$, $C \otimes_R T \simeq T\langle X_2; q_*(S_2)\rangle$ and $(B *_A C) \otimes_R T \simeq T\langle X_1 \cup X_2; q_*(S_1 \cup S_2 \cup S_3)\rangle$ where S_3 is as described in (3). But $(B \otimes_R T) *_{(A\otimes_R T)} (C \otimes_R T)$ is also isomorphic to the latter which enables us to conclude. □

Notation 2.7. When R is a Dedekind ring and M an R-module, let us denote by $q : M \twoheadrightarrow F(M)$ the unique quotient (cf. [5], Lemme (2.8.1.1)) of M which is R-flat and such the induced map $q_K : M_K \to F(M)_K$ is an isomorphism.

Let us analyze a few examples whose importance will be clear in the following sections:

Example 2.8. Let $f : A \to B$ and $g : A \to R$ be morphisms of R-algebras then the canonical morphism $\varphi : B *_A R \to B \star_A R$ is an isomorphism. Indeed we observe that $B *_R R = B$ and that the canonical morphisms $u : B \to B *_R R$ and $v : R \to B *_R R$ are nothing else but Id_B and the unit morphism u_B respectively, then for any $b \in B$ and any $r \in R$ we have $u(b)v(r) = v(r)u(b)$. Hence denoting by $u' : B \to B *_A R$ and $v' : R \to B *_A R$ the canonical morphisms we also have $u'(b)v'(r) = v'(r)u'(b)$ as $u' = \lambda u$ and $v' = \lambda v$ where $\lambda : B *_R R \to B *_A R$ is the universal morphism. Then φ can be inverted according to Proposition 2.4. Observe that $B *_A R$ is finite as an R-module if B is finite (it is indeed a quotient of B).

Example 2.9. Let R be a discrete valuation ring with uniformising element π. Let us fix a positive integer p and let us set $A := R[x]/x^p$, $B := R[y]/y^p$ and $C := R[z]/z^p$ (thus commutative R-algebras). Consider the morphisms $f : A \to B, x \mapsto \pi^n y$ and $g : A \to C, x \mapsto \pi^m z$ where $m > n > 0$ are integers. Then $B *_A C = R\langle y, z; y^p = z^p = 0, \pi^m z = \pi^n y\rangle$. Observe that, as an R-module, $B *_A C$ is not flat as $\pi^n(\pi^{m-n}z - y) = 0$ thus $\pi^{m-n}z - y$ is an R-torsion element. However if we add the relation $\pi^{m-n}z = y$ then we eliminate torsion from $B *_A C$ and what we obtain is (cf. Notation 2.7) $F(B *_A C) = R\langle y, z; y^p = z^p = 0, \pi^{m-n}z = y\rangle = R[z]/z^p$ thus finitely generated and flat and, in this particular case, it is isomorphic to $F(B \otimes_A C)$.

The following well-known result will be used several times in this paper:

Theorem 2.10. *Let R be a complete discrete valuation ring with fraction field K and residue field k. Let M be a torsion-free R-module of finite rank r (i.e. $r := dim_K(M \otimes_R K) < +\infty$). Then $M \simeq_{R\text{-mod}} K^{\oplus r-s} \oplus R^{\oplus s}$, where $s = dim_k(M \otimes_R k)$.*

Proof. This is [6], Chapter 16, Corollary 2, □

Theorem 2.10 is not true when R is not complete (cf. [6], Theorem 19) and this is why we will often need to restrict to complete discrete valuation rings. The following lemma is crucial in this paper:

Lemma 2.11. *Let R be a complete discrete valuation ring and assume that $f : A \to B$ and $g : A \to C$ are R-algebra morphisms where furthermore $g_K : A_K \to C_K$ is an isomorphism. Then if A, B and C are finitely generated and flat as R-modules then the same holds for $F(B *_A C)$. Moreover the canonical R-algebra morphism $B \to F(B *_A C)$ induces an isomorphism $B_K \to F(B *_A C)_K$.*

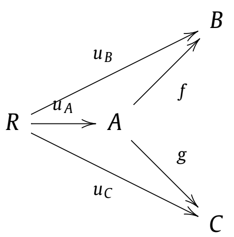
Proof. Let π be an uniformising element of R and let K and k be the fraction and residue fields respectively. As usual let us present by $R\langle X_0; S_0\rangle$, $R\langle X_1; S_1\rangle$ and $R\langle X_2; S_2\rangle$ respectively the R-

algebras A, B and C where for X_0, X_1, X_2 we take respectively bases of A, B, C as R-modules minus the identity elements so that the cardinality x_0, x_1, x_2 of those sets is the rank of A, B and C (respectively) minus one; of course $x_0 = x_2$. Now $B *_A C = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3\rangle$ where S_3 is as described in (3). In particular S_3 is a set made of x_0 R-linear relations relating the x_2 elements of X_2 and the x_1 elements of X_1. As in Example 2.9 the information of being R-torsion (if any) is contained in the set S_3, so if we want to cut out R-torsion we need to add another set of x_0 relations S_4 obtained as follows: for each relation $(s = 0) \in S_3$ add the relation $(t = 0)$ to S_4 where $s = \pi^v t$ and t has at least one coefficient equal to an invertible element of R. Thus $F(B *_A C) = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3 \cup S_4\rangle$. But since relations in S_3 are automatically satisfied if we add S_4 then $F(B *_A C) = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_4\rangle$. Now Lemma 2.6, point 1, implies that $F(B *_A C) \otimes_R k = k\langle X_1 \cup X_2; q_*(S_1 \cup S_2 \cup S_4)\rangle$ where $q : R \twoheadrightarrow k$ is the canonical surjection so $F(B *_A C) \otimes_R k$ is the quotient of $k\langle X_1 \cup X_2; q_*(S_1 \cup S_2)\rangle$ by the two-sided ideal generated by the relations $q_*(S_4)$. But in $k\langle X_1 \cup X_2; q_*(S_1 \cup S_2)\rangle$ the elements of the set $X_1 \cup X_2$ are $x_1 + x_2$ k-linearly independent vectors then if we add the $x_0 = x_2$ k-linear relations $q_*(S_4)$ what remains is a set of at least $x_1 = rk(B) - 1$ k-linearly independent elements which become $rk(B)$ if we add 1_B. Combining this with Theorem 2.10 we obtain that $F(B *_A C)$ is a finitely generated R-free module, as required, as $dim_k(F(B *_A C)) \otimes_R k = dim_K(F(B *_A C)) \otimes_R K$. The last assertion follows easily from Lemma 2.6, point 2. □

Remark 2.12. The construction in Lemma 2.11 does not depend on A. That means that if we take A', $f' : A' \to B$ and $g' : A' \to C$ satisfying similar assumptions then $F(B *_A C) \simeq F(B *_{A'} C)$. Indeed, again by [5], Lemme 2.8.1.1, we observe that $F(B *_A C)$ is isomorphic to the unique quotient of $B *_R C$ which is R-flat and whose tensor over K gives B_K; but the same property is satisfied by $F(B *_{A'} C)$ hence we conclude by unicity.

Proposition 2.13. *Let R be any commutative ring. Then the pushout in the category of R-bialgebras exists.*

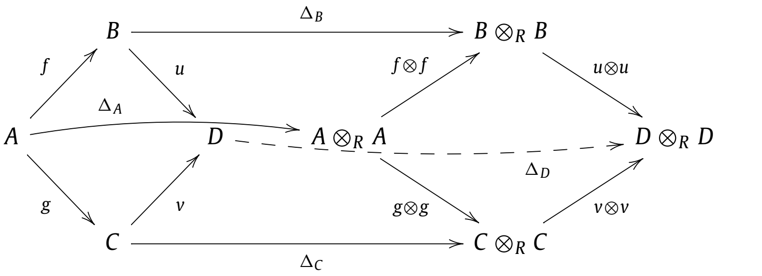
Proof. We follow[3] [7], Chapitre 5, §5.1, Proposition. Consider the diagram



(4)

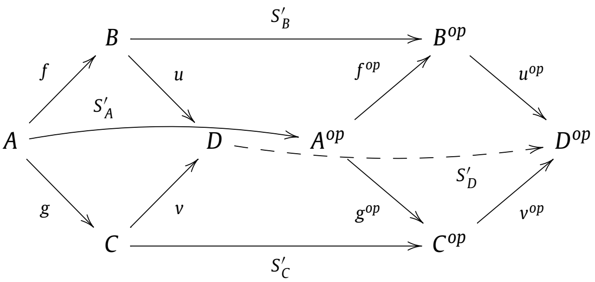
where we assume that A, B and C are R-bialgebras and the arrows are R-Hopf algebra morphisms. Let $D := B *_A C$ be the pushout of the diagram in the category of R-algebras and let m_D and u_D be respectively the multiplication and the unit morphism. Then we need to provide D with a comultiplication Δ_D and a counit ε_D such that $(D, m_D, u_D, \Delta_D, \varepsilon_D)$ is an R-bialgebra. We describe how to construct Δ_D, the costruction of ε_D being easier. The rest will be standard verification over complicated diagrams. The existence of Δ_D is explained in the following diagram, taking into account the universal property of D:

[3] In [7], however, Lemaire uses different notations.



□

Remark 2.14. Notation being as in Proposition 2.13 one observes that we can define an R-module morphism $S_D : D \to D$, candidate to be a coinverse, as follows: first construct the opposite algebras A^{op}, B^{op}, C^{op}, D^{op}, the opposite morphisms f^{op}, g^{op}, u^{op}, v^{op} and the morphisms of R-algebras S'_A, S'_B, S'_C, induced by the R-algebra anti-morphisms S_A, S_B, S_C. Then the existence of S'_D follows from the following diagram



exploiting the universal property of D then S_D is the anti-morphism induced by S'_D. However S_D may fail to be a coinverse for D as $m_D \circ (S_D \otimes id_D) \circ m_D$ may not be equal to $u_D \circ \varepsilon_D$ (same for $m_D \circ (id_D \otimes S_D) \circ m_D$).

In order to have an explicit description for S_D, constructed in Remark 2.14, set, as usual, $A = R\langle X_0; S_0\rangle$, $A = R\langle X_1; S_2\rangle$ and $C = R\langle X_2; S_2\rangle$ so $D = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3\rangle$ where S_3 is as described in (3); it is sufficient to set $S_D(x_1) := S_B(x_1)$ for any $x_1 \in X_1$, $S_D(x_2) := S_C(x_2)$ for any $x_2 \in X_2$ for any $x_1 \in X_1$, $S_D(x_1x_2) := S_D(x_2)S_D(x_1)$ and $S_D(x_2x_1) := S_D(x_1)S_D(x_2)$. It is well defined and is by construction an anti-isomorphism for the R-algebra D. A similar construction gives an explicit description of Δ_D, taking into account that Δ_D is a morphism of R-algebras and not an anti-morphism.

Corollary 2.15. *Let R be a complete discrete valuation ring and assume that $f : A \to B$ and $g : A \to C$ are R-algebra morphisms where furthermore $g_K : A_K \to C_K$ is an isomorphism. Then $F(B *_A C)$ has a natural structure of R-Hopf algebra. If moreover A, B and C are finitely generated and flat as R-modules then $F(B *_A C)$ is the pushout of B and C over A in R-$\mathcal{H}opf_{ff}$.*

Proof. By [5], (2.8.3) and of course Proposition 2.13 we obtain that $F(B *_A C)$ has a natural structure of R-bialgebra. We need to prove the existence of a coinverse $S_{F(B*_AC)}$ that gives $F(B *_A C)$ a natural structure of R-Hopf algebra. So let us take for $D := B *_A C$ the R-module morphism S_D defined in Remark 2.14. This morphism induces (by [5], Lemme 2.8.3) an R-module morphism $S_{F(D)} : F(D) \to F(D)$ which is the required coinverse: indeed

$$m_{F(D)} \circ (S_{F(D)} \otimes id_{F(D)}) \circ m_{F(D)} - u_D \circ \Delta_D \tag{5}$$

is the zero map 0_D and this is clear since $F(D) \subset B_K$ and (5) tensored over K gives rise to the equality

$$m_{B_K} \circ (S_{B_K} \otimes id_{B_K}) \circ m_{B_K} = u_{B_K} \circ \Delta_{B_K}$$

which holds as B_K is a K-Hopf algebra. The same is true for $m_{F(D)} \circ (id_{F(D)} \otimes S_{F(D)}) \circ m_{F(D)}$. Finally $F(B *_A C)$ is finitely generated and flat as an R-module when A, B and C are: this is Lemma 2.11. □

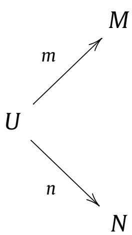
Remark 2.16. Notation being as in Proposition 2.13, we observe that $B *_A C$ is cocommutative if A, B, C are. The same conclusion holds, then, for $F(B *_A C)$ in Corollary 2.15. Moreover observe that if A, B, C are commutative then $F(B *_A C)$ is commutative too since it is contained in B_K. So in particular in this case $F(B *_A C) \simeq F(B \otimes_A C)$, as it happened in Example 2.9.

3. Pushout of group schemes

In this section R is any complete discrete valuation ring with fraction and residue fields respectively denoted by K and k.

3.1. The finite case

Let $M = Spec(B)$ and $N = Spec(C)$ be finite and flat R-group schemes, i.e. B and C are free over R and finitely generated as R-modules. Let us assume that there is a K-group scheme morphism $\psi : M_K \to N_K$. An *upper bound* for M and N is a finite and flat R-group scheme U, provided with a model map $U \to M$ and an R-group scheme morphism $\varphi : U \to N$ which generically coincides with $\psi : M_K \to N_K$. A *lower bound* for M and N is a finite and flat R-group scheme L, provided with a model map $N \to L$ and an R-group scheme morphism $\delta : M \to L$ which generically coincides with $\psi : M_K \to N_K$. The construction of an upper bound is easy: it is sufficient to set U as the schematic closure of M_K in $M \times N$ through the canonical closed immersion $M_K \hookrightarrow M_K \times N_K$ (and this holds when the base is any Dedekind scheme). Now consider the following commutative diagram

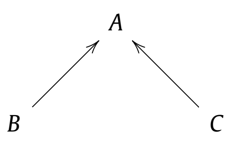


(6)

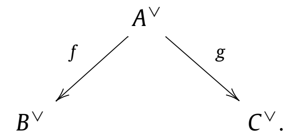
where $U = Spec(A)$ is any upper bound. We are now going to study the existence of a pushout $M \sqcup_U N$ in the category of finite and flat R-group schemes. We prove the following

Lemma 3.1. *The pushout of* (6) *in the category of finite and flat R-group schemes exists. Moreover $M \sqcup_U N$ is a lower bound for M and N.*

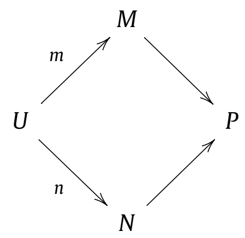
Proof. Notation being as in the beginning of this section, we have the following diagram of commutative R-Hopf algebras:



which, dualizing, gives rise to the following diagram of cocommutative (but possibly non-commutative) R-Hopf algebras:



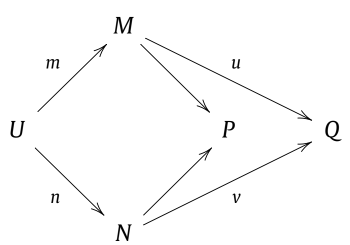
Let us consider the cocommutative R-Hopf algebra $F(B^\vee *_{A^\vee} C^\vee)$ constructed in Corollary 2.15. Now we take the spectrum of its dual $P := Spec(F(B^\vee *_{A^\vee} C^\vee)^\vee)$. First of all we observe that $F(B^\vee *_{A^\vee} C^\vee)^\vee$ is commutative as $F(B^\vee *_{A^\vee} C^\vee)$ is cocommutative so that taking its spectrum does make sense. It remains to prove that the commutative diagram



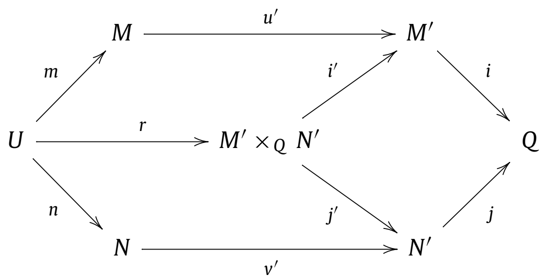
is in fact a pushout in the category of finite and flat R-group schemes. But this follows from the fact that $F(B^\vee *_{A^\vee} C^\vee)$ is a pushout in R-$\mathcal{H}opf_{ff}$. That $M \sqcup_U N := P$ is a lower bound for M and N is also clear by construction. □

Theorem 3.2. *The pushout of* (6) *in the category of affine R-group schemes exists.*

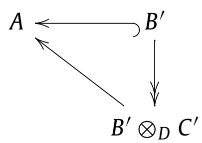
Proof. Consider the commutative diagram



where P is the pushout of (6) in the category of finite and flat R-group schemes constructed in Lemma 3.1 and $Q = Spec(D)$ is any affine R-group scheme. We are going to show that P is also the pushout of (6) in the category of affine R-group schemes. Let us factor u through $M' := Spec(B')$ via the morphisms $u' : M \to M'$ and $i : M' \to Q$ where i is a closed immersion and u' is a schematically dominant morphism (i.e. the induced morphism $B' \to B$ is injective) so that M' is a finite and flat R-group scheme since M is. Likewise we factor v through the finite and flat R-group scheme $N' := Spec(C')$ via the schematically dominant morphism $v' : N \to N'$ and the closed immersion $j : N' \to Q$. Now consider the finite R-group scheme (it needs not be flat a priori) $M' \times_Q N'$ and the natural closed immersions $i' : M' \times_Q N' \hookrightarrow M'$ and $j' : M' \times_Q N' \hookrightarrow N'$. So let us denote by $r : U \to M' \times_Q N'$ the universal morphism, then we have the following commutative diagram



and in particular we have the following commutative diagram of R-algebras



where $B' \to B' \otimes_D C'$ is surjective but also injective since $B' \hookrightarrow A$ is (recall that $m : U \to M$ is a model map). Hence i' is an isomorphism so there exists a universal morphism $P \to N'$ by Lemma 3.1 and we conclude. $\square$

Remark 3.3. If U' is any other upper bound for M and N then there is a canonical isomorphism $M \sqcup_U N \simeq M \sqcup_{U'} N$: this is a consequence of Remark 2.12. However this does not mean that the lower bound for M and N is unique, which is clearly not true in general.

3.2. The quasi-finite case

Let $M = Spec(B)$ and $N = Spec(C)$ be quasi-finite (by this we will always mean affine and of finite type over R, with finite special and generic fibers) and flat R-group schemes. It is known (see [1], §7.3, p. 179) that any quasi-finite R-group scheme H has a finite part H_f, that is an open and closed subscheme of H which consists of the special fibre H_k and of all points of the generic fibre which specialize to the special fibre. It is thus flat over R if H is.

Remark 3.4. If $H = Spec(A)$ is a quasi-finite and flat R-group scheme then its finite part coincides with $Spec(A^{\vee\vee})$ where $A^{\vee\vee}$ is the double dual of A: this follows from the fact that $A \simeq_{R\text{-}mod} K^{\oplus t} \oplus R^{\oplus s}$ (cf. Theorem 2.10) so $A^{\vee} \simeq_{R\text{-}mod} R^{\oplus s}$ is an R-Hopf algebra and not just an R-algebra. Hence the canonical surjection $A \twoheadrightarrow A^{\vee\vee}$ gives the desired closed immersion $H_f \hookrightarrow H$ of group schemes. However this fact will not be necessary in the remainder of this paper.

Let us assume that there is a K-group scheme morphism $\psi : M_K \to N_K$. We define upper and lower bounds exactly as in the finite case. One can easily construct an upper bound U for M and N simply proceeding as in Section 3.1. So U will be in general a quasi-finite and flat R-group scheme. For the lower bound it will be a little bit more complicated. So consider again the commutative diagram (6) where $U = Spec(A)$ is any upper bound. We are going to study the existence of a pushout $M \sqcup_U N$ in the category of affine R-group schemes. We prove the following

Theorem 3.5. *Assume that N_K admits a finite and flat model over R. Then the pushout of (6) in the category of affine R-group schemes exists. Moreover $M \sqcup_U N$ is a lower bound for M and N.*

Proof. Let N' denote a finite and flat R-model for N_K, i.e. a finite and flat R-group scheme whose generic fibre is isomorphic to N_K. Consider the finite part M_f and N_f of, respectively, M and N.

Compose the closed immersion $M_{f,K} \to M_K$ with $\psi_K : M_K \to N_K$ thus obtaining a morphism $M_{f,K} \to N_K$. By Theorem 3.2 we construct a lower bound L_1 for M_f and N', which is finite and flat over R, generically isomorphic to N_K, then it is already a lower bound for M and N'. Considering the closed immersion $N_{f,K} \to N_K$ we also construct a lower bound L_2 for N_f and N', which is finite and flat over R, generically isomorphic to N_K, then it is already a lower bound for N and N'. So a lower bound L for L_1 and L_2 (which are generically isomorphic) exists by Theorem 3.2 and is also a lower bound for M and N. We still need to compute the pushout of M and N over U: let us set $U_f := Spec(A_f)$, $M_f := Spec(B_f)$, $N_f := Spec(C_f)$ and $L := Spec(D)$. Consider the natural R-bialgebra morphism (cf. Proposition 2.13) $B_f^\vee *_{A_f^\vee} C_f^\vee \to D^\vee$ and factor it as follows

$$B_f^\vee *_{A_f^\vee} C_f^\vee \twoheadrightarrow E \hookrightarrow D^\vee$$

where E is a cocommutative R-bialgebra which is flat and finitely generated as an R-module because $D^\vee$ is. Consider the morphism $S_{B_f^\vee *_{A_f^\vee} C_f^\vee} : B_f^\vee *_{A_f^\vee} C_f^\vee \to B_f^\vee *_{A_f^\vee} C_f^\vee$ constructed in Remark 2.14; the commutative diagram

$$\begin{array}{ccccc} B_f^\vee *_{A_f^\vee} C_f^\vee & \twoheadrightarrow & E & \hookrightarrow & D^\vee \\ \downarrow{\scriptstyle S_{B_f^\vee *_{A_f^\vee} C_f^\vee}} & & & & \downarrow{\scriptstyle S_{D^\vee}} \\ B_f^\vee *_{A_f^\vee} C_f^\vee & \twoheadrightarrow & E & \hookrightarrow & D^\vee \end{array}$$

induces an anti-morphism of R-algebras

$$S_E : E \to E$$

which gives E a natural structure of R-Hopf algebra: indeed $m_E \circ (id_E \otimes S_E) \circ m_E = u_E \circ \varepsilon_E$ and $m_E \circ (S_E \otimes id_E) \circ m_E = u_E \circ \varepsilon_E$ since the same equalities hold for $D^\vee$. It is now sufficient to take the union of $Spec(E^\vee)$ and $N_K \simeq L_K$ in order to construct a quasi-finite and flat R-group scheme P which is certainly a pushout in the category of quasi-finite and flat R-group schemes. Arguing as in the proof of Theorem 3.2 we can deduce that P is also a pushout in the category of affine R-group schemes. □

Remark 3.6. As in Remark 3.3 one observes that if U' is any other upper bound for M and N then there is a canonical isomorphism $M \sqcup_U N \simeq M \sqcup_{U'} N$: indeed E, as constructed in the proof, is the only quotient of $B_f^\vee *_R C_f^\vee$, R-flat which over K gives $B_{f\,K}^\vee *_K C_{f\,K}^\vee \to D^\vee{}_K$ and this does not depend on A_f. The same will hold for Corollary 3.7 and will be used in Corollary 3.9.

Corollary 3.7. *When N_K is étale then after possibly a finite extension of scalars the pushout of (6) in the category of affine R-group schemes exists. Again $M \sqcup_U N$ is a lower bound for M and N.*

Proof. Clear since after possibly a finite extension K' of K the K-group scheme N_K becomes constant then it certainly admits a finite, constant (so flat) model over R', the integral closure of R in K'. □

Let K' be a finite extension of K and R' the integral closure of R in K' then R' is a complete discrete valuation ring. Assume that W is a torsion-free R module of finite rank n then we have the following

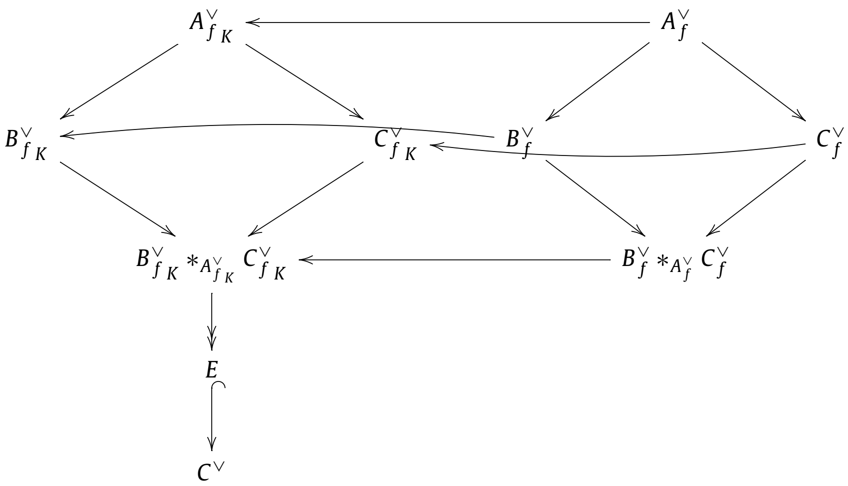
Lemma 3.8. *If $W \otimes_R R'$ is finitely generated as an R'-module then W is finitely generated as an R-module too (thus free).*

Proof. By Theorem 2.10 $W \simeq_{R\text{-}mod} K^{\oplus n-s} \oplus R^{\oplus s}$, where $s = dim_k(W_k)$, hence $W \otimes_R R' \simeq_{R'\text{-}mod} K'^{\oplus n-s} \oplus R'^{\oplus s}$ so if $W \otimes_R R'$ is finitely generated as an R'-module then $n-s=0$ and we conclude. $\square$

This will be used in the following

Corollary 3.9. *When R is complete and N_K is étale then the pushout of (6) in the category of affine R-group schemes exists. Again $M \sqcup_U N$ is a lower bound for M and N.*

Proof. Again $U_f = Spec(A_f)$, $M_f = Spec(B_f)$, $N_f = Spec(C_f)$ will denote the finite part of U, M and N respectively. Let us consider the duals $A_f^\vee = A^{\vee\vee}$, $B_f^\vee = B^{\vee\vee}$ and $C_f^\vee = C^{\vee\vee}$ and the commutative diagram



(7)

where E comes from the factorization of the universal morphism $B^\vee_{f\,K} *_{A^\vee_{f\,K}} C^\vee_{f\,K} \to C^\vee$. Arguing as in Theorem 3.5 we provide it with a natural structure of K-Hopf algebra. Using again [5], Lemme (2.8.1.1) we construct the unique quotient

$$B^\vee_f *_{A^\vee_f} C^\vee_f \twoheadrightarrow E'$$

which is R-flat and which generically gives

$$B^\vee_{f\,K} *_{A^\vee_{f\,K}} C^\vee_{f\,K} \twoheadrightarrow E.$$

Thus E' has naturally a structure of a cocommutative R-Hopf algebra: indeed it inherits from $B^\vee_f *_{A^\vee_f} C^\vee_f$ a cocommutative R-coalgebra structure and by means of [5], (2.8.3) an anti-morphism of R-algebras $S'_E : E' \to E'$ which is a coinverse since tensoring it over K we obtain $S_E : E \to E$ which is a coinverse for E. If we prove that E' is finitely generated as an R-module then $Spec(E'^\vee)$ glued to N is the desired pushout. So now it remains to prove that E' is finitely generated as an R-module: let $K \to K'$ be a finite field extension such that $N_{K'}$ admits a finite and flat model over R', the integral closure of R in K'. Then by Corollary 3.7 and Remark 3.6 $E' \otimes_R R'$ is R-finite and flat. Lemma 3.8 implies that E' is R-finite and flat too. $\square$

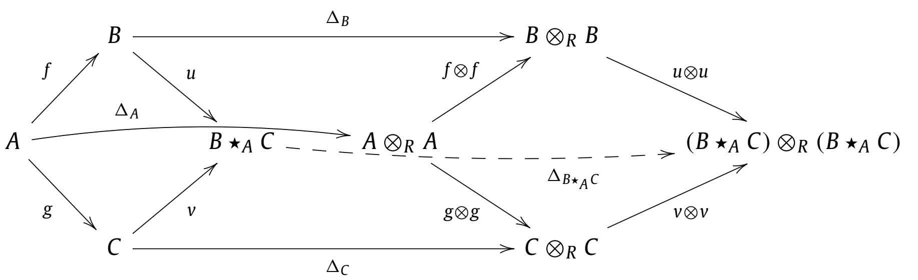
Remark 3.10. It is less elegant but still true that Corollary 3.9 holds for all those N_K that admits a finite and flat R-model after possibly a finite extension of scalars and étale ones are just a particular case. Observe furthermore that, following the proof, in the situation of both Theorem 3.5 and Corollary 3.7 one can find a finite and flat lower bound for M and N. This can be false in the situation of Corollary 3.9.

3.3. Cokernels and quotients

In a category $\mathcal{C}$ with zero object $0_{\mathcal{C}}$ (that is an object which is both initial and final), we can define the cokernel of a morphism $f : A \to B$ (see for instance [8], III, §3) which turns out to be the pushout $0_{\mathcal{C}} \sqcup_A B$ of the obvious diagram. As explained in the introduction in this section we are going to describe, in Proposition 3.12, a new and easy proof for a well-known result. First we need a lemma:

Lemma 3.11. *Let R be a Dedekind ring or a field, A, B and C R-Hopf algebras provided with R-Hopf algebra morphisms $f : A \to B$ and $g : A \to C$. Then the star product $B \star_A C$ defined in Definition 2.3 has a natural structure of R-Hopf algebra.*

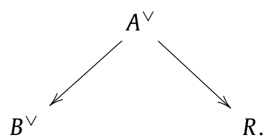
Proof. First we prove the existence of the comultiplication $\Delta_{B\star_A C}$: it is sufficient to consider the following diagram



where the existence of $\Delta_{B\star_A C}$ is ensured by Proposition 2.4. The existence of $\varepsilon_{B\star_A C}$ is easier and an argument similar to the one used in Remark 2.14 ensures the existence of an anti-morphism $S_{B\star_A C} : B \star_A C \to B \star_A C$ which is compatible with $S_{B\otimes_R C}$, i.e. if $\lambda : B \otimes_R C \twoheadrightarrow B \star_A C$ denotes the canonical projection then $\lambda \circ S_{B\otimes_R C} = S_{B\star_A C} \circ \lambda$. From this we deduce that $S_{B\star_A C}$ is the desired coinverse for $B \star_A C$. $\square$

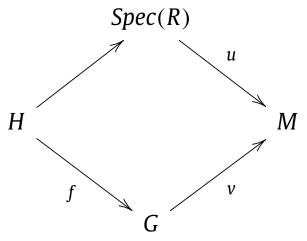
Proposition 3.12. *Let R be a Dedekind ring or a field, G and H two finite and flat R-group schemes and $f : H \to G$ a morphism of R-group schemes. Then the cokernel of f exists in the category of R-affine group schemes.*

Proof. The zero object in the category of R-affine group schemes is $Spec(R)$. Let us set $H = Spec(A)$ and $G = Spec(B)$. Then we first compute the pushout in the category of R-Hopf algebras of the diagram



(8)

In Example 2.8 we have observed that $W := B^\vee *_{A^\vee} R \simeq B^\vee \star_{A^\vee} R$ canonically. That W has a natural structure of R-Hopf algebra follows from Lemma 3.11. If R is a Dedekind ring and W is not flat then we consider $F(W)$ (cf. Notation 2.7) which is flat and finitely generated and inherits the R-Hopf algebra structure. Since the case of a field is similar let us just consider the case of a Dedekind ring R: we are now going to prove that $Spec(F(W)^\vee)$ is the desired pushout. So let M be any affine R-group scheme, $v : G \to M$ an R-group scheme morphism and $u : Spec(R) \to M$ the natural inclusion (the unity map). Let us assume we have a commutative diagram



(9)

Observe that we can assume M to be finite and flat, for if it is not we can factor v through a finite and flat (since G is) R-group scheme that makes a diagram similar to (9) commmute. When M is finite and flat it is easy to construct a universal morphism $Spec(F(W)^\vee) \to M$ since $F(W)$ is easily seen to be the pushout of diagram (8) in R-$\mathcal{H}opf_{ff}$. □

Corollary 3.13. *Let R be a Dedekind ring, G and H two finite and flat R-group schemes with H a closed and normal R-subgroup scheme of G. Then the quotient G/H exists in the category of R-affine group schemes.*

Proof. This follows directly from Proposition 3.12 where we take for $f : H \to G$ the given closed immersion. □

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