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Stability of planar shock fronts for multidimensional systems of relaxation equations

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ABSTRACT

We investigate stability of multidimensional planar shock profiles of a general hyperbolic relaxation system whose equilibrium model is a system, under the necessary assumption of spectral stability and a standard set of structural conditions that are known to hold for many physical systems. Our main result, generalizing the work of Kwon and Zumbrun in the scalar relaxation case, is to establish the bounds on the Green's function for the linearized equation and obtain nonlinear L^2 asymptotic behavior/sharp decay rate of perturbed weak shock profiles. To establish Green's function bounds, we use the semigroup approach in the low-frequency regime, and use the energy method for the high-frequency bounds, separately. For the system equilibrium case, the analysis of the linearized equation is complicated due to glancing phenomena. We treat this difficulty similarly as in the inviscid and viscous systems, under the constant multiplicity condition.

1. Introduction

In this paper, we study stability and the large-time behavior of multidimensional planar shocks of *hyperbolic relaxation systems* of general form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \sum_{j=1}^{d} \begin{pmatrix} f^j(u,v) \\ g^j(u,v) \end{pmatrix}_{x_j} = \begin{pmatrix} 0 \\ \tau^{-1} q(u,v) \end{pmatrix}, \tag{1.1}$$

where $u, f^j \in \mathbb{R}^n$, $v, g^j, q \in \mathbb{R}^r$, with the condition

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$$\operatorname{Re}\sigma\big(q_v\big(u, v^*(u)\big)\big) < 0, \tag{1.2}$$

along a smooth *equilibrium manifold* $v = v^*(u)$ defined by

$$E := \big\{(u, v) \in \mathbb{R}^{n+r} \,\big|\, q(u, v) = 0\big\}, \tag{1.3}$$

and $\tau > 0$ is a (typically small) parameter determining relaxation time. The first n equations and the second r equations represent a conservation law for u and relaxation rate equations for v, respectively.

The relaxation phenomenon arises in many physical situations. It is present in nonequilibrium gas dynamics, river flows, traffic flows, viscoelasticity with vanishing memory, phase transitions with small transition time, and kinetic theory of gases. Originally, the word "relaxation" was used to describe certain thermodynamic nonequilibrium phenomena in gas dynamics. When an equilibrated physical system is perturbed, for example, by a sudden change in temperature or pressure, the system tends to re-equilibrate for the new conditions in the adjustment of the rotational and vibrational energy. In our general system (1.1), the condition (1.2) implies that during this re-equilibrium process a perturbed solution eventually relaxes to the equilibrium state. Besides these physical models, some hyperbolic relaxation systems have been studied in a numerics point of view. The Jin–Xin model was proposed in [17] as a numerical scheme approximating discontinuous solutions of the corresponding equilibrium system.

An interesting phenomenon in the study of hyperbolic relaxation systems is the existence of smooth traveling wave solutions satisfying

$$\begin{aligned} &(u, v)(x, t) = (\bar{u}, \bar{v})(x_1 - at),\\ &\lim_{z\to\pm\infty}(\bar{u}, \bar{v})(z) = (u_\pm, v_\pm), \end{aligned} \tag{1.4}$$

where the end states $(u_\pm, v_\pm)$, necessarily satisfy $v^*(u_\pm) = v_\pm$, and $u_\pm$ is a shock solution of the corresponding equilibrium system:

$$u_t + \sum_{j=1}^{d} df^{j,*}(u)u_{x_j} = 0. \tag{1.5}$$

Here and after we denote

$$df^{j,*}(u) := \partial_u\big(f^j\big(u, v^*(u)\big)\big) = f_u^j\big(u, v^*(u)\big) - f_v^j\big(u, v^*(u)\big)q_v^{-1}\big(u, v^*(u)\big)q_u\big(u, v^*(u)\big). \tag{1.6}$$

By the classical Chapman–Enskog expansion, the equilibrium system for the conservation laws variable u can be further approximated, at a formal level, to its first order with respect to the relaxation time parameter τ as follows:

$$u_t + \sum_{k=1}^{d} df^{k,*}u_{x_k} = \tau \sum_{j,k=1}^{d}\big(B_{jk}^* u_{x_j}\big)_{x_k}, \tag{1.7}$$

where $B_{jk}^* = -f_v^k q_v^{-1}(g_u^j - g_v^j q_v^{-1} q_u + q_v^{-1} q_u (f_u^j - f_v^j q_v^{-1} q_u))$ is called the Chapman–Enskog viscosity. For hyperbolic–parabolic systems, the existence and stability of smooth traveling wave solutions have been studied by many authors. In light of the formal approximation, it is natural to ask if we can obtain similar results for hyperbolic relaxation systems. The existence of such traveling solutions is known for small amplitude profiles in several different contexts, see for example, [22,31,23,7].

However, profiles of large amplitude may develop "subshocks" or jump discontinuities. We restrict here to the smooth and small-amplitude case.

In the study of hyperbolic relaxation systems, the stability of the solution $(\bar{u}, \bar{v})$ of (1.1)–(1.4) has been investigated with respect to several different notions of stability. For example, one is the stability as the relaxation time $\tau \to 0$, the so-called "zero relaxation limit" problem, see [4,5] and references therein. Another one which we are concerned with in the present work is the time-asymptotic stability in the sense that a perturbed solution nearby the shock wave solution remains close in an appropriate norm as $t \to \infty$. For this purpose, we set, without loss of generality, $\tau = 1$, and we study the time-asymptotic stability.

The stability of traveling waves for a general 2×2 quasilinear relaxation system in one spatial dimension was initially studied by T.-P. Liu in his seminal work [22]. In his paper, nonlinear stability was established, under the main stability condition, the so-called "subcharacteristic" condition. This is a crucial condition implying that the system is dissipative. His result implies that the formal Chapman–Enskog approximation has a justifiable interpretation in terms of the large-time behavior of the solutions. Another fundamental example is the Jin–Xin model introduced as a numerical scheme approximating discontinuous solutions of the corresponding equilibrium system by S. Jin and Z. Xin. In [17] the stability for this 2×2 system in one spatial dimension with the linear transport term was obtained by showing L^1 contraction property. These two models served as a guideline to the fundamental ideas in the study of general relaxation systems. Later, Mascia and Zumbrun [23] showed the nonlinear stability for general $N \times N$ relaxation systems in one spatial dimension, under the necessary assumption of spectral stability, which was verified using singular perturbation argument in [28]. They applied the pointwise semigroup approach introduced by Zumbrun and Howard [36] to the relaxation problems. Establishing the Green function bounds together with shock tracking method, they proved nonlinear stability with sharp decay rate.

Loosely following [23], the stability of multidimensional planar shocks of the general relaxation system whose equilibrium model is scalar (i.e. $n = 1$) was studied in [20]. Under the necessary assumption of spectral stability together with dissipative structural assumption, nonlinear L^2 asymptotic behavior with sharp decay rate of perturbed weak shock was obtained. There are several nonlinear stability results for specific multidimensional scalar relaxation models. Nonlinear stability of the 3×3 Jin–Xin model in two spatial dimensions and two-dimensional shallow river model were obtained in [15] and [16], respectively. They both rely on the structure of specific models and give only stability without decay rates or the asymptotic behavior, whereas the result in [20] applies to general equations, yielding sharp decay rates. On the other hand, it relies on the assumption of spectral stability, which needs to be verified. However, to our knowledge, no stability result on the multidimensional general relaxation system whose equilibrium model is a system is obtained. System equilibrium cases include many important physical examples, such as gas dynamics in thermo-nonequilibrium as in [32], moment closure and discrete kinetic models obtained from Boltzman equation, and so on. For the system case, the linearized estimates are much more complicated due to glancing phenomena; see [6] for the related Kreiss–Majda theory in the inviscid case, and see also [33,34] for the viscous case. A significant difference from the scalar case considered in [20] is that there are more than one slow modes in the system case, in which complicated glancing modes and super-slow modes are present. Our proof is carried out by verifying that we can express these modes in the framework of [34], in which the author considers the similar case of multidimensional planar viscous shock profiles.

In the present paper, generalizing the results of [20] in the scalar relaxation case to the system equilibrium relaxation case, we prove stability with the asymptotic behavior and the decay rates of small-amplitude multidimensional planar shocks of (1.1), under the following assumptions. For notational convenience, we rewrite (1.1) as

$$U_t + \sum_{j=1}^{d} A_j(U) U_{x_j} = \tau^{-1} Q(U), \tag{1.8}$$

where $A_j(U) = (df^j(u,v), dg^j(u,v))^t$ and $Q(U) = (0, q(u,v))^t$.

Assumptions 1.1.

(H0) $f^j, g^j, q \in \mathcal{C}^m$, $m \geqslant [d/2]+3$.
(H1) (i) $\sigma(\sum_j \xi_j A_j(U_\pm))$ real, semi-simple for all $\xi \in \mathbb{R}^d$, and (ii) $\sigma(A_1(U_\pm))$ different from a, the speed of traveling waves.
(H2) (Non-strictly hyperbolic with constant multiplicity) The eigenvalues of the reduced system $\sum_j \xi_j f_u^{j,*}(u_\pm)$ are real, of constant multiplicity with respect to $\xi \in \mathbb{R}^d \setminus \{0\}$ and different from a.
(H3) (Dissipative condition)

$$\operatorname{Re}\sigma\left(-i\sum_{j=1}^{d} i\xi_j\big(df^j, dg^j\big)^t(u_\pm, v_\pm) + (0, dq)^t(u_\pm, v_\pm)\right) \leqslant -\theta|\xi|^2/\big(1+|\xi|^2\big)$$

for all $\xi \in \mathbb{R}^d$, $\theta > 0$.
(H4) The set of traveling wave solutions of (1.1) forms a smooth manifold $(\bar{u}_\delta, \bar{v}_\delta)$, $\delta \in \mathcal{U} \in \mathbb{R}^1$.

In the present work, we restrict our attention to the standard case of a classical, Lax-type shock (U_-, U_+, s). (H3) is guaranteed by our dissipative structural assumptions (A1) and (A2) below. (H2) is a generalized condition of the strictly hyperbolic condition (H2)' that requires all eigenvalues are real and simple for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Let us present here some technical definitions for our further assumptions on the non-strict hyperbolicity on the reduced system. The eigenvalues of $\sum_j \xi_j df^{j,*}(u_\pm)$, denoted by $a_r^\pm(\xi)$, $1 \leqslant r \leqslant n$, are necessarily real, positive homogeneous degree one, and by (H2), locally analytic on $\xi \in \mathbb{R}^d \setminus \{0\}$. Let

$$P_\pm(\tau, \xi) := i\tau + i\sum_j \xi_j\, df^{j,*}(u_\pm) \tag{1.9}$$

denote the frozen-coefficient symbols associated with the first-order conservation law at the equilibrium end states $u = u_\pm$. Then, $\det P_\pm(\tau, \xi)$ has n locally analytic, positive homogeneous degree one roots

$$\tau = -a_r^\pm(\xi), \quad r = 1, \ldots, n, \tag{1.10}$$

describing dispersion relations for the frozen-coefficient equation.

The relations, with a notation $\tilde{\xi} = (\xi_2, \ldots, \xi_d)$,

$$i\xi_1 = \mu_r(\tilde{\xi}, \tau), \quad r = 1, \ldots, n, \tag{1.11}$$

describe roots of $\det R_\pm(\tau, \xi) = 0$, where

$$R_\pm(\tau, \xi) := i\xi_1 + \left(i\tau + \sum_{j\neq 1} i\xi_j\, df^{j,*}\right)\big(df^{1,*}\big)_\pm^{-1}. \tag{1.12}$$

Evidently, graphs (1.10) and (1.11) describe the same sets, since $\det P_\pm = \det R_\pm \det df_\pm^{1,*}$ and $\det df_\pm^{1,*} \neq 0$; the roots $i\tau$ describe characteristic rates of temporal decay, whereas $\mu = i\xi_1$ describe characteristic rates of spatial decay in the x_1 direction.

Definition 1.2. We define the glancing sets $\mathcal{G}(P_\pm)$ as the set of all $(\tilde{\xi}, \tau)$ such that, for some real ξ_1 and $1 \leqslant r \leqslant n$, $\tau = -a_r^\pm(\xi_1, \tilde{\xi})$ and $(\partial a_r^\pm/\partial \xi_1)(\xi_1, \tilde{\xi}) = 0$: that is the projection onto $(\tilde{\xi}, \tau)$ of the set of real roots (ξ, τ) of $\det P_\pm = 0$ at which (1.10) is not analytically invertible as a function (1.11). The roots (ξ, τ) are called glancing points.

Here is an additional assumption on the glancing set.

Assumption 1.3.

(H5) Each glancing set $\mathcal{G}(P_\pm)$ is the union of (possibly intersecting) finitely many smooth curves $\tau = \eta_q^\pm(\tilde{\xi})$ on which the root ξ_1 of $\tau + a_r^\pm(\cdot, \tilde{\xi}) = 0$ has constant multiplicity $s_q \geqslant 2$, defined as the order of the first nonvanishing partial derivative $\partial^s a_r/\partial \xi_1^s$ with respect to ξ_1, i.e., the associate inverse function $\xi_1^q(\tilde{\xi}, \tau)$ has constant degree of singularity s_q.

(H5) is a technical condition introduced in [33] in the context of the hyperbolic–parabolic system. Here we assume this condition on the reduced system (1.9). It is automatic in dimensions $d = 1, 2$ and in any dimension for rotationally invariant problems. In one dimension, the glancing set is empty. In the two-dimensional case, the homogeneity of a_r and its derivatives implies that the ray through $(\tilde{\xi}, \lambda)$ is the graph of $\tau(\tilde{\xi})$ and that (H5) holds there. By the implicit function theorem, (H5) holds also in the case all branch singularities are of square root type, degree $s_q = 2$ with η_q defined implicitly by the requirement $\partial a_q/\partial \xi_1 = 0$. In particular, it holds in the case that all eigenvalues are either linear or else strictly convex/concave in ξ_1 for $\tilde{\xi} \neq 0$. Thus, it holds always for the equations of gas dynamics in all dimensions.

Now let us state our necessary assumption of spectral stability. Let $D(\lambda, \tilde{\xi})$ as in Definition 2.6, Section 2, denote the Evans function associated with Fourier transformed operator $L_{\tilde{\xi}}$ of the linearized operator about the wave $\bar{U}$. Introduced by Evans in the context of nerve impulse equations [8], the Evans function serves as a characteristic function for the operator $L_{\tilde{\xi}}$. We study the point spectrum of $L_{\tilde{\xi}}$ via the Evans function $D(\lambda, \tilde{\xi})$, an analytic function measuring the angle of the nontrivial intersection between the stable manifold at $+\infty$ and the unstable manifold at $-\infty$. For further discussion of the Evans function, see [1,11,28] and references therein. We assume

Assumption 1.4 *(Strong spectral stability conditions).*

(D1) $D(\lambda, \tilde{\xi}) \neq 0$, $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\operatorname{Re}\lambda \geqslant 0$, $(\lambda, \tilde{\xi}) \neq (0, 0)$, and additionally,
(D2) $|D(\lambda, \tilde{\xi})| \geqslant c|(\lambda, \tilde{\xi})|$, $c > 0$ for $\operatorname{Re}\lambda \geqslant 0$ and $|(\lambda, \tilde{\xi})|$ sufficiently small.

This spectral stability condition has been successfully verified in the viscous case, analytically and numerically. Especially, it is analytically verified for small amplitudes by [28] for one-dimensional case, and by [9,10] for both one- and multidimensional cases. It can also be verified numerically for large amplitudes as in [3,12–14]. For one-dimensional general relaxation system, it is verified using Evans function techniques (singular perturbation argument) as in [28]. This verification problem can be also considered using energy estimates as in [16,15] for specific models and as in [26] for small amplitude shocks in the general model, or a combination of asymptotic ODE methods and numerical methods as in [12].

During the course of our analysis, we will often find it convenient to work with the Evans function in polar coordinates, and for reference we re-state Assumption 1.4 in this context. Let $D_{\lambda_0,\tilde{\xi}_0}(\rho) := D(\rho\lambda_0, \rho\tilde{\xi}_0)$ for $(\lambda_0, \tilde{\xi}_0) \in S_+^d := \{(\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1}\colon \operatorname{Re}\lambda \geqslant 0\}$ and $\rho > 0$. This Evans function in polar coordinates will be discussed in more detail later in Section 2. With this definition, (D1)–(D2) can simply be re-stated as $D_{\lambda_0,\tilde{\xi}_0}(\rho)$ vanishes precisely to first order at $\rho = 0$ and has no other zeroes in $S_+^d \times \{\rho > 0\}$. One can check that (D1)–(D2) is equivalent to the condition that the shock profile (1.4) is a transversal connection of the traveling wave ODE and it satisfies the uniform Lopatinski condition. In the analysis of low-frequency bounds in Section 2, we will make use of the bounds in (D2) for $\lambda \in \Lambda_{\tilde{\xi}} := \{\operatorname{Re}\lambda \geqslant -\theta_0|\tilde{\xi}|^2$ for some $\theta_0 > 0\}$. This slightly more general condition can be obtained from (D2) together with the analytic extendability of $D_{\lambda_0,\tilde{\xi}_0}(\rho)$ as described in Lemma 2.7.

For relaxation shock problems, we encounter more singular high-frequency behavior associated with the hyperbolic nature of the equations. In the viscous case, the linearized operator about the wave is sectorial, generating an analytic semigroup, and high-frequency contributions are essentially negligible, whereas in the relaxation case, the linearized operator generates a C^0 semigroup, and there is substantial high-frequency contribution which complicates the analysis for small time. This difficulty is overcome by "high-frequency energy estimate" obtained by carrying out Kawashima-type estimates. This idea was initially suggested in [34] in the context of hyperbolic–parabolic systems, and also carried out in the context of scalar relaxation system in [20]. In the viscous case, the short-time well-posedness theory is standard, whereas we encounter more singular short-time behavior in the relaxation systems. In order to overcome this, we establish the "damping estimate", based on Kawashima-type energy estimate under the following structural assumptions.

(A1) (Symmetric dissipative condition) (1.8) is simultaneously symmetrizable in the sense of Friedrichs. That is, there exists A^0 such that (i) A^0 is symmetric, positive definite, (ii) A^0A^j are symmetric for $1 \leqslant j \leqslant d$, and (iii) $A^0 d_U Q$ is symmetric, negative semi-definite.
(A2) (Genuine-coupling condition) No eigenvector of $i\sum_{j=1}^d \xi_j A_j(U_\pm)$ lies in the kernel of $d_U Q(U_\pm)$.

Note that a combination of assumptions (A1) and (A2) implies (A3) below. It is called the skew-symmetrizer theorem essentially due to [29]. See [18,25,34,35] for more about general skew-symmetrizers in the several different contexts. Moreover, conditions (A1)–(A2) also imply (H3), see [18,29,32].

(A3) (Compensation matrix) There exists a differential matrix operator $K(\partial_x)$ satisfying

$$\widehat{K(\partial_x)f}(\xi) = i\bar{K}(\xi)\hat{f}(\xi), \tag{1.13}$$

where $\bar{K}(\xi)$ is a skew-symmetric operator which is smooth and homogeneous degree one in ξ satisfying

$$\operatorname{Re}\sigma\left(|\xi|^2 A^0 d_U Q - \bar{K}(\xi)\sum_{j=1}^d \xi_j A^j\right)_\pm \leqslant -\theta|\xi|^2 \quad \text{for all } \xi \text{ in } \mathbb{R}^d. \tag{1.14}$$

Remark 1.5. 1. This is the standard set of structural assumptions proposed by W.-A. Yong in [30], as adapted to the shock problem by Mascia and Zumbrun [23] in the one spatial dimension case.

2. As described in [23,25], (A1)–(A2) are satisfied for a wide variety of relaxation systems: the extended thermodynamic models in the moment closure hierarchies of Levermore [21]; the discrete velocity kinetic models of Platkowski and Illner [27]; the BGK models of Bouchut [2]; the numerical scheme of Jin and Xin [17].

3. There is a more restrictive notion of symmetrizability by a nonlinear change of variables in terms of a convex entropy function. This is guaranteed by the existence of a convex entropy function to (1.1). This nonlinear version of symmetrizability implies the symmetrizability in the sense of Friedrichs. For further discussion of the existence of such an entropy function for (1.1), and its relation to symmetrization, see [19].

Finally, to simplify later discussion for the low-frequency bounds in Lemma 2.12, we assume (without loss of generality):

(S1) Nonzero eigenvalues of $d_U Q A_1^{-1}$ are distinct.
(S2) Eigenvalues of $(i\,df^{\tilde{\xi}_0,*} + i\tau_0 I_n)(df^{1,*})^{-1}$ with nonzero real part are semi-simple and locally analytic.

Before we state our main result, we introduce our notations. $\bar{U}(x_1 - at)$ is a traveling wave solution of (1.1), and $\tilde{U}(x,t)$ is a solution of (1.1) with the initial data $\tilde{U}_0(x)$. For $s \geqslant 0$, $H^s(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \mid (1+|\xi|^s)\hat{f}(\xi) \in L^2(\mathbb{R}^d)\}$ denotes the sth-order Sobolev space in the L^2 sense, equipped with the norm

$|\cdot|_{H^s}$. Let $s_0 := \max\{[d/2]+2, 4\}$ where $d \geqslant 2$ is a space dimension. Now we state the main theorem of this paper:

Theorem 1.6. *Let $\bar{U}(x_1 - at)$ be a relaxation shock profile* (1.4) *of* (1.1) *with its amplitude $|U_+ - U_-| < \delta_S$ sufficiently small. Under assumptions* (H0)–(H5), (A1)–(A2), (D1)–(D2), *for $s \geqslant s_0$, we obtain the asymptotic $L^1 \cap H^s \to H^s$ stability with decay rate*

$$\left|\tilde{U}(\cdot,t) - \bar{U}\right|_{H^s} \leqslant C(1+t)^{-\frac{d-1}{4}} |\tilde{U}_0 - \bar{U}|_{L^1\cap H^s} \tag{1.15}$$

provided that the initial perturbation $|\tilde{U}_0 - \bar{U}|$ is sufficiently small in $L^1 \cap H^s$.

Remark 1.7. It is also possible to establish L^p stability, $p \geqslant 2$ as follows. First we can obtain L^∞ stability with decay rate of $(1+t)^{-\frac{d-1}{2}}$ under a further regularity assumption, $s \geqslant [d/2]+5$. This can be proven in a similar way as L^2 stability by using the Sobolev inequality, $|\cdot|_{L^\infty} \leqslant c|\cdot|_{H^s}$, and the high-frequency estimate as in Lemma 3.2, see [20] for more details. Then the interpolation yields an L^p estimate with decay rate of $(1+t)^{\frac{-(d-1)}{2}(1-\frac{1}{p})}$ for $p \geqslant 2$. Note also that L^∞ stability without any further assumption on regularity is possible to obtain, but with a less optimal decay rate. We can obtain L^∞ stability with the same decay rate as that of L^2, simply using the Sobolev inequality, $|\cdot|_{L^\infty} \leqslant c|\cdot|_{H^s}$, together with the estimate (1.15). Then, again by interpolation, L^p stability with the same decay rate as that of L^2 is obtained.

Outline of the paper. In Section 2, we establish L^p bounds on the Green's function associated with the linearized equations about the shock wave. First we construct the resolvent kernel and its bound in Laplace–Fourier frequency domain, and obtain L^p Green's function bounds via inverse Laplace–Fourier transform. Section 3 is devoted to establishing the damping estimate and the high-frequency estimate via Kawashima-type energy estimates. In Section 4, we prove Theorem 1.6. In Appendix A, we carry out the detailed computation of block-diagonalization.

2. Green function bounds

In this section, we construct the resolvent kernel for the linearized equation about the wave, and establish its bounds, under the necessary assumption of spectral stability. Using these, we obtain the low-frequency contribution of Green function and its derivatives via inverse Laplace–Fourier transform.

2.1. Spectral resolution formula

Let $\bar{U}(x_1 - at) = (\bar{u}, \bar{v})^t(x_1 - at)$ be a traveling wave solution of (1.1) satisfying (1.4). Letting $a = 0$, without loss of generality, $\bar{U}(x_1)$ is a stationary shock wave solution. Linearizing (1.1) about $\bar{U}(x_1)$, we obtain

$$U_t + \sum_{j=1}^{d} \left(\bar{A}^j U\right)_{x_j} - d_U \bar{Q}\, U = (0_n, I_r) N_0(U) + \sum_{j=1}^{d} N_j(U)_{x_j}, \tag{2.1}$$

where $d_U \bar{Q} = d_U Q(\bar{U}(x_1))$, $\bar{A}^j = (df^j, dg^j)^t(\bar{U}(x_1))$, and $N_j(U) = \mathcal{O}(|U|^2)$ for $0 \leqslant j \leqslant n$. We consider the linear initial value problem associated with (2.1):

$$U_t = LU := -\sum_j \left(\bar{A}^j U\right)_{x_j} + d_U \bar{Q}\, U, \qquad U(0) = U_0. \tag{2.2}$$

Taking the Fourier transform in the transverse directions $\tilde{x} := (x_2, \dots, x_d)$, we reduce to a family of partial differential equations (PDE)

$$\hat{U}_t = L_{\tilde{\xi}} \hat{U} := -\left(\bar{A}^1 U\right)' - i \sum_{j=2}^{d} \xi_j \bar{A}^j \hat{U} + d_U \bar{Q} \hat{U}, \qquad \hat{U}(0) = \hat{U}_0$$

in (x_1, t) indexed by frequency $\tilde{\xi} \in \mathbb{R}^{d-1}$, where $\hat{U} = \hat{U}(x_1, \tilde{\xi}, t)$ denotes the Fourier transform of $U = U(x, t)$ in $\tilde{x}$ and "$'$" denotes d/dx_1. Taking the Laplace transform in t, we obtain the resolvent equation:

$$(\lambda - L_{\tilde{\xi}}) \hat{\hat{U}} = \hat{U}_0, \tag{2.3}$$

where $\hat{\hat{U}}(x_1, \tilde{\xi}, \lambda)$ denotes the Laplace–Fourier transform of $U = U(x, t)$.

Definition 2.1. (a) The Green function $G(x, t; y)$ associated with the linearized equations (2.2) is defined by

(i) $(\partial_t - L)G = 0$ in the distributional sense for all $t > 0$, and
(ii) $G(x, t; y) \rightharpoonup \delta(x - y)$ as $t \to 0$.

(b) The resolvent kernel $G_{\lambda, \tilde{\xi}}(x_1, y_1)$ associated with the resolvent equation (2.3) is defined as a distributional solution of

$$(\lambda - L_{\tilde{\xi}}) G_{\lambda, \tilde{\xi}}(x_1, y_1) = \delta(x_1 - y_1).$$

Formally, one can write

$$G(x, t; y) := e^{Lt} \delta(x - y),$$

and

$$G_{\lambda, \tilde{\xi}}(x_1, y_1) := (\lambda - L_{\tilde{\xi}})^{-1} \delta(x_1 - y_1).$$

In the following proposition, we observe that L generates a $\mathcal{C}^0$ semigroup, and we obtain the spectral resolution formula. This inverse Laplace–Fourier transform will be used to convert the resolvent kernel $G_{\tilde{\xi}, \lambda}(x_1, y_1)$ in the low-frequency regime to obtain the low-frequency contribution for $G(x, t; y)$.

Proposition 2.2. *Under assumptions* (H0)–(H4), (A1)–(A2), *L generates a $\mathcal{C}^0$ semigroup $|e^{Lt}| \leqslant C e^{\eta_0 t}$ on L^2 with domain $\mathcal{D}(L) := \{U\colon\ U, LU \in L^2\}$, satisfying the generalized spectral resolution formula, for some $\eta > \eta_0$,*

$$G(x, t; y) = \frac{1}{(2\pi i)^d} \text{P.V.} \int_{\eta - i\infty}^{\eta + i\infty} \int_{\mathbb{R}^{d-1}} e^{i\tilde{\xi} \cdot (\tilde{x} - \tilde{y}) + \lambda t} G_{\lambda, \tilde{\xi}}(x_1, y_1) \, d\tilde{\xi} \, d\lambda. \tag{2.4}$$

For the proof, we refer to [20] for the multidimensional scalar relaxation system. See also [23,34] for one-dimensional relaxation system and real-viscosity hyperbolic–parabolic system, respectively.

2.2. The Evans function

Consider the homogeneous eigenvalue ordinary differential equation (ODE):

$$(L_{\tilde{\xi}} - \lambda)W = \left(d_U Q - i\sum_{j=2}^{d} \xi_j A_j - \lambda\right)W - (A_1 W)' = 0, \tag{2.5}$$

and its limiting constant-coefficient equation at $x_1 = \pm\infty$:

$$(L_{\tilde{\xi},\pm} - \lambda)W = \left(d_U Q_\pm - \sum_{j=2}^{d} i\xi_j A_{j,\pm} - \lambda\right)W - (A_{1,\pm} W)' = 0. \tag{2.6}$$

By the change of variable $V := A_{1,\pm}W$, we have

$$\begin{aligned} V' &= \left(d_U Q_\pm - \sum_{j=2}^{d} i\xi_j A_{j,\pm} - \lambda\right)(A_{1,\pm})^{-1} V \\ &=: \mathbb{A}_\pm V. \end{aligned} \tag{2.7}$$

Definition 2.3. The domain of consistent splitting Λ is defined as the connected component of $(\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1}$ containing $\tilde{\xi} = 0$ and λ going to real $+\infty$, for which the coefficients

$$\left(d_U Q_\pm - \sum_{j=2}^{d} i\xi_j A_{j,\pm} - \lambda\right)(A_{1,\pm})^{-1} \tag{2.8}$$

in (2.7) have p eigenvalues of negative real part and $(N-p)$ eigenvalues of positive real part, with no pure imaginary eigenvalues.

Lemma 2.4. *Under assumptions* (H0)–(H1), (H3), *there holds*

$$\Lambda \subset \left\{(\lambda, \tilde{\xi})\colon \operatorname{Re}\lambda > -\theta|\tilde{\xi}|^2/\left(1+|\tilde{\xi}|^2\right) \text{ for some } \theta > 0\right\}. \tag{2.9}$$

In particular, for $|(\lambda, \tilde{\xi})| \geqslant r > 0$, arbitrary, $\Lambda \subset \{\lambda\colon \operatorname{Re}\lambda \geqslant -\eta\} \times \mathbb{R}^{d-1}$, where $\eta(r) := \theta r^2 > 0$, r sufficiently small.

Proof. Noting that eigenvalues $\mu(\lambda, \tilde{\xi})$ of coefficient (2.8) relate to solutions of the dispersion-relation

$$\lambda(\xi) \in \sigma\left(d_U Q_\pm - \sum_{j=1}^{d} i\xi_j A_{j,\pm}\right)$$

by the relation $\mu = i\xi_1$, we find by assumption (H3) that the coefficient has no pure imaginary eigenvalues when $\operatorname{Re}\lambda > -\theta|\xi|^2/(1+|\xi|^2)$ for $\xi = (\xi_1, \tilde{\xi})$, all $\xi_1 \in \mathbb{R}$, or equivalently $\operatorname{Re}\lambda > -\theta|\tilde{\xi}|^2/(1+|\tilde{\xi}|^2)$. A straightforward homotopy argument taking λ to real plus infinity then gives the result; see [23,33,34]. □

Proposition 2.5. *Under assumptions* (H0)–(H1), (H3), *for* $(\lambda,\tilde{\xi})$ *in the domain of consistent splitting* Λ, *there are* $N := n + r$ *solutions of* (2.5)

$$\{\varphi_1^+(x_1;\lambda,\tilde{\xi}),\dots,\varphi_p^+(x_1;\lambda,\tilde{\xi})\}$$

and

$$\{\varphi_{p+1}^-(x_1;\lambda,\tilde{\xi}),\dots,\varphi_N^-(x_1;\lambda,\tilde{\xi})\},$$

which are locally analytic (*in* $(\lambda,\tilde{\xi})$) *bases for the stable and unstable manifolds as* $x_1 \to +\infty$ *and* $x_1 \to -\infty$, *respectively, that is, the* (*unique*) *manifolds of solutions decaying exponentially as* $x_1 \to \pm\infty$. *There are also solutions of* (2.5)

$$\{\psi_1^-(x_1;\lambda,\tilde{\xi}),\dots,\psi_p^-(x_1;\lambda,\tilde{\xi})\}$$

and

$$\{\psi_{p+1}^+(x_1;\lambda,\tilde{\xi}),\dots,\psi_N^+(x_1;\lambda,\tilde{\xi})\}$$

which are locally analytic (*in* $(\lambda,\tilde{\xi})$) *bases for stable and unstable manifolds as* $x_1 \to -\infty$ *and* $x_1 \to +\infty$, *respectively, that is, manifolds of solutions blowing up exponentially as* $x_1 \to \pm\infty$ (*not unique*).

Proof. This standard result holds for general variable-coefficient systems whose coefficients converge exponentially as $x_1 \to \pm\infty$ (a consequence of the gap and conjugation lemmas; see [23,24,33,34]). □

Note that $\bar{U}'(x_1)$, a derivative of the traveling wave, is a solution fast decaying both at $+\infty$ and $-\infty$. Hereafter we let, without loss of generality, $\varphi_1^+(x_1;0,0) = \bar{U}'(x_1) = \varphi_N^-(x_1;0,0)$.

Definition 2.6 *(Evans function).* For $(\lambda,\tilde{\xi})$ in the domain of consistent splitting Λ, we define the Evans function as

$$D(\lambda,\tilde{\xi}) := \det\big(\varphi_1^+,\dots,\varphi_p^+,\varphi_{p+1}^-,\dots,\varphi_N^-\big)\big|_{x_1=0}. \tag{2.10}$$

Evidently, the Evans function is locally analytic in $(\lambda,\tilde{\xi})$ in the domain of consistent splitting, with zeros of $D(\cdot,\tilde{\xi})$ corresponding to eigenvalues of $L_{\tilde{\xi}}$ and it can in fact be extended continuously along rays through the origin using a polar coordinate. It is obvious that the condition

$$D(\lambda,\tilde{\xi}) \neq 0 \quad \text{for } \tilde{\xi} \in \mathbb{R}^{d-1},\ \operatorname{Re}\lambda > 0$$

is a necessary condition for stability.

Define the Lopatinski determinant $\Delta(\lambda,\tilde{\xi})$ for the equilibrium system by

$$\Delta(\lambda,\tilde{\xi}) := \det\big(r_1^-,\dots,r_{n-i_-}^-, i\big[f^{\tilde{\xi},*}\big] + \lambda[u], r_{i_++1}^+,\dots,r_n^+\big), \tag{2.11}$$

where

$$f^{\tilde{\xi},*}(u) := \sum_{j=2}^{d} \xi_j f^j\big(u, v^*(u)\big),$$

and $r_j^{\pm}(\lambda, \tilde{\xi})$ are defined as bases for the unstable/stable, respectively, subspaces of the matrix

$$\left(\lambda I + i\, df^{\tilde{\xi},*}\right)\left(df^{1,*}\right)_{\pm}^{-1}.$$

We have the primary relation between D and Δ in the limit as frequency goes to zero:

$$D(\lambda, \tilde{\xi}) = \gamma \Delta(\lambda, \tilde{\xi}) + \mathcal{O}\left(|\tilde{\xi}| + |\lambda|\right)^2, \tag{2.12}$$

where γ is a constant measuring transversality of stable/unstable manifolds in the traveling wave ODE. For the details of the proof, see Proposition A.1 in [33]. The proof of the relation (2.12) is considerably simpler in the relaxation than in the viscous case. Note also that Δ is linear on rays, but not linear. More specifically, it has a conical singularity at $(\tilde{\xi}, \lambda) = (0, 0)$ and is degree one homogeneous, with a gradient discontinuity at the origin. In light of this, we can blow up the singularity in D, Δ at the origin using polar coordinates

$$(\lambda, \tilde{\xi}) = (\rho\lambda_0, \rho\tilde{\xi}_0), \qquad \left|(\lambda_0, \tilde{\xi}_0)\right| = 1.$$

Define

$$D_{\lambda_0,\tilde{\xi}_0}(\rho) := D(\rho\lambda_0, \rho\tilde{\xi}_0),$$

for λ_0, $\tilde{\xi}_0$ held fixed. Evidently $D_{\lambda_0,\tilde{\xi}_0}(\rho)$ is analytic in all coordinates for $\operatorname{Re}\lambda_0 > 0, \tilde{\xi}_0 \in \mathbb{R}^{d-1}, \rho > 0$. We can remove the singularity at $(\lambda, \tilde{\xi}) = (0, 0)$ as follows.

Lemma 2.7. *$D_{\lambda,\tilde{\xi}}(\rho)$ can be extended analytically onto $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\rho > -\eta$, for some $\eta > 0$.*

Proof. Loosely following [33] for the viscous case, we shall extend the bases $\{w_j^{\pm}\}$ described in Proposition 2.5 so that their wedge products are analytic. The difficulty is that the limiting coefficient equations lose hyperbolicity at $\pm\infty$ for $\rho = 0$. That is, the equation

$$A_{\pm}^1 V' - d_U Q_{\pm} V = 0, \tag{2.13}$$

has an n-fold center manifold consisting of all constant solutions. Thus we cannot use the spectral separation argument directly. To overcome this, we will appeal to the gap lemma of [11]. We first show the existence of extensions $\bar{w}_j^{\pm}$ for the limiting coefficient equations. It is evident that the stable/unstable manifolds extend analytically by their spectral separation from other modes. The bifurcation of the center manifold near $\rho = 0$ is crucial. Substituting the Ansatz $W = e^{\mu x_1} V$ into the limiting equations, we obtain the characteristic equations,

$$\left[d_U Q_{\pm} + \rho\left(-i\sum_{j=2}^{d} \xi_{0j} A_{j,\pm} - \lambda_0 I\right) - \mu A_{1,\pm}\right] V = 0. \tag{2.14}$$

Positing the Taylor expansion,

$$\begin{cases} \mu = 0 + c_1\rho + \cdots, \\ V = V^0 + \rho V^1 + \cdots, \end{cases} \tag{2.15}$$

and matching terms of order ρ, we obtain

$$(-iA_{\tilde{\xi}_0} - \lambda_0 I - c_1 A_1)V^0 + d_U Q V^1 = 0. \tag{2.16}$$

Left-multiplying $[I, 0]$ on (2.16), we obtain

$$\big(i\big(df^{\tilde{\xi}_0,*}\big) + \lambda_0 I_n + c_1 df^{1,*}\big)r = 0. \tag{2.17}$$

Substituting $c_1 = i\xi_{0,1}$ in (2.17), we have $\lambda_0 \in \sigma(i(df^{\xi_0,*}))$, pure imaginary. This is a contradiction. Thus, the stable/unstable spectrum splits to first order, and we obtain the analytic extension by the standard matrix perturbation theory. Note that the analyticity of individual eigenvalues μ may fail. The desired result follows by the gap lemma of [11] provided

- (i) the coefficients of the limiting equations decay at exponential rate $e^{-\alpha|x_1|}, \alpha > 0$; and
- (ii) the spectral gap of the subspaces $\{\bar{w}_1^+, \ldots, \bar{w}_p^+\}$ and $\{\bar{w}_{p+1}^-, \ldots, \bar{w}_N^-\}$ is greater than $-\alpha$ (equivalently, spectral overlap is less than α).

We have (i) by structure of $\bar{U}(\cdot)$. The gap condition (ii) follows for small ρ provided that such extension exists, since the gap is zero at $\rho = 0$. Moreover, the gap lemma implies that $w_j^\pm$ converges to the corresponding $\bar{w}_j^\pm$ at rate $e^{-\frac{\alpha}{2}|x_1|}|\bar{w}_j^\pm|$ as $x_1 \to \pm\infty$, respectively. The proof is similar to that of positive spectral gap case, see [11,36]. □

Remark 2.8. The function $\bar{w}_j^\pm$ may be chosen within groups of r fast modes bounded away from the center manifold of coefficient $\mathbb{A}_\pm$, analytic in $(\rho, \tilde{\xi}_0, \lambda_0)$ for $\rho \geqslant 0$, $\tilde{\xi}_0 \in \mathbb{R}^{d-1}$, $\operatorname{Re}\lambda_0$, and n slow modes approaching the center manifold as $\rho \to 0$, analytic in $(\rho, \tilde{\xi}_0, \lambda_0)$ for $\rho > 0$, $\tilde{\xi}_0 \in \mathbb{R}^{d-1}$, $\operatorname{Re}\lambda_0 > 0$ and continuous at the boundary $\rho = 0$.

2.3. Construction of the resolvent kernel

We next derive explicit representation formulae for the resolvent kernel $G_{\lambda,\tilde{\xi}}$. We seek a solution of form

$$G_{\lambda,\tilde{\xi}}(x_1, y_1) = \begin{cases} \Phi^+(x_1; \lambda, \tilde{\xi})N^+(y_1; \lambda, \tilde{\xi}), & x_1 > y_1, \\ \Phi^-(x_1; \lambda, \tilde{\xi})N^-(y_1; \lambda, \tilde{\xi}), & x_1 < y_1 \end{cases}$$

where

$$\Phi^+(x_1; \lambda, \tilde{\xi}) = \big(\varphi_1^+(x_1; \tilde{\xi}, \lambda), \ldots, \varphi_p^+(x_1; \tilde{\xi}, \lambda)\big) \in \mathbb{R}^{N\times p}$$

and

$$\Phi^-(x_1; \lambda, \tilde{\xi}) = \big(\varphi_{p+1}^-(x_1; \tilde{\xi}, \lambda), \ldots, \varphi_N^-(x_1; \tilde{\xi}, \lambda)\big) \in \mathbb{R}^{N\times (N-p)}.$$

From the jump condition of the Green kernel $G_{\lambda,\tilde{\xi}}$, we have

$$\begin{pmatrix} \Phi^+(y_1; \lambda, \tilde{\xi}) & \Phi^-(y_1; \lambda, \tilde{\xi}) \end{pmatrix} \begin{pmatrix} N^+(y_1; \lambda, \tilde{\xi}) \\ -N^-(y_1; \lambda, \tilde{\xi}) \end{pmatrix} = -\big(A^1\big)^{-1}(y_1), \tag{2.18}$$

and inverting (2.18), we express for the resolvent kernel $G_{\lambda,\tilde\xi}$:

$$G_{\lambda,\tilde\xi}(x_1,y_1)=\begin{cases}-\left(\Phi^+(x_1;\lambda,\tilde\xi)\quad 0\right)\left(\Phi^+\quad \Phi^-\right)^{-1}(y_1;\lambda,\xi)(A^1)^{-1}(y_1), & x_1>y_1,\\ \left(0\quad \Phi^-(x_1;\lambda,\tilde\xi)\right)\left(\Phi^+\quad \Phi^-\right)^{-1}(y_1;\lambda,\xi)(A^1)^{-1}(y_1), & x_1<y_1.\end{cases}$$

Now, consider the dual equation of (2.5)

$$\left(L_{\tilde\xi}^*-\lambda^*\right)\tilde W=0, \tag{2.19}$$

where

$$L_{\tilde\xi}^*\tilde W:=\left(A^1\right)^*\tilde W'+\left(d_U Q^*+iA_{\tilde\xi}^*\right)\tilde W=\left(A^1\right)^*\tilde W'+\left(d_U Q^*+i\sum_{j=2}^d \xi_j A_j^*\right)\tilde W.$$

Lemma 2.9. *For any W, $\tilde W$ solutions such that $(L_{\tilde\xi}-\lambda)W=0$ and $(L_{\tilde\xi}^*-\lambda^*)\tilde W=0$, there holds*

$$\left\langle \tilde W, A^1 W\right\rangle \equiv \mathit{constant}, \tag{2.20}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual complex inner product.

Proof.

$$\begin{aligned}\left\langle \tilde W, A^1W\right\rangle' &=\left\langle \left(A^1\right)^*\tilde W', W\right\rangle+\left\langle \tilde W,\left(A^1W\right)'\right\rangle\\ &=\left\langle\left(\lambda^* I-d_U Q^*-iA_{\tilde\xi}^*\right)\tilde W, W\right\rangle+\left\langle \tilde W,\left(-\lambda I+d_U Q-iA_{\tilde\xi}\right)W\right\rangle=0. \qquad \square\end{aligned}$$

From (2.20), it follows that if there are p independent solutions $\varphi_1^+,\dots,\varphi_p^+$ of $(L_{\tilde\xi}-\lambda I)W=0$ decaying at $+\infty$ and $N-p$ independent solutions $\varphi_{p+1}^-,\dots,\varphi_N^-$ of the same equation decaying at $-\infty$, then there exist $N-p$ independent solutions $\tilde\psi_{p+1}^+,\dots,\tilde\psi_N^+$ of $(L_{\tilde\xi}^*-\lambda^* I)\tilde W=0$ decaying at $+\infty$ and p independent solutions $\tilde\psi_1^-,\dots,\tilde\psi_p^-$ of the same equation decaying at $-\infty$. Similarly as with our definitions for $\Phi^\pm$, we set

$$\begin{aligned}\Psi^+(x_1;\lambda,\tilde\xi)&=\left(\psi_{p+1}^+(x_1;\lambda,\tilde\xi)\quad\cdots\quad \psi_N^+(x_1;\lambda,\tilde\xi)\right)\in\mathbb{R}^{N\times(N-p)},\\ \Psi^-(x_1;\lambda,\tilde\xi)&=\left(\psi_1^-(x_1;\lambda,\tilde\xi)\quad\cdots\quad \psi_p^-(x_1;\lambda,\tilde\xi)\right)\in\mathbb{R}^{N\times p},\end{aligned}$$

and

$$\Psi(x_1;\lambda,\tilde\xi)=\left(\Psi^-(x_1;\lambda,\tilde\xi)\quad \Psi^+(x_1;\lambda,\tilde\xi)\right)\in\mathbb{R}^{N\times N},$$

where $\psi_j^\pm$ are the exponentially growing solutions at $\pm\infty$, respectively, of $(L_{\tilde\xi}-\lambda I)W=0$ as described above. In light of this, we may define dual exponentially decaying and growing solutions $\tilde\psi_j^\pm$ and $\tilde\varphi_j^\pm$, respectively, via

$$\left(\tilde\Psi^\pm\quad \tilde\Phi^\pm\right)^* A^1\left(\Psi^\pm\quad \Phi^\pm\right)\equiv I.$$

Let us denote

$$\tilde{\Phi} := (\,\tilde{\Phi}^- \quad \tilde{\Phi}^+\,),$$

and

$$\tilde{\Psi} := (\,\tilde{\Psi}^- \quad \tilde{\Psi}^+\,).$$

Now we express for $G_{\lambda,\tilde{\xi}}(x_1, y_1)$ in terms of dual solutions:

$$G_{\lambda,\tilde{\xi}}(x_1, y_1) = \begin{cases} -(\Phi^+(x_1;\lambda,\tilde{\xi}), 0)\, M(\lambda,\tilde{\xi})\, (\tilde{\Psi}^-(y_1;\lambda,\xi), 0)^*, & x_1 > y_1, \\ (0, \Phi^-(x_1;\lambda,\tilde{\xi}))\, M(\lambda,\tilde{\xi})\, (0, \tilde{\Psi}^+(y_1;\lambda,\xi))^*, & x_1 < y_1 \end{cases} \tag{2.21}$$

where

$$M(\lambda,\tilde{\xi}) := \begin{pmatrix} -M^+(\lambda,\tilde{\xi}) & 0 \\ 0 & M^-(\lambda,\tilde{\xi}) \end{pmatrix} = \Phi(z;\lambda,\tilde{\xi})^{-1} \left(A^1\right)^{-1}(z)\, \tilde{\Psi}(z;\lambda,\tilde{\xi})^{*-1}. \tag{2.22}$$

Note that $M(\lambda,\tilde{\xi})$ is independent of z thanks to Lemma 2.9. Using these dual solutions, we have the following expressions:

Proposition 2.10. *On $\Lambda \cap \rho(L_{\tilde{\xi}})$, there hold*

$$G_{\lambda,\tilde{\xi}}(x_1, y_1) = \sum_{k,j} M^+_{jk}(\lambda,\tilde{\xi})\varphi^+_j(x_1;\lambda,\tilde{\xi})\tilde{\psi}^-_k(y_1;\lambda,\tilde{\xi})^*, \tag{2.23}$$

for $y_1 \leqslant 0 \leqslant x_1$;

$$G_{\lambda,\tilde{\xi}}(x_1, y_1) = \sum_{k,j} d^+_{jk}(\lambda,\tilde{\xi})\varphi^-_j(x_1;\lambda,\tilde{\xi})\tilde{\psi}^-_k(y_1;\lambda,\tilde{\xi})^* - \sum_{j} \psi^-_j(x_1;\lambda,\tilde{\xi})\tilde{\psi}^-_j(y_1;\lambda,\tilde{\xi})^*, \tag{2.24}$$

for $y_1 \leqslant x_1 \leqslant 0$; and

$$G_{\lambda,\tilde{\xi}}(x_1, y_1) = \sum_{k,j} d^-_{jk}(\lambda,\tilde{\xi})\varphi^-_j(x_1;\lambda,\tilde{\xi})\tilde{\psi}^-_k(y_1;\lambda,\tilde{\xi})^* + \sum_{j} \varphi^-_j(x_1;\lambda,\tilde{\xi})\tilde{\varphi}^-_j(y_1;\lambda,\tilde{\xi})^* \tag{2.25}$$

for $x_1 \leqslant y_1 \leqslant 0$, where

$$M^+ = (-I, 0)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,\Psi^-$$

and

$$d^\pm = (0, I)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,\Psi^-.$$

Symmetric representations hold for $y_1 \geqslant 0$.

Remark 2.11. Representation (2.21) together with uniform exponential decay of $\Phi^{\pm}$, $\tilde{\Psi}^{\pm}$, Proposition 2.5, and the fact that $d^{\pm}$ are bounded when the Evans function $D(\lambda,\tilde{\xi}) := \det(\Phi^+,\Phi^-)$ does not vanish yields uniform bounds

$$\left|G_{\lambda,\tilde{\xi}}(x,y)\right| \leqslant Ce^{-\theta|x-y|},$$

$\theta > 0$, on the resolvent set $\rho(L_{\tilde{\xi}})$, in particular (by assumption $(\mathcal{D})$) for $\operatorname{Re}\lambda \geqslant -\eta$, $\eta > 0$ on intermediate frequencies $1/R \leqslant |(\lambda,\tilde{\xi})| \leqslant R$, $R > 0$ arbitrary. However, we shall not use this in our analysis, carrying out instead energy-based resolvent estimates for intermediate and high frequencies. We shall use (2.21) only in the *low-frequency* regime $|(\lambda,\tilde{\xi})| \ll 1$.

2.4. Low-frequency bounds

We now investigate spatial growth/decay in modes $\varphi_j^{\pm}$, $\psi_j^{\pm}$, $\tilde{\varphi}_j^{\pm}$, $\tilde{\psi}_j^{\pm}$ for $\rho := |(\lambda,\tilde{\xi})|$ small. By the gap lemma, applied to individual modes, there holds

$$\varphi_j^{\pm} = \bar{\varphi}_j^{\pm} + \mathcal{O}\big(e^{-\theta|x_1|}\big)\big|\bar{\varphi}_j^{\pm}\big|,$$

and similarly for $\tilde{\varphi}_j^{\pm}$, $\psi_j^{\pm}$, $\tilde{\psi}_j^{\pm}$, where $\bar{\varphi}_j^{\pm}$ denote the associated solutions of the limiting constant coefficient equations at $x_1 = \pm\infty$. In light of this, we will read off decay/growth of $\varphi_j^{\pm}$ from the explicitly available solutions $\bar{\varphi}_j^{\pm}$, and similarly for $\tilde{\varphi}_j^{\pm}, \psi_j^{\pm}, \tilde{\psi}_j^{\pm}$. Let us define a parabolic surface

$$\Gamma_{\tilde{\xi}} := \big\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda = -\theta_1\big(|\operatorname{Im}\lambda|^2 + |\tilde{\xi}|^2\big)\big\},$$

where $\theta_1 > 0$ sufficiently small. Now restrict our attention to the surface in the low-frequency regime, that is, for $\rho > 0$ sufficiently small,

$$(\lambda,\tilde{\xi}) \in \Gamma_{\tilde{\xi}} \cap B_{\rho}(0,0).$$

Lemma 2.12. *Under assumptions* (H0)–(H5), *Assumption* 1.4, *for* $\lambda \in \Gamma_{\tilde{\xi}}$ *and* $\rho := |(\tilde{\xi},\lambda)|$ *and* θ, $\theta_1 > 0$ *sufficiently small, there exists a choice of bases consisting of* $(n+r)$ *solutions* $\{\varphi_j^{\pm}\}$, $\{\tilde{\varphi}_j^{\pm}\}$, $\{\psi_j^{\pm}\}$, $\{\tilde{\psi}_j^{\pm}\}$ *to the eigenvalue equation* (2.5) *such that, at* $z = 0$, *their wedge products,*

$$(\phi_1 \wedge \cdots \wedge \phi_l)(z),$$

where

$$\begin{cases} l = p, & \text{for } \phi_j = \varphi_j^+, \tilde{\varphi}_j^-, \psi_j^-, \tilde{\psi}_j^+, \\ l = n + r - p, & \text{for } \phi_j = \varphi_j^-, \tilde{\varphi}_j^+, \psi_j^+, \tilde{\psi}_j^-, \end{cases}$$

and determinants

$$\det(\Phi \quad \Psi)_{\pm}(z),$$

and

$$\det(\tilde{\Phi} \quad \tilde{\Psi})_{\pm}(z)$$

are uniformly bounded above and below, and there hold bounds

$$\varphi_j^{\pm} = \gamma_{21,\varphi_j^{\pm}}\big[e^{\mu_j^{\pm} x_1} V_j^{\pm} + \mathcal{O}\big(e^{-\theta|x_1|}\big)\big], \quad x_1 \gtrless 0, \tag{2.26}$$

$$\tilde{\varphi}_j^{\pm} = \gamma_{21,\tilde{\varphi}_j^{\pm}}\big[e^{-\mu_j^{\pm} x_1} \tilde{V}_j^{\pm} + \mathcal{O}\big(e^{-\theta|x_1|}\big)\big], \quad x_1 \gtrless 0, \tag{2.27}$$

$$\psi_j^{\pm} = \gamma_{21,\psi_j^{\pm}}\big[e^{\nu_j^{\pm} x_1} V_j^{\pm} + \mathcal{O}\big(e^{-\theta|x_1|}\big)\big], \quad x_1 \gtrless 0, \tag{2.28}$$

and

$$\tilde{\psi}_j^{\pm} = \gamma_{21,\tilde{\psi}_j^{\pm}}\big[e^{-\nu_j^{\pm} x_1} \tilde{V}_j^{\pm} + \mathcal{O}\big(e^{-\theta|x_1|}\big)\big], \quad x_1 \gtrless 0, \tag{2.29}$$

where $|V_j^{\pm}|$, $|\tilde{V}_j^{\pm}|$ *are uniformly bounded above and below, and*:

(i) *The decay/growth rates* $\mu_j^{\pm}/\nu_j^{\pm}$ *satisfy*

$$\big|\mathrm{Re}\,\mu_j^{\pm}\big|, \big|\mathrm{Re}\,\nu_j^{\pm}\big| \sim 1, \tag{2.30}$$

for fast modes,

$$\big|\mathrm{Re}\,\mu_j^{\pm}\big|, \big|\mathrm{Re}\,\nu_j^{\pm}\big| \sim \rho, \tag{2.31}$$

for intermediate-slow modes, and

$$\big|\mathrm{Re}\,\mu_j^{\pm}\big|, \big|\mathrm{Re}\,\nu_j^{\pm}\big| \sim \rho^2, \tag{2.32}$$

for super-slow modes; *moreover,*

$$\big|\mu_j^{\pm}\big|, \big|\nu_j^{\pm}\big| = \mathcal{O}(\rho), \tag{2.33}$$

for both intermediate- and super-slow modes.

(ii) *The factors* $\gamma_{21,\beta}$ *satisfy*

$$\gamma_{21,\beta} \sim 1, \quad \beta = \varphi_j^{\pm}, \tilde{\varphi}_j^{\pm}, \psi_j^{\pm}, \tilde{\psi}_j^{\pm} \tag{2.34}$$

for fast and intermediate-slow modes, and for super-slow modes for which $\mathrm{Im}\,\lambda$ *is bounded distance* $\theta_1\epsilon$ *away from any associated branch singularities* $\eta_j(\tilde{\xi})$, *and*

$$\gamma_{21,\beta} \sim \big(\rho + \rho^{-1}\big(\mathrm{Im}\,\lambda - \eta_j(\tilde{\xi})\big)\big)^{-t_\beta}, \tag{2.35}$$

for super-slow modes for which $\mathrm{Im}\,\lambda$ *is within* $\theta_1\epsilon$ *of an associated branch singularity* $\eta_j(\tilde{\xi})$ *with*

$$(t_{\varphi_j^{\pm}}, t_{\psi_j^{\pm}}, t_{\tilde{\varphi}_j^{\pm}}, t_{\tilde{\psi}_j^{\pm}}) := \begin{cases} (\frac{r-1}{4r}, \frac{3r-1}{4r}, \frac{3r-1}{4r}, \frac{r-1}{4r}) & \text{for } s = 2r, \\ (\frac{r}{2(2r+1)}, \frac{3r+1}{2(2r+1)}, \frac{3r}{2(2r+1)}, \frac{r-1}{2(2r+1)}) & \text{for } s = 2r+1,\ p > 0, \\ (\frac{r-1}{2(2r+1)}, \frac{3r}{2(2r+1)}, \frac{3r+1}{2(2r+1)}, \frac{r}{2(2r+1)}) & \text{for } s = 2r+1,\ p < 0; \end{cases} \tag{2.36}$$

here $s = K_j^{\pm}$ *is the order of the associated branch singularity* $\eta_j^{\pm}(\tilde{\xi})$.

(iii) *The left-/right-eigenvectors* $\tilde{V}_j^{\pm}$ *and* $V_j^{\pm}$ *are of the form*

$$\tilde{V}_j^{\pm} = \left(s_j^{\pm}, 0\right) + \mathcal{O}(\rho), \tag{2.37}$$

and

$$V_j^{\pm} = \begin{pmatrix} r_j^{\pm} \\ -q_v^{-1} q_u r_j^{\pm} \end{pmatrix} + \mathcal{O}(\rho), \tag{2.38}$$

for intermediate- and super-slow modes, where r_j *are eigenvectors of* $f_u^{1,*}$ *and* s_j *are defined by the relation* $\langle s_j, f_u^{1,*} r_k\rangle = \delta_j^k$*, and*

$$\tilde{V}_j^{\pm} = L_j^{\pm} + \mathcal{O}(\rho), \tag{2.39}$$

and

$$V_j^{\pm} = R_j^{\pm} + \mathcal{O}(\rho), \tag{2.40}$$

for fast modes, where $L_j^{\pm}$ *and* $R_j^{\pm}$ *are the left- and right-eigenvectors of* (2.6) *respectively, at zero frequency.*

Proof. Appealing to the gap lemma, modulo an exponentially decaying error, we read off the decay/growth of solutions of the variable ODE (2.5) from the solutions for the corresponding limiting constant equations. Thus, we consider the limiting coefficient eigenvalue equation:

$$-A_1 U' + \left(-i\sum_{j=2}^{d} \xi_j A_j + d_U Q - \lambda I\right) U = 0. \tag{2.41}$$

Here we drop $\pm$ signs for the notational convenience. Parameterizing the curve $\Gamma_{\tilde{\xi}}$ in the low-frequency regime, we introduce

$$(\lambda, \tilde{\xi})(\rho, \tilde{\xi}_0, \tau_0) := \left(\rho\tau_0 i - \theta_1\rho^2, \rho\tilde{\xi}_0\right),$$

where $(\tilde{\xi}_0, \tau_0) \in S^d$ held fixed. Evidently, $(\lambda, \tilde{\xi})$ traces out the portion of the surface $\Gamma_{\tilde{\xi}}$ in the small frequency regime as $(\rho, \tau_0, \tilde{\xi}_0)$ ranges in the compact set $[0, \delta] \times S^d$. Let $\bar{\bar{U}}(\rho)$ denote the solutions of the limiting constant coefficient equations at $(\lambda, \tilde{\xi})$ from which $\varphi_j^{\pm}(\rho)$, $\psi_j^{\pm}(\rho)$ are constructed by the gap lemma. Making as usual the Ansatz

$$\bar{\bar{U}}^{\pm} = e^{\mu x_1} V,$$

substituting it into (2.41), we obtain the characteristic equation:

$$\left[d_U Q + \rho\left(-i\sum_{j=2}^{d} \xi_{0j} A_j - \lambda_0 I\right) - \mu A_1\right] V = 0, \tag{2.42}$$

or equivalently, with $W = A_1 V$,

$$\mu W = \left(d_U Q A_1^{-1} + \rho \left(-i \sum_{j=2}^{d} \xi_{0j} A_j - \lambda_0 I \right) A_1^{-1} \right) W. \tag{2.43}$$

This is a matrix perturbation problem with an eigenvalue μ and a parameter ρ near zero. Here we assume:

(S1) Nonzero eigenvalues of $d_U Q A_1^{-1}$ are distinct.

(S2) Eigenvalues of $(i\, df^{\tilde{\xi}_0,*} + i\tau_0 I_n)(df^{1,*})^{-1}$ with nonzero real part are semi-simple and locally analytic.

For nonzero eigenvalues $\mu \neq 0$ at $\rho = 0$, they are said to be fast-mode eigenvalues. It is easy to check that $|\mathrm{Re}\,\mu| \sim 1$ as $\rho \to 0$, and they are spectrally separated by (S1). On the other hand, the zero eigenvalues at $\rho = 0$ are called slow-mode eigenvalues. To investigate the slow modes, we introduce the curves

$$(\tilde{\xi}, \lambda)(\rho, \tilde{\xi}_0, \tau_0) := \left(\rho\tilde{\xi}_0, i\rho\tau_0 - \theta_1\rho^2\right),$$

where $\tilde{\xi}_0 \in \mathbb{R}^{d-1}$ and $\tau_0 \in \mathbb{R}$ are restricted to the unit sphere S^{d-1}. Positing the Puisieux expansion, we have

$$\begin{cases} \mu = 0 + c_1\rho + \cdots, \\ V = V^0 + \rho V^1 + \cdots. \end{cases}$$

Matching terms of order 0 and ρ, we have

$$d_U Q A_1^{-1} V^0 = 0$$

and

$$\left((-iA_{\tilde{\xi}_0} - i\tau_0 I)A_1^{-1} - c^1 I_n\right)V^0 + d_U Q A_1^{-1} V^1 = 0. \tag{2.44}$$

Left-multiplying $[I, 0]$ on (2.44), we obtain

$$\left(\left(i\, df^{\tilde{\xi}_0,*} + i\tau_0 I_n\right)\left(df^{1,*}\right)^{-1} - \alpha_0 I_n\right) r = 0, \tag{2.45}$$

with $c_1 = -\alpha_0$. Note that α_0 and r are an eigenvalue and an eigenvector of the reduced system, respectively. For eigenvalues α_0 of nonzero real part, denoted as intermediate-slow modes, we have growth or decay at rate $\mathcal{O}(\rho)$, and spectral separation by assumption (S2).

For the case that $\alpha_0 = i\xi_{01}$ is pure imaginary, denoted as super-slow modes, we need to consider the next order correction. Let $\tilde{\alpha}$ and $\tilde{V}$ be the next order correction to α and V^0, respectively. Here $\tilde{\alpha} = i\xi_{01} + \mathcal{O}(\rho)$ and $\mu = i\rho\xi_{01} + o(\rho)$.

For further analysis of super-slow modes, we block-diagonalize (2.42) with substitution $\mu = i\rho\xi_{01} + o(\rho)$. By the block-diagonalization carried out in Appendix A, we find an analytic invertible matrix $\mathcal{T}(\rho; \tilde{\xi}_0, \mu)$ near $\rho = 0$, such that

$$\mathcal{T}^{-1}L\mathcal{T} = \mathcal{T}^{-1}\left(-\mu A_1 - i\rho\sum_{j\neq 1}^{d}\xi_{0j}A_j + d_U Q\right)\mathcal{T}$$
$$= \begin{pmatrix} -S(\mu,\rho\tilde{\xi}_0) & 0 \\ 0 & -F(\mu,\rho\tilde{\xi}_0)\end{pmatrix} + \mathcal{O}(\rho^3), \tag{2.46}$$

where

$$S(\mu,\rho\tilde{\xi}_0) = \mu\, df^{1,*} + i\rho\sum_{j\neq 1}\xi_{0j}\,df^{j*} + \mu^2 B_{11}^* + \rho^2\sum_{j,k\neq 1}\xi_{0j}\xi_{0k}B_{jk}^*$$
$$+ \mu\rho\left(i\sum_{k\neq 1}\xi_{0k}B_{1k}^* + i\sum_{j\neq 1}\xi_{0j}B_{j1}^*\right), \tag{2.47}$$

and

$$F(\mu,\rho\tilde{\xi}_0) = -q_v + \mu\left(g_v^1 + q_v^{-1}q_u f_v^1\right) + \rho\sum_{j\neq 1}\xi_{0j}\left(g_v^j + q_v^{-1}q_u f_v^j\right) + \mathcal{O}\left(\rho^2\right).$$

Note that $f_u^{j,*}$, B_{jk}^* we found here are the same coefficients obtained in the Chapman–Enskog expansion. See Appendix A for the details of this block-diagonalization. To find the slow-mode eigenvalues μ to the second order in ρ, we consider

$$\left(S(\mu,\rho\tilde{\xi}_0) + \lambda I_n\right)V_I = 0.$$

Substituting

$$\begin{cases} \mu = 0 + \tilde{\alpha}\rho + \cdots, \\ V_I = \tilde{V}_I + \cdots \end{cases}$$

in (2.48) with $\tilde{\alpha} = i\xi_{01} + \mathcal{O}(\rho)$, and matching terms of order ρ, we obtain the next order correction equations:

$$\left[i\rho\sum_{j\neq 1}^{d}\xi_{0j}\,df^{j,*} + \rho^2\sum_{j,k=1}^{d}\xi_{0j}\xi_{0k}B_{jk}^* + \mu\, df^{1,*} + \lambda I_n\right]\tilde{V}_I = 0. \tag{2.48}$$

Substituting $\lambda = i\rho\tau_0 - \theta_1\rho^2$ in (2.48), and left-multiplying $(df^{1,*})^{-1}$, we obtain the modified equation at the second order:

$$\left[\left(df^{1,*}\right)^{-1}\left(i\sum_{j\neq 1}\xi_{0j}\,df^{j,*} + \left(i\tau_0 + \rho\left(B_{\xi_0\xi_0}^* - \theta_1\right)\right)I_n\right) - \tilde{\alpha}I_n\right]\tilde{V}_I = 0, \tag{2.49}$$

where $\tilde{\alpha} = i\xi_{01} + \mathcal{O}(\rho)$. From the dissipative condition (H3), we find that $B_{\xi_0\xi_0}^* \geqslant \theta$ for some $\theta > 0$, and that $(B_{\xi_0\xi_0}^* - \theta_1)$ is positive definite by choosing sufficiently small $\theta_1 > 0$ so that $\theta_1 < \theta$. Then (2.49) is exactly the same equations arising in the super-slow modes analysis for the viscous system in [33], and this is regarded as a connection to the inviscid analysis of Kreiss–Majda. Thus, the super-slow modes in the relaxation system are the same as the ones in the corresponding viscous system.

We refer readers to [33] for the detailed proof for the viscous case. Combining the fast, intermediate-slow and the super-slow modes analysis, together with the gap lemma as in [33,35], we obtain (2.26)–(2.36). In light of the eigenvalue analysis, we can verify that (2.37)–(2.40) are the eigenvectors of (2.8) by inspection. □

The following remark is a key observation made in [23] (Lemma 5.8, p. 834) to obtain the scattering coefficients bounds in the following lemma.

Remark 2.13. For transverse Lax shocks, with a suitable choice of basis at $\rho = 0$, fast-growing modes $\psi_j^{\pm}$ are fast-decaying at $\mp\infty$. Equivalently, fast-decaying dual modes $\tilde{\psi}_j^{\pm}$ are fast-growing at $\mp\infty$: i.e. the only bounded solutions of the adjoint eigenvalue equation are constant solutions. It follows that all fast-growing solutions $\psi_j^{\pm}$ at $\pm\infty$ can be expressed as linear combinations of fast-decaying solutions $\varphi_j^{\mp}$ at $\mp\infty$, respectively.

Lemma 2.14. *Under the same assumptions as in Lemma* 2.12*, for $\lambda \in \Gamma_{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, θ, $\theta_1 > 0$ sufficiently small, there hold*

$$\left|M_{jk}^{+}\right|, \left|d_{jk}^{+}\right| \leqslant \begin{cases} C\rho^{-1}\gamma_{22,\beta}, & j = 1, \\ C\gamma_{22,\beta}, & j \neq 1 \end{cases} \quad \text{for } \beta = M_{jk}^{+}, d_{jk}^{+}, \tag{2.50}$$

and

$$\left|M_{jk}^{-}\right|, \left|d_{jk}^{-}\right| \leqslant \begin{cases} C\rho^{-1}\gamma_{22,\beta}, & j = N, \\ C\gamma_{22,\beta}, & j \neq N \end{cases} \quad \text{for } \beta = M_{jk}^{-}, d_{jk}^{-}, \tag{2.51}$$

where

$$\gamma_{22,M_{jk}^{\pm}}\gamma_{21,\varphi_j^{\pm}}\gamma_{21,\tilde{\psi}_k^{\pm}} \leqslant \left(1 + \sum_j \left(\rho + \left|\sigma_j^{+}\right|\right)^{\frac{1}{2}(1-\frac{1}{K_j^{+}})}\right)\left(1 + \sum_k \left(\rho + \left|\sigma_k^{-}\right|\right)^{-\frac{1}{2}(1-\frac{1}{K_k^{-}})}\right), \tag{2.52}$$

$$(\gamma_{22,d_{jk}^{\pm}}\gamma_{21,\varphi_j^{\pm}}\gamma_{21,\tilde{\psi}_k^{\pm}}) \leqslant 1 + \sum_j \left(\rho + \left|\sigma_j^{\pm}\right|\right)^{(1-\frac{1}{K_j^{\pm}})}, \tag{2.53}$$

with $\gamma_{21,\beta}$ as defined in (2.34)–(2.35)*, $\sigma_j^{\pm} := \rho^{-1}(\operatorname{Im}\lambda - \eta_j^{\pm}(\tilde{\xi}))$, $\eta_j(\tilde{\xi})$ and $K_j^{\pm}$ as in Lemma* 2.12*. Moreover, with slow dual modes taken identically constant, there hold*

$$\left|M_{jk}^{\pm}\right|, \left|d_{jk}^{\pm}\right| \leqslant C\gamma_{22} \tag{2.54}$$

if $\tilde{\psi}_k$ is a fast mode, and

$$\left|M_{jk}^{\pm}\right|, \left|d_{jk}^{\pm}\right| \leqslant C\gamma_{22}\rho \tag{2.55}$$

if $\tilde{\psi}_k$ is a fast mode, and additionally, φ_j is a slow mode.

Proof. Since proofs of each cases are similar, we provide the details only for d_{jk}^{+} here. Recall the expression for $d^{+} = (d_{jk}^{+})$ in Proposition 2.10. By Cramer's rule, we express

$$d_{jk}^+ = \frac{\det(\varphi_1^+, \ldots, \varphi_p^+, \varphi_{p+1}^-, \ldots, \overbrace{\psi_k^-}^{j\text{th slot}}, \ldots, \varphi_N^-)}{\det(\Phi^+, \Phi^-)}, \tag{2.56}$$

where $\varphi_1^+(x_1, 0, 0) = \bar{U}'(x_1) = \varphi_N^-(x_1, 0, 0)$. By the strong spectral condition $(\mathcal{D})$ as in Assumption 1.4, (2.56) shows immediately that

$$|d_{jk}^+| \leqslant C\rho^{-1}.$$

If $j \notin \{1, N\}$, by linear dependency of $\{\varphi_1^+, \varphi_N^-\}$, we have

$$\det(\varphi_1^+, \ldots, \varphi_p^+, \varphi_{p+1}^-, \ldots, \overbrace{\psi_k^-}^{j\text{th slot}}, \ldots, \varphi_N^-) \leqslant C\rho.$$

In turn, $|d_{jk}^+| \leqslant C$. Furthermore, if ψ_k^- is a fast mode, we have $|d_{jk}^+| \leqslant C$ for all j. The fact stated in Remark 2.13 that the fast growth mode ψ_k^- is a linear combination of fast decay solutions at $+\infty$, i.e., $\psi_k^- \in \mathrm{Span}\{\varphi_i^+ \mid \varphi_i \in \mathcal{F}\}$, together with the linear dependency of fast decaying solutions at $+\infty$ yields the result. If, additionally, φ_j^- is a slow mode, we have $|d_{jk}^+| \leqslant C\rho$. This can be verified by calculating the first derivative of the numerator in (2.56) as follows:

$$\begin{aligned}
&\partial_\rho \det(\varphi_1^+, \ldots, \varphi_p^+, \varphi_{p+1}^-, \ldots, \overbrace{\psi_k^-}^{j\text{th slot}}, \ldots, \varphi_N^-)\big|_{\rho=0} \\
&\quad = \det(\partial_\rho \varphi_1^+, \ldots, \varphi_N^-)\big|_{\rho=0} + \cdots + \det(\varphi_1^+, \ldots, \partial_\rho \varphi_N^-)\big|_{\rho=0} = 0.
\end{aligned}$$

This implies that $\det(\varphi_1^+, \ldots, \varphi_p^+, \varphi_{p+1}^-, \ldots, \overbrace{\psi_k^-}^{j\text{th slot}}, \ldots, \varphi_N^-)| \leqslant C\rho^2$, in turn $|d_{jk}^+| \leqslant C\rho$. □

The following lemma gives the refined derivative bounds. This is essentially the same as viscous case, so we omit the proof here.

Lemma 2.15. *Under the same assumptions as in Lemma* 2.12*, for* $\lambda \in \Gamma_{\tilde{\xi}}$*,* $\rho := |(\tilde{\xi}, \lambda)|$ *and* $\theta, \theta_1 > 0$ *sufficiently small, there exists a choice of slow modes* $\{\tilde{\varphi}_j^\pm\}$*,* $\{\tilde{\psi}_j^\pm\}$ *satisfying all properties in Lemma* 2.12*, and*

$$\begin{aligned}
|(\partial/\partial y_1)\tilde{\varphi}_j^\pm| &\leqslant C\rho|\tilde{\varphi}_j^\pm|, \\
|(\partial/\partial y_1)\tilde{\psi}_j^\pm| &\leqslant C\rho|\tilde{\psi}_j^\pm|.
\end{aligned} \tag{2.57}$$

2.5. Resolvent kernel bounds

In the following lemma, we establish the resolvent kernel bounds in the low-frequency regime using the expressions for $G_{\tilde{\xi},\lambda}(x_1, y_1)$ in Proposition 2.10 together with the decay/growth rates and the spatial decay bounds in the previous lemmas.

Lemma 2.16. *Under the same assumptions as in Lemma* 2.12*, for* $(\tilde{\xi}, \lambda) \in \Gamma_{\tilde{\xi}} \cap B_\rho(0,0)$*,* $\rho > 0$ *sufficiently small, there holds*

$$\left|G_{\tilde{\xi},\lambda}(x_1, y_1)\right| \leqslant C\gamma_2\left(\rho^{-1}e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + e^{-\theta\rho^2|x_1-y_1|}\right), \tag{2.58}$$

$$\left|G_{\tilde{\xi},\lambda}(x_1, y_1)(0, I)^t\right| \leqslant C\gamma_2\left(e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + \rho e^{-\theta\rho^2|x_1-y_1|}\right), \tag{2.59}$$

and

$$\left|(\partial/\partial y_1)G_{\tilde{\xi},\lambda}(x_1, y_1)\right| \leqslant C\gamma_2\left(e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + \rho e^{-\theta\rho^2|x_1-y_1|}\right), \tag{2.60}$$

where

$$\gamma_2 := \begin{cases} 1, & \textit{strictly hyperbolic case}, \\ 1 + \sum_j [\rho^{-1}|\mathrm{Im}\,\lambda - \eta_j^{\pm}(\tilde{\xi})| + \rho]^{\frac{1}{s_j}-1}, & \text{(H2)}\ \textit{holds}. \end{cases} \tag{2.61}$$

Here $s_j \geqslant 2$ *is the multiplicity of branch singularity* $\tau = \eta_j^{\pm}(\tilde{\xi})$*, as defined in Definition* 1.2.

Proof. Let us present explicit formulae for the solutions of the eigenvalue ODE (2.5). By Lemma 2.12, the left-eigenvectors can be expressed as

$$\tilde{W}_j^{\pm,*}(y_1, \rho) = \gamma_{21,\beta}\left(e^{-\mu_j^{\pm}|y_1|}\tilde{V}^{\pm,*}(\rho) + e^{-c|y_1|}\right), \tag{2.62}$$

and the right-eigenvectors are

$$W_j^{\pm}(x_1, \rho) = \gamma_{21,\beta}\left(e^{\mu_j^{\pm}|x_1|}V^{\pm}(\rho) + e^{-c|x_1|}\right) \tag{2.63}$$

for slow modes, where $\tilde{V}^{\pm,*}(\rho) = (s_j, 0) + \mathcal{O}(\rho)$, $V^{\pm}(\rho) = \binom{r_j}{-q_v^{-1}q_u r_j} + \mathcal{O}(\rho)$, and $\gamma_{21,\beta}$, $\mu_j^{\pm}$, s_j, r_j are defined in Lemma 2.12. On the other hand, the left- and right-eigenvectors can be expressed, respectively as

$$\tilde{W}_j^{\pm,*}(y_1, \rho) = e^{-\mu_j^{\pm}(\rho)y_1}\tilde{V}_j^{\pm,*}(\rho)\left(I + \mathcal{O}\left(e^{-c|y_1|}\right)\right), \tag{2.64}$$

and

$$W_j^{\pm}(x_1, \rho) = e^{\mu_j^{\pm}(\rho)x_1}V_j^{\pm}(\rho)\left(I + \mathcal{O}\left(e^{-c|x_1|}\right)\right) \tag{2.65}$$

for fast modes. Here, the slow-mode and fast-mode eigenvalues μ_j are as described in Lemma 2.12.

Since proofs of each case are similar, we provide details only for the case of $y_1 \leqslant x_1 \leqslant 0$. Recall the expression for the resolvent kernel;

$$G_{\lambda,\tilde{\xi}}(x_1, y_1) = \sum_{k,j} d_{jk}^{+}\varphi_j^{-}(x_1)\tilde{\psi}_k^{-}(y_1)^* - \sum_j \psi_j^{-}(x_1)\tilde{\psi}_j^{-}(y_1)^*. \tag{2.66}$$

Using the expressions (2.62)–(2.65),

$$\begin{aligned}
&\sum_{k,j} d^+_{jk}\varphi^-_j(x_1)\tilde{\psi}^-_k(y_1)^* \\
&\quad = \sum_{\tilde{\psi}_k\in\mathcal{S}} \left(d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|}\begin{pmatrix} c_1 & 0 \\ c_2 & 0\end{pmatrix} + \mathcal{O}(\rho)\, d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|}\right) \\
&\qquad + \sum_{\varphi_j\in\mathcal{S},\, \tilde{\psi}_k\in\mathcal{F}} d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|} + \mathcal{O}\big(e^{-\theta|x_1-y_1|}\big).
\end{aligned} \tag{2.67}$$

By right-multiplying $(0, I)^{tr}$ on (2.67), we have

$$\begin{aligned}
\sum_{k,j} d^+_{jk}\varphi^-_j(x_1)\tilde{\psi}^-_k(y_1)^*(0,I)^{tr} &= \sum_{\tilde{\psi}_k\in\mathcal{S}} \mathcal{O}(\rho)\, d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|} \\
&\quad + \sum_{\varphi_j\in\mathcal{S},\, \tilde{\psi}_k\in\mathcal{F}} d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|} \\
&\quad + \mathcal{O}\big(e^{-\theta|x_1-y_1|}\big).
\end{aligned} \tag{2.68}$$

Using the scattering coefficients bounds (2.50)–(2.55), we obtain

$$\left|\sum_{k,j} d^+_{jk}\varphi^-_j(x_1)\tilde{\psi}^-_k(y_1)^*(0,I)^{tr}\right| \leqslant C\gamma_2 e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + C\gamma_2\rho e^{-\theta\rho^2|x_1-y_1|}. \tag{2.69}$$

Similarly, by more careful grouping of the fast and slow modes together with the refined derivative bounds in Lemma 2.15, for $\alpha = 0, 1$, we obtain

$$\begin{aligned}
&(\partial/\partial y_1)^\alpha \sum_{k,j} d^+_{jk}\varphi^-_j(x_1)\tilde{\psi}^-_k(y_1)^* \\
&\quad = \mathcal{O}\big(\rho^\alpha\big) \sum_{\varphi_j\in\mathcal{S},\, \tilde{\psi}_k\in\mathcal{S}} d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|} \\
&\qquad + \mathcal{O}\big(\rho^\alpha\big) \sum_{\varphi_j\in\mathcal{F},\tilde{\psi}_k\in\mathcal{S}} d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|} \\
&\qquad + \mathcal{O}\big(\rho^\alpha\big) \sum_{\varphi_j\in\mathcal{S},\, \tilde{\psi}_k\in\mathcal{F}} d^+_{jk}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k} e^{\mu_j|x_1|}e^{-\mu_k|y_1|} + \mathcal{O}\big(e^{-\theta|x_1-y_1|}\big).
\end{aligned}$$

Using the scattering coefficients bounds (2.50)–(2.55), we obtain

$$\left|(\partial/\partial y_1)^\alpha \sum_{k,j} d^+_{jk}\varphi^-_j(x_1)\tilde{\psi}^-_k(y_1)^*\right| \leqslant C\gamma_2\rho^{\alpha-1} e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + C\gamma_2\rho^\alpha e^{-\theta\rho^2|x_1-y_1|}.$$

Now consider the second term $\sum_j \psi^-_j \tilde{\psi}^{-*}_j$ in (2.66). Grouping the terms into the slow mode and the fast mode, we have

$$\begin{aligned}\sum_j \psi_j^- \tilde{\psi}_j^{-*} &= \sum_{\psi_j \in \mathcal{S}} \psi_j^- \tilde{\psi}_j^{-*} + \sum_{\psi_j \in \mathcal{F}} \psi_j^- \tilde{\psi}_j^{-*} \\ &= \sum_{\psi_j \in \mathcal{S}} \left(\gamma_{21,\psi_j} \gamma_{21,\tilde{\psi}_j} e^{-\theta\rho^2|x_1-y_1|} \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \end{pmatrix} + \mathcal{O}(\rho) \gamma_{21,\psi_j} \gamma_{21,\tilde{\psi}_j} e^{-\theta\rho^2|x_1-y_1|} \right) \\ &\quad + \mathcal{O}\big(e^{-\theta|x_1-y_1|}\big). \end{aligned} \tag{2.70}$$

By right-multiplying $(0, I)^{tr}$ on (2.70), we have

$$\sum_j \psi_j^- \tilde{\psi}_j^{-*}(0, I)^{tr} = \mathcal{O}(\rho) \gamma_{21,\psi_j} \gamma_{21,\tilde{\psi}_j} e^{-\theta\rho^2|x_1-y_1|} + \mathcal{O}\big(e^{-\theta|x_1-y_1|}\big). \tag{2.71}$$

Similarly, using the refined derivative bounds in Lemma 2.15, for $\alpha = 0, 1$, we have

$$\begin{aligned}(\partial/\partial y_1)^\alpha \sum_j \psi_j^- \tilde{\psi}_j^{-*} &= (\partial/\partial y_1)^\alpha \Bigg[\sum_{\psi_j \in \mathcal{S}} \psi_j^- \tilde{\psi}_j^{-*} + \sum_{\psi_j \in \mathcal{F}} \psi_j^- \tilde{\psi}_j^{-*} \Bigg] \\ &= \mathcal{O}\big(\rho^\alpha\big) \sum_{\psi_j \in \mathcal{S}} \gamma_{21,\psi_j} \gamma_{21,\tilde{\psi}_j} e^{\mu_j^-(x_1-y_1)} + \mathcal{O}\big(e^{-\theta|x_1-y_1|}\big). \end{aligned} \tag{2.72}$$

Using the expressions (2.72)–(2.68) together with the bounds on d_{jk}^+, $\gamma_{21,\beta}$ as in Lemma 2.14, we have the bounds:

$$\Bigg| \sum_j \psi_j^- \tilde{\psi}_j^{-*}(0, I)^{tr} \Bigg| \leqslant C\rho\gamma_2 e^{-\theta\rho^2|x_1-y_1|}, \tag{2.73}$$

and

$$\Bigg| (\partial/\partial y_1)^\alpha \sum_j \psi_j^- \tilde{\psi}_j^{-*} \Bigg| \leqslant C\rho^\alpha \gamma_2 e^{-\theta\rho^2|x_1-y_1|}. \tag{2.74}$$

Combining (2.69)–(2.74) together with the expression (2.66), we have the desired bounds for $y_1 \leqslant x_1 \leqslant 0$. The remaining cases can be established similarly. □

2.6. Decomposition of the Green function

For fixed small δ_1, $r > 0$ to be chosen later, define a "low-frequency" part G^I and a "high-frequency" part G^{II} of G, respectively by

$$G^I(x,t;y) := \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leqslant r} \oint_{\Gamma_{\tilde{\xi}} \cap \{|\lambda| < r\}} e^{i\tilde{\xi}\cdot(\tilde{x}-\tilde{y})+\lambda t} G_{\lambda,\tilde{\xi}}(x_1, y_1)\, d\lambda\, d\tilde{\xi},$$

where $\Gamma_{\tilde{\xi}} := \{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda = -\theta_1(|\operatorname{Im}\lambda|^2 + |\tilde{\xi}|^2)\}$, and

$$G^{II}(x,t;y) := \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int \chi_{\{|\tilde{\xi}| \geqslant \delta_1 \text{ or } |\operatorname{Im}\lambda| \geqslant r\}} e^{i\tilde{\xi}\cdot(\tilde{x}-\tilde{y})+\lambda t} G_{\lambda,\tilde{\xi}}(x_1, y_1)\, d\tilde{\xi}\, d\lambda.$$

Then, by the spectral resolution formula (2.4) together with Cauchy's theorem, we have a decomposition formula:

$$G(x,t;y) = G^I(x,t;y) + G^{II}(x,t;y). \tag{2.75}$$

In the present paper we will not use the high-frequency formula, instead we will employ the Kawashima-type energy method for the high-frequency estimate in Lemma 3.2.

2.7. Green function bounds

Now we establish bounds on G^I, the low-frequency contribution of G using inverse Laplace–Fourier transform and the resolvent kernel bounds we obtained in Lemma 2.16.

Lemma 2.17. *For multi-index α with $|\alpha| \leqslant 1$, there holds*

$$\left| \int_{y\in\mathbb{R}^d} \partial_y^{|\alpha|} G^I(\cdot,t;y) f(y)\,dy \right|_{L^p} \leqslant Ct^{-\frac{(d-1)}{2}(1-1/r)-\frac{|\alpha|}{2}} |f|_{L^q}, \tag{2.76}$$

$$\left| \int_{y\in\mathbb{R}^d} G^I(\cdot,t;y)(0_n, I_r)^t f(y)\,dy \right|_{L^p} \leqslant Ct^{-\frac{(d-1)}{2}(1-1/r)-1/2} |f|_{L^q}, \tag{2.77}$$

for all $t > 0$ and $f \in L^q$, where $1/r + 1/q = 1 + 1/p$ and $p \geqslant 2$.

Proof. Let $\tilde{G}_{\lambda,\tilde{\xi}}(x_1, y_1)$ be a parabolic extension of $\chi_{\{(\lambda,\tilde{\xi})\in\Gamma_{\tilde{\xi}}^{LF}\}} G_{\lambda,\tilde{\xi}}(x_1, y_1)$, the low-frequency part of $G_{\lambda,\tilde{\xi}}(x_1, y_1)$. Here $\Gamma_{\tilde{\xi}}$ is defined by the parametrization

$$\lambda(\tilde{\xi}, k) = ik - \theta_1\big(k^2 + |\tilde{\xi}|^2\big), \quad k \in \mathbb{R}$$

and $\Gamma_{\tilde{\xi}}^{LF} := \Gamma_{\tilde{\xi}} \cap B_r(0,0)$ for $r > 0$ sufficiently small. Recall that

$$\frac{1}{2\pi i} \oint_{\lambda\in\Gamma_{\tilde{\xi}}} e^{\lambda t} \tilde{G}_{\lambda,\tilde{\xi}}(x_1, y_1)\,d\lambda$$

is the Fourier transform of the Green's function $\tilde{G}(x,t;y)$. Using the Hausdorff–Young inequality, we obtain

$$\big|\tilde{G}(\cdot,t;y)\big|_{L^p(\tilde{x})} \leqslant \left| \frac{1}{2\pi i} \oint_{\lambda\in\Gamma_{\tilde{\xi}}} e^{\lambda t} \tilde{G}_{\lambda,\tilde{\xi}}(x_1, y_1)\,d\lambda \right|_{L^q(\tilde{\xi})}, \tag{2.78}$$

where $1/p + 1/q = 1$ and $p \geqslant 2$. Applying the resolvent kernel bounds, the right side of (2.78) can be estimated as

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left|e^{\lambda(\tilde{\xi},k)t}\right|\left|\tilde{G}_{\tilde{\xi},\lambda(\tilde{\xi},k)}(x_1,y_1)\right||d\lambda/dk|\,dk\right|_{L^q(\tilde{\xi})}\\
&\quad\leqslant C\left|\int_{-\infty}^{+\infty}e^{-\theta(k^2+|\tilde{\xi}|^2)t}\gamma_2\big(\rho^{-1}e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|}+e^{-\theta\rho^2|x_1-y_1|}\big)\,dk\right|_{L^q(\tilde{\xi})}\\
&\quad=:|A+B|_{L^q(\tilde{\xi})}
\end{aligned}$$

where γ_2 is defined as in Lemma 2.16. Let $\varepsilon := \frac{1}{\max_j s_j}$ ($0<\varepsilon<1$ chosen arbitrarily if there are no singularities), $r<1/2$, and $s>1-\varepsilon\geqslant 1/2$ such that $r+s=1$. Noting that $\rho\sim(|k|+|\tilde{\xi}|)$, we have

$$\rho^{-1}\gamma_2\leqslant\left[\big(|k|+|\tilde{\xi}|\big)^{-1}\left(1+\sum_j\big(\rho^{-1}\big|\operatorname{Im}\lambda-\eta_j^{\pm}(\tilde{\xi})\big|\big)^{\varepsilon-1}\right)\right]. \tag{2.79}$$

By standard parabolic scaling together with (2.79), we obtain the bounds

$$\begin{aligned}
|A|_{L^q(\tilde{\xi})}&\leqslant\left|e^{-\theta|x_1|}\int_{-\infty}^{+\infty}\left(|\tilde{\xi}|^{-r}|k|^{-s}+\sum_j|\tilde{\xi}|^{-r}\big|k-\eta_j^{\pm}\big|^{-s}\right)e^{-\theta(k^2+|\tilde{\xi}|^2)t}\,dk\right|_{L^q(\tilde{\xi})}\\
&=e^{-\theta|x_1|}\int_{-\infty}^{+\infty}\left(|k|^{-s}+\sum_j\big|k-\eta_j^{\pm}\big|^{-s}\right)e^{-\theta k^2t}\,dk\left(\int_{\tilde{\xi}\in\mathbb{R}^{d-1}}|\tilde{\xi}|^{-qr}e^{-q\theta|\tilde{\xi}|^2t}\,d\tilde{\xi}\right)^{1/q}\\
&=Ce^{-\theta|x_1|}t^{-\frac12+\frac{s}{2}}t^{-\frac{d-1}{2q}+\frac{r}{2}}=Ce^{-\theta|x_1|}t^{-\frac{d-1}{2}(1-\frac1p)},
\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}
|B|_{L^q(\tilde{\xi})}&\leqslant\left|\int_{-\infty}^{+\infty}\left(1+\sum_j|\tilde{\xi}|^{1-\varepsilon}\big|k-\eta_j^{\pm}(\tilde{\xi})\big|^{\varepsilon-1}\right)e^{-\theta(k^2+|\tilde{\xi}|^2)(t+|x_1-y_1|)}\,dk\right|_{L^q(\tilde{\xi})}\\
&\leqslant\left(\int_{-\infty}^{+\infty}e^{-\theta|k|^2(t+|x_1-y_1|)}\,dk\right)\left(\int_{\tilde{\xi}\in\mathbb{R}^{d-1}}e^{q\theta|\tilde{\xi}|^2(t+|x_1-y_1|)}\,d\tilde{\xi}\right)^{1/q}\\
&\quad+\left(\int_{-\infty}^{+\infty}|k|^{1-\varepsilon}e^{-\theta|k|^2(t+|x_1-y_1|)}\,dk\right)\left(\int_{\tilde{\xi}\in\mathbb{R}^{d-1}}|\tilde{\xi}|^{q(\varepsilon-1)}e^{q\theta|\tilde{\xi}|^2(t+|x_1-y_1|)}d\tilde{\xi}\right)^{1/q}\\
&=C\big(t+|x_1-y_1|\big)^{-\frac{d-1}{2}(1-\frac1p)-\frac12}.
\end{aligned} \tag{2.81}$$

Note that the contribution from $|(\tilde{\xi},\lambda)|>R$ for sufficiently large $R>0$ is bounded by

$$e^{-\theta(t+|x_1-y_1|)} \tag{2.82}$$

for some $\theta > 0$. Combining (2.80)–(2.82) with Green function decomposition (2.75), we obtain

$$\left|G^I(\cdot,t;y)\right|_{L^p(x)} = \left|\left|G^I(\cdot,t;y)\right|_{L^p(\tilde{x})}\right|_{L^p(x_1)} \leqslant Ct^{-\frac{d-1}{2}(1-\frac{1}{p})}. \tag{2.83}$$

Similar computations for $G^I(\cdot,t;y)(0,I)^t$ and $(\partial/\partial y)G^I(\cdot,t;y)$ yield the following bounds:

$$\left|(\partial/\partial y)G^I(\cdot,t;y)\right|_{L^p(x)} \leqslant Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \tag{2.84}$$

and

$$\left|G^I(\cdot,t;y)(0,I)^{tr}\right|_{L^p(x)} \leqslant Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \tag{2.85}$$

By the convolution estimates together with (2.83)–(2.85), we obtain the desired results. □

3. Energy estimates

In this section, we establish the damping estimate and high-frequency estimate. These estimates both rely on the Kawashima condition (A3), which is implied by a combination of the dissipative structural assumptions (A1) and (A2).

3.1. Damping estimate

Let $U(x,t) := \tilde{U}(x,t) - \bar{U}(x_1)$ be a nonlinear perturbation. For the proof of the damping estimate, it is favorable to use a quasilinear form for $U(x,t)$:

$$U_t + \sum_{j=1}^{d}\left(\tilde{A}^j U_{x_j}\right) - (d_U\tilde{Q}U) = f, \tag{3.1}$$

where $f := \mathcal{O}(|\bar{U}_{x_1}||U| + |U|^2)$, $\tilde{A}^j := A^j(\tilde{U})$ and $d_U\tilde{Q} := d_U Q(\tilde{U})$.

Let $\varepsilon_0 > 0$ be a sufficiently small number which will be specified in Lemma 3.1, and $s \geqslant [\frac{d}{2}] + 2$. By the standard H^s local existence theory for a quasilinear symmetric hyperbolic system, there are $T > 0$ such that there exists a unique solution of (3.1) satisfying

$$U \in \mathcal{C}^0\left([0,T]; H^s\right), \tag{3.2}$$

and

$$\left|U(\cdot,t)\right|_{H^s} < \varepsilon_0, \quad t \in [0,T] \tag{3.3}$$

provided that $|U_0|_{H^s} < \epsilon_0/2$.

Proposition 3.1 *(Damping estimate). Assume* (A1), (A2) *hold* (*thus* (A3) *holds*)*. If* $|U|_{H^s}(t) \leqslant \varepsilon_0$ *sufficiently small for* $0 \leqslant t \leqslant T$ *where* $s \geqslant [\frac{d}{2}] + 2$*, there holds*

$$|U|^2_{H^s}(t) \leqslant e^{-\tilde{\theta}t}|U|^2_{H^s}(0) + C\int_0^t e^{-\tilde{\theta}(t-s)}|U|^2_{L^2}(s)\,ds \tag{3.4}$$

for $0 \leqslant t \leqslant T$*, and some* $\tilde{\theta} > 0$*.*

Proof. Let α be a multi-index with $|\alpha| = r \geqslant 1$ and $\partial_x^r := \sum_{|\alpha|=r} \partial_x^\alpha$. Taking a differential operator ∂_x^r on Eq. (3.1), we have

$$\partial_x^r U_t + \sum_{j=1}^d \tilde{A}^j \partial_x^r U_{x_j} - d_U \tilde{Q}\, \partial_x^r U = -\big[\partial_x^r, \tilde{A}^j \partial_j\big]U + \big[\partial_x^r, d_U \tilde{Q}\,\big]U + \partial_x^r f, \tag{3.5}$$

where $[A, B]U := A(BU) - B(AU)$ is a commutator operator. By left-multiplying $\tilde{A}^0$ on (3.5), we have

$$\tilde{A}^0 \partial_x^r U_t + \sum_{j=1}^d \tilde{A}^0 \tilde{A}^j \partial_x^r U_{x_j} - \tilde{A}^0 d_U \tilde{Q}\, \partial_x^r U = -\tilde{A}^0\big[\partial_x^r, \tilde{A}^j \partial_j\big]U + \tilde{A}^0\big[\partial_x^r, d_U \tilde{Q}\,\big]U + \tilde{A}^0 \partial_x^r f. \tag{3.6}$$

Taking the L^2 inner product of $\tilde{A}^0 \partial_x^r U$ against $\partial_x^r U$, we have the energy estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\langle \tilde{A}^0 \partial_x^r U, \partial_x^r U\big\rangle + \big\langle \tilde{A}^0 d_U \tilde{Q}\, \partial_x^r U, \partial_x^r U\big\rangle \\
&\quad = \frac{1}{2}\left\langle \left(\partial_t \tilde{A}^0 + \sum_{j=1}^d \partial_j\big(\tilde{A}^0 \tilde{A}^j\big)\right)\partial_x^r U, \partial_x^r U \right\rangle \\
&\qquad + \sum_{j=1}^d \big\langle \tilde{A}^0\big[\partial_x^r, \tilde{A}^j \partial_j\big]U, \partial_x^r U\big\rangle + \big\langle \tilde{A}^0\big[\partial_x^r, d_U \tilde{Q}\,\big]U, \partial_x^r U\big\rangle + \big\langle \tilde{A}^0 \partial_x^r f, \partial_x^r U\big\rangle \\
&\quad \leqslant C(\varepsilon_0 + \delta_S)|\partial_x U|_{H^{r-1}}^2 + C|U|_{L^2}^2.
\end{aligned} \tag{3.7}$$

Here $\delta_S := |U_+ - U_-|$ is the shock amplitude, and it is assumed to be sufficiently small. For the last inequality, we invoked Moser's inequalities and Sobolev inequalities. On the other hand, taking the L^2 inner product of $K(\partial_x)\partial_x^{r-1} U$ against $\partial_x^{r-1} U$, we have the auxiliary energy estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\langle K(\partial_x)\partial_x^{r-1} U, \partial_x^{r-1} U\big\rangle \\
&\quad = \frac{1}{2}\big\langle K(\partial_x)\partial_x^{r-1} U_t, \partial_x^{r-1} U\big\rangle + \frac{1}{2}\big\langle K(\partial_x)\partial_x^{r-1} U, \partial_x^{r-1} U_t\big\rangle \\
&\quad \leqslant c_0 \operatorname{Re}\big\langle iK(\xi)|\xi|^{r-1}\hat{U}_t, |\xi|^{r-1}\hat{U}\big\rangle \\
&\quad \leqslant c_0 \operatorname{Re}\big\langle K(\xi)A_-(\xi)|\xi|^{r-1}\hat{U}, |\xi|^{r-1}\hat{U}\big\rangle + c_0 \operatorname{Re}\big\langle iK(\xi)|\xi|^{r-1}\hat{H}, |\xi|^{r-1}\hat{U}\big\rangle.
\end{aligned} \tag{3.8}$$

Here we used Plancherel's identity and the following Fourier transformed equations

$$\hat{U}_t = -\sum_j i\xi_j A_-^j \hat{U} + \hat{H}, \tag{3.9}$$

where

$$H := \sum_j \big(A_-^j - \tilde{A}^j\big)U_{x_j} + (d_U \tilde{Q}\,)U + f. \tag{3.10}$$

By the Moser inequality together with the small amplitude assumption, the last term in (3.8) can be bounded as

$$\begin{aligned}\left|\mathrm{Re}\left\langle iK(\xi)|\xi|^{r-1}\hat{H}, |\xi|^{r-1}\hat{U}\right\rangle\right| &\leqslant C\left|\partial_x^{r-1}H\right|_{L^2}\left|\partial_x^r U\right|_{L^2} \\ &\leqslant C(\varepsilon_0+\delta_S)|\partial_x U|_{H^{r-1}}^2 + C|U|_{L^2}^2. \qquad (3.11)\end{aligned}$$

By (3.7)–(3.11), for $r \geqslant 1$, there holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\langle\tilde{A}^0\partial_x^r U, \partial_x^r U\right\rangle - \left\langle K(\partial_x)\partial_x^{r-1}U, \partial_x^{r-1}U\right\rangle\right) \\ &\quad\leqslant c_0\,\mathrm{Re}\left\langle\left(|\xi|^2 A_-^0 d_U Q_- - K(\xi)A_-(\xi)\right)|\xi|^{r-1}\hat{U}, |\xi|^{r-1}\hat{U}\right\rangle \\ &\qquad + C(\varepsilon_0+\delta_S)\left|\partial_x^r U\right|_{L^2}^2 + C(\varepsilon_0+\delta_S)\sum_{j=1}^{r}\left|\partial_x^{r-j}U\right|_{L^2}^2 + C|U|_{L^2}^2 \\ &\quad\leqslant -c_0\theta\left\langle|\xi|^r\hat{U}, |\xi|^r\hat{U}\right\rangle + C(\varepsilon_0+\delta_S)\left|\partial_x^r U\right|_{L^2}^2 + C(\varepsilon_0+\delta_S)\sum_{j=1}^{r}\left|\partial_x^{r-j}U\right|_{L^2}^2 + C|U|_{L^2}^2 \\ &\quad\leqslant -\theta\left|\partial_x^r U\right|_{L^2}^2 + C(\varepsilon_0+\delta_S)\left|\partial_x^r U\right|_{L^2}^2 + C(\varepsilon_0+\delta_S)\sum_{j=1}^{r}\left|\partial_x^{r-j}U\right|_{L^2}^2 + C|U|_{L^2}^2, \qquad (3.12)\end{aligned}$$

as long as $|U|_{H^s} < \varepsilon_0$. The second last inequality is true by (1.14).

We define

$$\mathcal{E}(t) := \left\langle\tilde{A}^0 U, U\right\rangle + \sum_{r=1}^{s} c^{-r}\left(\left\langle\tilde{A}^0\partial_x^r U, \partial_x^r U\right\rangle - \left\langle K(\partial_x)\partial_x^{r-1}U, \partial_x^{r-1}U\right\rangle\right). \qquad (3.13)$$

It is easy to check that $\mathcal{E}(t)$ is equivalent to $|U|_{H^s}^2(t)$ for a suitable choice of $c > 0$. By (3.12) and the smallness assumptions, $\varepsilon_0 + \delta_S < \frac{\theta}{2C}\frac{c-1}{c}$, there is a $\tilde{\theta}_1 > 0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\langle\tilde{A}^0 U, U\right\rangle + \sum_{r=1}^{s} c^{-r}\left(\left\langle\tilde{A}^0\partial_x^r U, \partial_x^r U\right\rangle - \left\langle K(\partial_x)\partial_x^{r-1}U, \partial_x^{r-1}U\right\rangle\right)\right) \\ &\quad\leqslant -\tilde{\theta}_1|U|_{H^s}^2 + C|U|_{L^2}^2. \qquad (3.14)\end{aligned}$$

Using (3.14) and (3.13), we have the Gronwall-type inequality:

$$\frac{d}{dt}\mathcal{E}(t) \leqslant -\tilde{\theta}\mathcal{E}(t) + C|U|_{L^2}^2(t). \qquad (3.15)$$

Therefore, we have the desired result. □

3.2. High-frequency solution operator analysis

We present the high-frequency estimates, which can be proved via a simpler linear version of Kawashima-type estimate. Since the proof is essentially the same as that of the scalar relaxation case, we refer the readers to [20] for the proof.

Lemma 3.2 *(High-frequency operator estimate). Let G^{II} be the high-frequency part of Green function associated with $(\partial_t - L)$. For any $f \in H^3$, there holds*

$$\left| \int G^{II}(x,t;y) f(y)\,dy \right|_{L^2} \leqslant Ce^{-\theta t}|f|_{H^3} \tag{3.16}$$

for some $\theta > 0$.

4. Nonlinear stability

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Defining the nonlinear perturbation

$$U(x,t) := \tilde{U}(x,t) - \bar{U}(x_1), \tag{4.1}$$

and taking Taylor expansion, we obtain the nonlinear perturbation equation

$$U_t - LU = (0_n, I_r)^{tr} N_0(U) + \sum_{j=1}^{d} N_j(U)_{x_j}, \tag{4.2}$$

where

$$N_j(U) = \mathcal{O}\big(|U|^2\big) \quad \text{for } j = 0, 1, \ldots, d, \tag{4.3}$$

as long as $|U|$ remains bounded by some fixed bounded constant. Applying Duhamel's principle, we can express

$$\begin{aligned} U(x,t) = {} & \int_{\mathbb{R}^d} G(x,t;y) U_0(y)\,dy \\ & - \int_0^t \int_{\mathbb{R}^d} G^I(x,t-s;y) \bigg((0_n, I_r)^{tr} N_0(U) + \sum_j N_j(U)_{x_j} \bigg)\,dy\,ds \\ & - \int_0^t \int_{\mathbb{R}^d} G^{II}(x,t-s;y) \bigg((0_n, I_r)^{tr} N_0(U) + \sum_j N_j(U)_{x_j} \bigg)\,dy\,ds \\ = {} & I_1 + I_2 + I_3. \end{aligned} \tag{4.4}$$

Define

$$\zeta(t) := \sup_{0 \leqslant s \leqslant t} \big| U(\cdot,s) \big|_{L^2} (1+s)^{\frac{d-1}{4}}. \tag{4.5}$$

Let $\varepsilon_0 > 0$ be a sufficiently small number as in Proposition 3.1. By the standard local existence theory for a quasilinear symmetric hyperbolic system, there are $T^* > 0$ such that there exists a unique solution of (3.1) satisfying

$$U \in \mathcal{C}^0\big(\big[0, T^*\big]; H^s\big), \tag{4.6}$$

and

$$\big|U(\cdot, t)\big|_{H^s} < \varepsilon_0, \quad t \in \big[0, T^*\big] \tag{4.7}$$

provided that $|U_0|_{H^s} < \varepsilon_0/2$. Thus, $\zeta(t)$ is continuous for $t \in [0, T^*]$. Let $\zeta_0 \leqslant \varepsilon_0/2$ be a small number which will be determined later. We shall establish

Claim. *If $|U_0|_{H^s} < \zeta_0$, then*

$$\zeta(t) \leqslant C_2\big(\zeta_0 + \zeta(t)^2\big) \quad \textit{for } t \geqslant 0. \tag{4.8}$$

From this result, it follows by continuous induction that $\zeta(t) \leqslant 2C_2\zeta_0$ for $t \geqslant 0$, provided that $\zeta_0 < 1/4C_2$. Definition (4.5) then yields the desired result

$$\big|U(\cdot, t)\big|_{L^2} \leqslant 2C_2\zeta_0(1+t)^{-\frac{d-1}{4}} \quad \text{for } t \geqslant 0. \tag{4.9}$$

It is remained to establish the claim above.

Proof of Claim. Combining the low- and high-frequency bounds in Lemmas 2.17 and 3.2, we can bound

$$|I_1|_{L^2(x)} = \bigg|\int_{\mathbb{R}^d} G(x, t; y)U_0(y)\,dy\bigg|_{L^2(x)} \leqslant C\zeta_0(1+t)^{-\frac{d-1}{4}}. \tag{4.10}$$

The above inequality is true by Minkowski's inequality and the local well-posedness of the linear problem. Using the low-frequency bounds (2.76)–(2.77) in Lemma 2.17 together with the definition of $\zeta(t)$, we obtain

$$\begin{aligned} |I_2|_{L^2(x)} &\leqslant C\int_0^t (1+t-s)^{-(d-1)/4}(t-s)^{-1/2}|U|_{L^2}^2(s)\,ds \\ &\leqslant C\zeta^2(t)\int_0^t (1+t-s)^{-(d-1)/4}(t-s)^{-1/2}(1+s)^{-(d-1)/2}\,ds \\ &\leqslant C\zeta^2(t)(1+t)^{-(d-1)/4}. \end{aligned} \tag{4.11}$$

By Lemma 3.2 and Proposition 3.1, we obtain

$$|I_3|_{L^2(x)} \leqslant Ce^{-\theta t}\zeta_0^2 + C(1+t)^{-(d-1)/2}\zeta^2(t) \leqslant C(1+t)^{-(d-1)/4}\big(\zeta_0^2 + \zeta^2(t)\big). \tag{4.12}$$

Note that the estimate (4.12) is true as long as $|U|_{H^s}(t) < \varepsilon_0$ since it depends on the damping estimate in Proposition 3.1. Thus we will use the continuation argument to prove the Claim above. Define

$$T := \sup\Big\{t > 0\colon\ U \in \mathcal{C}^0\big([0, t]; H^s\big),\ \sup_{0 \leqslant \tau \leqslant t} |U|_{H^s}(\tau) < \varepsilon_0\Big\}.$$

This is well defined and $T > 0$ thanks to (4.6). Combining (4.10)–(4.12), we have

$$\zeta(t) \leqslant C_2\big(\zeta_0 + \zeta(t)^2\big), \quad 0 \leqslant t \leqslant T. \tag{4.13}$$

It follows by continuous induction that

$$\zeta(t) \leqslant 2C_2\zeta_0, \quad 0 \leqslant t \leqslant T, \tag{4.14}$$

provided that $\zeta_0 < \frac{1}{4C_2}$. Using the damping estimate again together with (4.14), we have

$$\begin{aligned} |U|_{H^s}^2(T) &\leqslant C\left(e^{-\theta T}\zeta_0^2 + \int_0^T e^{-\theta(T-s)}|U|_{L^2}^2(s)\,ds\right) \\ &\leqslant C\left(e^{-\theta T} + \frac{4C_2^2}{\theta}\right)\zeta_0^2 < \frac{\varepsilon_0^2}{4} \end{aligned} \tag{4.15}$$

if we choose $\zeta_0 < \min\{\frac{\varepsilon_0}{2}, \frac{1}{4C_2}, \frac{\varepsilon_0}{2\sqrt{C(1+4C_2^2/\theta)}}\}$. Note that the choice of ζ_0 is independent of T. This implies that $T = +\infty$ if ζ_0 is chosen as above. This completes the proof of Claim. □

Therefore, the desired L^2 stability is established. Moreover, by the global existence of the solution together with (4.15), we have the estimates (4.14) and (3.4) for $t \in [0, \infty)$. These estimates give the H^s estimate for the global solution:

$$|U|_{H^s}(t) \leqslant C\zeta_0(1+t)^{-(d-1)/4} \quad \text{for } t \in [0, \infty). \tag{4.16}$$

This completes the proof. □

Appendix A. Block-diagonalization

In this section, we block-diagonalize the Fourier–Laplace transformed operator:

$$\mathcal{L}(\rho; \lambda_0, \xi_0) := L_\xi - \lambda I = d_U Q - i\rho\sum_{j=1}^d \xi_{0j}A_j - \rho\lambda_0 I, \tag{A.1}$$

with polar coordinates $(\lambda, \xi) = (\rho\lambda_0, \rho\xi_0)$ for each $(\lambda_0, \xi_0) \in \mathbb{S}^d$. By standard matrix perturbation theory, there is an invertible matrix $\mathcal{T}(\rho)$ defined in a neighborhood of $\rho = 0$, such that

$$\mathcal{T}^{-1}(\rho)\mathcal{L}(\rho)\mathcal{T}(\rho) = \operatorname{diag}\big(S(\rho), F(\rho)\big), \tag{A.2}$$

for sufficiently small $\rho > 0$. Here, $\mathcal{T}(\rho)$ and $\mathcal{T}^{-1}(\rho)$ are analytic at $\rho = 0$. Let $\tilde{T}(\rho; \xi_0)$ and $T(\rho; \xi_0)$ be second-order expansions of $\mathcal{T}^{-1}(\rho)$ and $\mathcal{T}(\rho)$, respectively. Dropping the subscripts of (λ_0, ξ_0) for our notational convenience, we let

$$\tilde{T} := \tilde{T}_0 + i\rho\sum_{j=1}^d \xi_j\tilde{T}_{1,j} + \rho^2\sum_{j,k=1}^d \xi_j\xi_k\tilde{T}_{2,jk}, \tag{A.3}$$

and

$$T := T_0 + i\rho \sum_{j=1}^{d} \xi_j T_{1,j} + \rho^2 \sum_{j,k=1}^{d} \xi_k \xi_k T_{2,jk}. \tag{A.4}$$

Then without loss of generality, we can choose

$$T_0 = \begin{pmatrix} I & 0 \\ -q_v^{-1} q_u & I \end{pmatrix}, \qquad \tilde{T}_0 = \begin{pmatrix} I & 0 \\ q_v^{-1} q_u & I \end{pmatrix},$$

and let

$$\tilde{T}_{1j} = \begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ \tilde{c}_j & \tilde{d}_j \end{pmatrix}, \qquad T_{1j} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix},$$
$$\tilde{T}_{2,jk} = \begin{pmatrix} \tilde{a}_2^{jk} & \tilde{b}_2^{jk} \\ \tilde{c}_2^{jk} & \tilde{d}_2^{jk} \end{pmatrix}, \qquad T_{2,jk} = \begin{pmatrix} a_2^{jk} & b_2^{jk} \\ c_2^{jk} & d_2^{jk} \end{pmatrix}.$$

We expand $\tilde{T}(\lambda - L_\xi)T$ up to the second order in ρ. It is easy to find the zeroth-order term:

$$-\tilde{T}_0 d_U Q T_0 = \begin{pmatrix} 0 & 0 \\ 0 & -q_v \end{pmatrix}. \tag{A.5}$$

Collecting all the first-order terms, we have

$$\begin{aligned} & i\xi_j(\tilde{T}_0 A_j T_0 - \tilde{T}_{1j} d_U Q T_0 - \tilde{T}_0 d_U Q T_{1j}) \\ &= \begin{pmatrix} df^{j*} & -\tilde{b}_j q_v + f_v^j \\ -(q_u a_j + q_v c_j) + q_v^{-1} q_u df^{j*} + dg^{j*} & -(q_u b_j + q_v d_j) - \tilde{d}_j q_v + q_v^{-1} q_u f_v^j + g_v^j \end{pmatrix}, \end{aligned} \tag{A.6}$$

where $df^{j*} := f_u^j - f_v^j q_v^{-1} q_u$, and similarly, $dg^{j*} := g_u^j - g_v^j q_v^{-1} q_u$. We impose the block-diagonal condition which implies that the non-diagonal blocks in (A.6) are zero blocks. Thus we have

$$\begin{cases} \tilde{b}_j = f_v^j q_v^{-1}, \\ q_u a_j + q_v c_j = q_v^{-1} q_u df^{j*} + dg^{j*}. \end{cases} \tag{A.7}$$

From the inversion relation, we have

$$\tilde{T}_{1j} T_0 + \tilde{T}_0 T_{1j} = 0 \quad \text{for } j = 1, 2, \ldots, d.$$

This yields the following four sub-matrix equations:

$$\begin{cases} \tilde{b}_j = f_v^j q_v^{-1} \quad \Rightarrow \quad b_j = -f_v^j q_v^{-1}, \\ a_j = \tilde{b}_j q_v^{-1} q_u - \tilde{a}_j, \\ q_v^{-1} q_u a_j + c_j = -(\tilde{c}_j - \tilde{d}_j q_v^{-1} q_u) \quad \Rightarrow \quad q_u a_j + q_v c_j = -(q_v \tilde{c}_j - q_v \tilde{d}_j q_v^{-1} q_u), \\ q_v^{-1} q_u b_j + d_j = -\tilde{d}_j \quad \Rightarrow \quad q_u b_j + q_v d_j = -q_v \tilde{d}_j. \end{cases} \tag{A.8}$$

From (A.8), we find that

$$\begin{cases} a_j = 0, \\ b_j = -f_v^j q_v^{-1}, \\ c_j = q_v^{-1}\big(q_v^{-1} q_u df^{j*} + dg^{j*}\big), \quad f_v^j c_k = B_{jk}^*, \\ d_j = q_v^{-1} q_u f_v^j q_v^{-1}, \\ \tilde{a}_j = f_v^j q_v^{-1} q_v^{-1} q_u, \\ \tilde{b}_j = f_v^j q_v^{-1}, \\ \tilde{c}_j = -c_j = -q_v^{-1}\big(q_v^{-1} q_u df^{j*} + dg^{j*}\big), \\ \tilde{d}_j = 0. \end{cases} \tag{A.9}$$

Plugging these in (A.6), we have the following first-order terms:

$$i\xi_j(\tilde{T}_0 A_j T_0 - \tilde{T}_{1j} d_U Q T_0 - \tilde{T}_0 d_U Q T_{1j}) = \begin{pmatrix} df^{j*} & 0 \\ 0 & q_v^{-1} q_u f_v^j + g_v^j \end{pmatrix}. \tag{A.10}$$

Calculating the second-order terms, we have

$$\tilde{T}_{1j} d_U Q T_{1k} = \begin{pmatrix} \tilde{b}_j(q_u a_k + q_v c_k) & \tilde{b}_j(q_u b_k + q_v d_k) \\ \tilde{d}_j(q_u a_k + q_v c_k) & \tilde{d}_j(q_u b_k + q_v d_k) \end{pmatrix} = \begin{pmatrix} -B_{jk}^* & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.11}$$

where $\tilde{b}_j(q_u a_k + q_v c_k) = f_v^j q_v^{-1}(q_v^{-1} q_u df^{k*} + dg^{k*}) = -B_{jk}^*$ is the Chapman–Enskog viscous term. On the other hand, we have

$$\tilde{T}_{1j} A_k T_0 = \begin{pmatrix} \tilde{a}_j\, df^{k*} + \tilde{b}_j\, dg^{k*} & \tilde{a}_j f_v^k + \tilde{b}_j g_v^k \\ \tilde{c}_j\, df^{k*} + \tilde{d}_j\, dg^{k*} & \tilde{c}_j f_v^k + \tilde{d}_j g_v^k \end{pmatrix} = \begin{pmatrix} -B_{jk}^* & * \\ * & * \end{pmatrix}, \tag{A.12}$$

and

$$\begin{aligned} \tilde{T}_0 A_j T_{1k} &= \begin{pmatrix} f_u^j a_k + f_v^j c_k & f_u^j b_k + f_v^j d_k \\ (q_v^{-1} q_u f_u^j + g_u^j) a_j + (q_v^{-1} q_u f_v^j + g_v^j) c_k & (q_v^{-1} q_u f_u^j + g_u^j) b_k + (q_v^{-1} q_u f_v^j + g_v^j)\, d_k \end{pmatrix} \\ &= \begin{pmatrix} -B_{jk}^* & * \\ * & * \end{pmatrix}. \end{aligned} \tag{A.13}$$

Combining (A.11)–(A.13), we have

$$\xi_j \xi_k (\tilde{T}_{1j} d_U Q T_{1k} - \tilde{T}_0 A_j T_{1k} - \tilde{T}_{1j} A_k T_0) = \xi_j \xi_k \begin{pmatrix} B_{jk}^* & * \\ * & * \end{pmatrix}. \tag{A.14}$$

Expanding $\tilde{T}$ and T up to the second order, we have the additional resulting second-order terms:

$$\xi_j \xi_k \tilde{T}_0 d_U Q T_{2,jk} = \xi_j \xi_k \begin{pmatrix} 0 & 0 \\ q_u a_2^{jk} + q_v c_2^{jk} & q_u b_2^{jk} + q_v d_2^{jk} \end{pmatrix}, \tag{A.15}$$

and

$$\xi_j \xi_k \tilde{T}_{2,jk} d_U Q T_0 = \xi_j \xi_k \begin{pmatrix} 0 & \tilde{b}_2^{jk} q_v \\ 0 & \tilde{d}_2^{jk} q_v \end{pmatrix}. \tag{A.16}$$

Collecting all the second-order terms (A.14)–(A.16), we can choose $\tilde{T}_{2,jk}$ and $T_{2,jk}$ in order to have no second-order terms in non-diagonal blocks with the left-upper block unchanged. This is possible since the left-upper blocks in (A.15)–(A.16) are zero blocks. Combining (A.5), (A.10), and (A.15)–(A.16), we have the second-order block-diagonal matrix as follows:

$$\begin{aligned} &\mathcal{T}^{-1}(\rho)\left(i\rho\sum_{j=1}^{d}\xi_j A_j - d_U Q\right)\mathcal{T}(\rho) \\ &\quad = \begin{pmatrix} i\rho\sum_j \xi_j df^{j*} + \rho^2\sum_{j,k}\xi_j\xi_k B^*_{jk} & 0 \\ 0 & -q_v + i\rho\sum_j \xi_j (g^j_v + q_v^{-1} q_u f^j_v) + \mathcal{O}(|\rho|^2) \end{pmatrix} \\ &\qquad + \mathcal{O}\big(|\rho|^3\big). \end{aligned} \tag{A.17}$$

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