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A uniformly convergent scheme on a nonuniform mesh for convection–diffusion parabolic problems

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Abstract

In this paper we construct a numerical method to solve one-dimensional time-dependent convection–diffusion problem with dominating convection term. We use the classical Euler implicit method for the time discretization and the simple upwind scheme on a special nonuniform mesh for the spatial discretization. We show that the resulting method is uniformly convergent with respect to the diffusion parameter. The main lines for the analysis of the uniform convergence carried out here can be used for the study of more general singular perturbation problems and also for more complicated numerical schemes. The numerical results show that, in practice, some of the theoretical compatibility conditions seem not necessary.

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1. Introduction

In this paper we consider parabolic problems of type

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x)u = f(x,t),\\
&(x,t) \in D \equiv \Omega \times (0,T) \equiv (0,1) \times (0,T),\\
&u(x,0) = u_0(x), \quad x \in \Omega,\\
&u(0,t) = u(1,t) = 0, \quad t \in [0,T],
\end{aligned}
\tag{1}
$$

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which represent the linear model, in fluid mechanics, of some convection–diffusion processes with dominating convection.

In these problems, the diffusion parameter ε can be very small in size with respect to the velocity term a or the reaction term b. In this singularly perturbed case, the solutions have, in general, a multiscale character, even for smooth data, with rapid variations in some narrow regions (layers). In this paper we will only consider the case $a(x) > \alpha > 0$, $b(x) \geqslant \tilde{b} \geqslant 0$ (a, b are smooth functions); in this situation, it is well known that a regular boundary layer appears in the outflow boundary $x = 1$ (see [9,12]). Classical discretization methods (standard finite difference or finite element schemes) fail to approach the exact solution $u(x,t)$ of (1), unless a large (ε-dependent) number of mesh points is used. This drawback can be overcome by the development of uniformly convergent methods, i.e., methods in which the rate of convergence and the error constant of the method are independent of the size of ε. One of the simplest ways to derive such methods consists of using a class of special piecewise uniform meshes, introduced in [13], which are constructed a priori in function of sizes of parameter ε, the convection term and the number of points N used in the spatial mesh.

The analysis of the uniform convergence with respect to ε of numerical methods based on Shishkin meshes is an interesting subject; for stationary problems see, for instance, the books [5,9,12] and references given there and the papers [1,4,6,15,16], which include numerical experiences for this type of meshes. For time-dependent problems we refer to papers [3,7,10], which give results for a finite difference scheme used for a parabolic problem of type (1) without convective term.

The main key for carrying out the analysis of the numerical method in this paper is focused in decomposing the global error in two components which are analyzed separately. Such idea starts by viewing the totally discrete algorithm as the result of a two-stage discretization process. The first stage consists of discretizing the time variable with the backward Euler method (for simplicity in our presentation we will consider constant time step). This produces a set of stationary singularly perturbed problems of type

$$\begin{aligned} &-\varepsilon\,\Delta t\, u''_{n+1}(x) + \Delta t\, a(x)u'_{n+1}(x) + (1 + \Delta t\, b(x))u_{n+1}(x) \\ &\qquad = u(x,t_n) + \Delta t\, f(x,t_{n+1}), \quad 0 < x < 1, \\ &u_{n+1}(0) = 0, \quad u_{n+1}(1) = 0. \end{aligned} \tag{2}$$

where $u_{n+1}(x) \equiv u_{n+1}(x, \Delta t, \varepsilon)$ is the approximation to $u(x, t_{n+1}, \varepsilon)$. In Section 3 we will show that the solution of (2) preserves the asymptotic behavior of the original problem; as well, we will prove the uniform convergence (with respect to both parameters Δt and ε) of the time semidiscretization. Some of the tasks, which we analyze there, are easily generalized for higher order schemes, like diagonally implicit Runge–Kutta methods.

To arrive to the totally discrete scheme, we must discretize the family of elliptic problems resulting from the time semidiscretization stage. In this work, we consider the simple upwind finite difference scheme defined on an appropriate piecewise uniform mesh of Shishkin type. At first sight, the parameter Δt appears to be another singular perturbation parameter into problems (2), but, under enough smoothness and compatibility requirements on the data of the continuous problem, we have seen that this parameter does not affect so severely as ε to the multiscale character of the semidiscrete solutions. In fact, in Section 4 we will prove that, for problems (2), the upwind method provides uniform convergence in ε and Δt. Then, joining both the results, we will deduce the uniform convergence of the totally discrete method.

In the proof of the convergence results, we will need a set of bounds for the derivatives of the solution of (1), obtained in Section 2. Finally, in Section 5, we will give some numerical examples which confirm the theoretical results. The technique of analysis that we develop here can be extended to different types of singularly perturbed parabolic problems and also to those numerical methods which can be deduced as the combination of a (time) integrator of differential equations and a (spatial) discretization scheme for elliptic problems.

Throughout the paper we will denote by C a generic positive constant independent of ε, the spatial mesh parameters and the time step.

2. The continuous problem: behavior with respect to the singular perturbation parameter

We will assume enough smoothness and compatibility conditions on data (f, u_0), in order to ensure the continuity and ε-uniform bound for the solution of (1) and its derivatives up to third order; this regularity is required to obtain, in the maximum norm on the domain $\bar{D}$, the appropriate space and time accuracy.

It is known that if $u_0 \in C^0(\bar{\Omega})$, $f \in C^0(\bar{D})$ and $u_0(0) = u_0(1) = 0$, then $u \in C^0(\bar{D})$. The maximum principle together with ε-uniform bounds for f and u_0 give the uniform bound $|u(x,t)| \leqslant C$ for all $(x,t) \in \bar{D}$. For simplicity, let us denote

$$L_{x,\varepsilon} \equiv -\varepsilon \frac{\partial^2}{\partial x^2} + a(x)\frac{\partial}{\partial x} + b(x)I. \tag{3}$$

To obtain additional sufficient conditions for $\partial^i u/\partial t^i \in C^0(\bar{D})$ we proceed inductively in i, differentiating problem (1) with respect to time variable t up to $i = 2$. So, we use the fact that function $v \equiv \partial u/\partial t \in C^0(\bar{D})$ is the solution of the problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + L_{x,\varepsilon} v = \frac{\partial f}{\partial t} && \text{in } D, \\ &v(x,0) = v_0 \equiv -L_{x,\varepsilon}u_0 + f(x,0), && x \in \bar{\Omega}, \\ &v(0,t) = v(1,t) = 0, && t \in [0,T], \end{aligned} \tag{4}$$

and assuming the compatibility conditions

$$\frac{\partial f}{\partial t}(0,0) + (L^2_{x,\varepsilon}u_0(x))(0,0) - (L_{x,\varepsilon}f(x))(0,0) = 0,$$

$$\frac{\partial f}{\partial t}(1,0) + (L^2_{x,\varepsilon}u_0(x))(1,0) - (L_{x,\varepsilon}f(x))(1,0) = 0,$$

we have that $v \in C^0(\bar{D})$ and it is ε-uniformly bounded (see [11]). For second derivative, in the same way we can prove that $\partial^2 u/\partial t^2 \in C^0(\bar{D})$ and it is ε-uniformly bounded.

Note that, in general, the condition

$$\left\{ \frac{\partial f^{i-1}}{\partial t^{i-1}},\ L_{x,\varepsilon}\frac{\partial f^{i-2}}{\partial t^{i-2}}, \ldots, L^{i-1}_{x,\varepsilon} f \right\} \subset C^0(\bar{D}) \quad (\varepsilon\text{-uniformly})$$

makes that $\partial u^i/\partial t^i \in C^0(\bar{D})$ and $L^i_{x,\varepsilon}u \in C^0(\bar{D})$ (ε-uniformly) are equivalent ones.

In the study of the uniform convergence of the spatial discretization stage, we also need to know the ε-asymptotic behavior of the solution of (1) and its spatial derivatives. To establish appropriate bounds, we rewrite (1) in the form

$$L_{x,\varepsilon}u = f(x,t) - \frac{\partial u}{\partial t} \equiv g(x,t), \quad (x,t)\in D,$$
$$u(0,t) = u(1,t) = 0, \tag{5}$$

where, under the assumed smoothness and compatibility conditions on u_0 and f, the function $g(x,t)$ is continuous and ε-uniformly bounded. Using the results of Kellogg and Tsan [8] for (5), it is straightforward to prove that

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leqslant C\left(1+\varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x\in[0,1],\ i=0,1 \tag{6}$$

for any $t\in[0,T]$.

For higher values of i, differentiating (5) with respect to x, it is possible to deduce similar bounds. For example, the function $w(x,t)\equiv \partial u(x,t)/\partial x$ satisfies

$$L_{x,\varepsilon}w = h(x,t) \equiv \frac{\partial f(x,t)}{\partial x} - \frac{\partial v}{\partial x} - a'(x)\frac{\partial u(x,t)}{\partial x} - b'(x)u, \quad (x,t)\in D,$$
$$w(0,t) = C_1, \quad w(1,t) = C_2\varepsilon^{-1}.$$

Then, if the function h satisfies

$$|h(x,t)| \leqslant C\left(1+\varepsilon^{-1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad (x,t)\in\bar{D},$$

we can deduce (6) for $i=2$. To show this bound for $h(x,t)$, it is sufficient to rewrite (4) in the form (5), and apply again the Kellogg–Tsan technique to obtain

$$\left|\frac{\partial v}{\partial x}\right| \leqslant C\left(1+\varepsilon^{-1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x\in[0,1],\ t\in[0,T].$$

From this bound the result immediately follows.

3. The time semidiscretization

As we have pointed out previously, in the first stage, we discretize only the time variable by means of the Euler implicit rule with uniform stepsize Δt:

$$\text{(a)}\quad u^0 = u_0(x), \tag{7}$$

$$\text{(b)}\quad (I+\Delta t\, L_{x,\varepsilon})u^{n+1} = u^n + \Delta t\, f(t_{n+1}),$$
$$u^{n+1}(0) = u^{n+1}(1) = 0, \tag{8}$$

which gives semidiscrete approximations $u^n(x)$ to the exact solution $u(x,t)$ of (1) at the time levels $t_n = n\,\Delta t$. Clearly, the operator $(I + \Delta t\, L_{x,\varepsilon})$ satisfies a maximum principle and consequently

$$\|(I + \Delta t\, L_{x,\varepsilon})^{-1}\|_\infty \leqslant \frac{1}{1 + \tilde{b}\,\Delta t}. \tag{9}$$

This ensures the stability of the method (8).

The local truncation error of the time semidiscretization method (8) is given by $e_{n+1} \equiv u(t_{n+1}) - \hat{u}^{\,n+1}$, where $\hat{u}^{\,n+1}$ is the solution of

$$\begin{aligned} &(I + \Delta t\, L_{x,\varepsilon})\hat{u}^{\,n+1}(x) = u(x,t_n) + \Delta t\, f(x,t_{n+1}),\\ &\hat{u}^{\,n+1}(0) = \hat{u}^{\,n+1}(1) = 0. \end{aligned} \tag{10}$$

This error measures the contribution of each time step to the global error of the time semidiscretization which is defined, at the instant t_n, as $E_n \equiv u(x,t_n) - u^n(x)$. Then, the following accuracy result follows.

Lemma 1. *If*

$$\left|\frac{\partial^i}{\partial t^i}u(x,t)\right| \leqslant C, \quad (x,t)\in\bar{\Omega}\times[0,T],\ 0\leqslant i\leqslant 2, \tag{11}$$

then the local error satisfies

$$\|e_{n+1}\|_\infty \leqslant C(\Delta t)^2. \tag{12}$$

Proof. Since the function $\hat{u}^{\,n+1}$ satisfies

$$(I + \Delta t\, L_{x,\varepsilon})\hat{u}^{\,n+1}(x) - \Delta t\, f(x,t_{n+1}) = u(x,t_n),$$

and as the solution of (1) is smooth enough, it holds

$$\begin{aligned} u(t_n) &= u(t_{n+1}) + \Delta t\, L_{x,\varepsilon}u(t_{n+1}) - \Delta t\, f(t_{n+1})\\ &\quad + \int_{t_n}^{t_{n+1}}(t_n - s)\frac{\partial^2 u}{\partial t^2}(s)\,\mathrm{d}s = (I + \Delta t\, L_{x,\varepsilon})u(t_{n+1}) - \Delta t\, f(t_{n+1}) + \mathrm{O}(\Delta t^2). \end{aligned}$$

Then, e_{n+1} is the solution of a boundary value problem of type

$$\begin{aligned} &(I + \Delta t\, L_{x,\varepsilon})e_{n+1} = \mathrm{O}(\Delta t^2),\\ &e_{n+1}(0) = e_{n+1}(1) = 0, \end{aligned}$$

and now, applying the stability result (9), (12) follows. □

Combining the stability and consistency properties of scheme (7) and (8), in the classical way we deduce the following result.

Theorem 2. *Under the hypotheses of Lemma* 1, *it holds*

$$\|E_n\|_\infty \leqslant C\Delta t \quad \forall\ n \leqslant T/\Delta t. \tag{13}$$

Therefore, the time semidiscretization process is uniformly convergent of first order.

In the analysis of the total discretization, we also need to know the asymptotic behavior, with respect to the singular perturbation parameter ε, of the exact solution $\hat{u}^{n+1}(x)$ of (10) and their derivatives, with respect to x, up to order 3. This behavior is given by the following result.

Lemma 3. *The solution of* (10) *satisfies*

$$\left|\frac{\mathrm{d}^i\hat{u}^{n+1}(x)}{\mathrm{d}x^i}\right| \leqslant C\left(1+\varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x\in\bar{\Omega},\ i=0,1,2,3. \tag{14}$$

Proof. The maximum principle (9) directly gives $\|\hat{u}^{n+1}\|_\infty \leqslant C$. Now, we consider the auxiliary boundary value problem

$$(I+\Delta t\, L_{x,\varepsilon})\delta(x) = -L_{x,\varepsilon}u(x,t_n)+f(x,t_{n+1}),$$
$$\delta(0)=\delta(1)=0,$$

whose solution is given by

$$\delta(x)=\frac{\hat{u}^{n+1}(x)-u(x,t_n)}{\Delta t}.$$

From $|L_{x,\varepsilon}u(x,t_n)|\leqslant C$ in $\bar{\Omega}$ and (9), we can deduce $|\delta(x)|\leqslant C$ in $\bar{\Omega}$. Writing (10) in the form

$$L_{x,\varepsilon}\hat{u}^{n+1}(x) = -\delta(x)+f(x,t_{n+1}),$$
$$\hat{u}^{n+1}(0)=\hat{u}^{n+1}(1)=0, \tag{15}$$

and using the technique of [8], it is straightforward to prove that

$$\left|\frac{\mathrm{d}^i\hat{u}^{n+1}(x)}{\mathrm{d}x^i}\right| \leqslant C\left(1+\varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x\in\bar{\Omega},\ i=0,1. \tag{16}$$

For higher derivatives, similar bounds are obtained inductively. Firstly, if we differentiate with respect to x problem (15), we have

$$L_{x,\varepsilon}\hat{w} = -\frac{\partial\delta(x)}{\partial x}+\frac{\partial f(x,t_{n+1})}{\partial x}-a'(x)w-b'(x)\hat{u}^{n+1}(x)\equiv h(x),$$
$$\hat{w}(0)=C_1, \quad \hat{w}(1)=C_2\varepsilon^{-1}, \tag{17}$$

where

$$\hat{w}=\frac{\mathrm{d}\hat{u}^{n+1}(x)}{\mathrm{d}x}, \quad |h(x)|\leqslant C\left(1+\varepsilon^{-1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x\in\bar{\Omega},$$

if we suppose that

$$\left|\frac{\partial\delta(x)}{\partial x}\right| \leqslant C\left(1+\varepsilon^{-1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x\in\bar{\Omega}. \tag{18}$$

Applying the same methodology of Kellogg and Tsan [8] to (17) we deduce (16) also for $i=2$.

To prove (18), we introduce the function $\delta_1 \equiv L_{x,\varepsilon}\delta$, which is the solution of

$$(I + \Delta t\, L_{x,\varepsilon})\delta_1(x) = -L^2_{x,\varepsilon}u(x,t_n) + L_{x,\varepsilon}f(x,t_{n+1}),$$

$$\delta_1(0) = \frac{1}{\Delta t}(f(0,t_{n+1}) - (L_{x,\varepsilon}u)(0,t_n)),$$

$$\delta_1(1) = \frac{1}{\Delta t}(f(1,t_{n+1}) - (L_{x,\varepsilon}u)(1,t_n)).$$

Taking into account the compatibility conditions

$$f(0,t_n) = (L_{x,\varepsilon}u)(0,t_n), \quad f(1,t_n) = (L_{x,\varepsilon}u)(1,t_n),$$

imposed to ensure that $\partial u/\partial t \in C^0(\bar{D})$, we know that the source term of the last boundary value problem is ε-uniformly bounded, and also that the boundary conditions are $(\Delta t, \varepsilon)$-uniformly bounded. Again, from (9) we obtain $|\delta_1(x)| \leqslant C$. The same technique applied on the problem

$$L_{x,\varepsilon}\delta(x) = \delta_1,$$

$$\delta(0) = 0, \quad \delta(1) = 0,$$

permit us to deduce (18).

To establish (16) for $i = 3$, we follow a similar procedure. Firstly, we differentiate (17) with respect to x, and rewrite it in the form

$$L_{x,\varepsilon}w_1 = h_1(x),$$

$$w_1 0) = C_1, \quad w_1(1) = C_2\varepsilon^{-2},$$

where

$$w_1 = \frac{\mathrm{d}^2\hat{u}^{\,n+1}(x)}{\mathrm{d}x^2}, \quad |h_1(x)| \leqslant C\left(1 + \varepsilon^{-2}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad x \in \bar{\Omega}.$$

The same technique used previously proves (16) for $i = 3$, if the function $h_1(x)$ is appropriately bounded. In this function, only the term $|\partial^2\delta(x)/\partial x^2|$ is not immediately bounded by

$$\left(1 + \varepsilon^{-2}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right).$$

To obtain this bound, we introduce the function $\delta_2 \equiv L^2_{x,\varepsilon}\delta$, which is the solution of

$$(I + \Delta t\, L_{x,\varepsilon})\delta_2(x) = -L^3_{x,\varepsilon}u(x,t_n) + L^2_{x,\varepsilon}f(x,t_{n+1}),$$

$$\delta_2(0) = D_0, \quad \delta_2(1) = D_1.$$

From hypotheses on data of problem (1), it is straightforward to see that D_0 and D_1 are evaluations of the function

$$\frac{1}{\Delta t}L_{x,\varepsilon}\left(-\frac{\hat{u}^{\,n+1}(x) - u(x,t_n)}{\Delta t} + f(x,t_{n+1}) + \frac{\partial u}{\partial t}(x,t_n) - f(x,t_n)\right)$$

$$\equiv \frac{1}{\Delta t} L_{x,\varepsilon}(f(x,t_{n+1}) - f(x,t_n)) + \frac{1}{\Delta t^2}\left[\frac{\hat{u}^{\,n+1}(x) - u(x,t_n)}{\Delta t}\right.$$
$$\left. -f(x,t_{n+1}) - \frac{\partial u}{\partial t}(x,t_n) + f(x,t_n) + \Delta t\, L_{x,\varepsilon}\frac{\partial u}{\partial t}(x,t_n)\right]$$

at the points $x=0$ and 1, respectively. Therefore, D_0 and D_1 are bounded independent of ε and Δt. Besides the smoothness and compatibility requirements imposed on problem (1), prove that the source term of the last boundary value problem is ε-uniformly bounded. Then, from (9) it follows $|\delta_2(x)| \leqslant C$. Using similar arguments for the problem

$$L_{x,\varepsilon}\delta_1(x) = \delta_2,$$
$$\delta_1(0) = C_0, \quad \delta_1(1) = C_1,$$

we can prove the required bound for $\partial^2\delta(x)/\partial x^2$. □

Nevertheless, to prove the uniform convergence of the simple upwind scheme, we will need a more precise decomposition of the exact solution $\hat{u}^{\,n+1}$ of (10). Similar to [2] we can obtain the following result.

Lemma 4. *The solution of* (10) *can be written in the form* $\hat{u}^{\,n+1}(x) = \hat{e}(x) + z(x)$, *where*

$$\hat{e}(x) = \gamma \exp\left(-\frac{a(1)(1-x)}{\varepsilon}\right), \quad \gamma = \frac{\varepsilon}{a(1)}\frac{\mathrm{d}\hat{u}^{\,n+1}}{\mathrm{d}x}(1), \tag{19}$$

$$\left|\frac{\mathrm{d}^i z(x,\varepsilon)}{\mathrm{d}x^i}\right| \leqslant C\left(1 + \varepsilon^{-i+1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad 0 \leqslant i \leqslant 3. \tag{20}$$

4. The spatial discretization

In this section, we study the totally discrete scheme obtained after the spatial discretization of (10). Let us introduce a nonnecessarily uniform mesh $\bar{\Omega}_N$ which will be generated as follows. Let $N \geqslant 4$ be an even number. We define the transition parameter σ as

$$\sigma = \min\left\{\frac{1}{2}, m\varepsilon \log N\right\}, \tag{21}$$

where m is a constant which we choose satisfying $m \geqslant 1/\alpha$. We divide the interval $[0,1]$ into two subintervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$ and we define

$$\bar{\Omega}_N \equiv \{x_0, x_1, \ldots, x_{N/2} = 1-\sigma, \ldots, x_N\}, \tag{22}$$

where

$$x_j = j\,\frac{2(1-\sigma)}{N}, \; j = 0, \ldots, \frac{N}{2},$$

$$x_j = 1 - \sigma + \left(j - \frac{N}{2}\right)\frac{2\sigma}{N}, \quad j = \frac{N}{2}+1, \ldots, N.$$

Let us denote $h_j = x_j - x_{j-1},\ j = 1,\dots,N,\ \tilde{h}_j = (h_j + h_{j+1})/2,\ j = 1,\dots,N-1$. Clearly,

$$h_j = \frac{2(1-\sigma)}{N}, \quad j = 1,\dots,\frac{N}{2}, \qquad h_j = \frac{2\sigma}{N}, \quad j = \frac{N}{2}+1,\dots,N.$$

Using the notation $[.]_h$ for the restriction of a function defined in $[0,1]$ to $\bar{\Omega}_N$, we will compute the totally discrete approximations u_h^n to $[u(t_n)]_h$ by

$$\text{(a)} \quad u_h^0 = [u_0]_h, \tag{23}$$

$$\begin{aligned} \text{(b)} \quad & (I + \Delta t\, L_{x,\varepsilon,h})u_h^{n+1} = u_h^n + \Delta t[f(x,t_{n+1})]_h, \\ & u_h^{n+1}(0) = u_h^{n+1}(1) = 0, \end{aligned} \tag{24}$$

where $L_{x,\varepsilon,h}$ is the discretization of the differential operator $L_{x,\varepsilon}$ using the simple upwind finite difference scheme. So, on the previous mesh the scheme is given by

$$(I + \Delta t\, L_{x,\varepsilon,h})u_j^{n+1} \equiv r_j^- u_{j-1}^{n+1} + r_j^c u_j^{n+1} + r_j^+ u_{j+1}^{n+1} = g_j, \quad j = 1,\dots,N-1, \tag{25}$$

$$u_0^{n+1} = 0, \quad u_N^{n+1} = 0, \tag{26}$$

where

$$r_j^- = \frac{-\varepsilon\,\Delta t}{h_j\tilde{h}_j} - \frac{a_j\Delta t}{h_j}, \quad r_j^+ = \frac{-\varepsilon\Delta t}{h_{j+1}\tilde{h}_j}, \quad r_j^c = 1 + \Delta t\, b_j - r_j^- - r_j^+, \tag{27}$$

$$a_j = a(x_j),\ b_j = b(x_j),\ g_j = u_j^n + \Delta t\, f(x_j,t_{n+1}). \tag{28}$$

Then, we can prove (see [2]) the following result.

Theorem 5. *Let us assume that* $\hat{u}^{\,n+1}$ *have the asymptotic behavior given by* (19) *and* (20), *and let* $\hat{u}_h^{n+1}$ *be its discretization, obtained by using* (25)–(28) *taking* $u_h^n \equiv [u(x,t_n)]_h$. *Then,*

$$\|[\hat{u}^{\,n+1}]_h - \hat{u}_h^{n+1}\|_\infty \leqslant CN^{-1}\log N. \tag{29}$$

Remark 6. Note that if we take $N^{-q} \leqslant C\,\Delta t$ with $0 < q < 1$, from Theorem 5 we can deduce that

$$\|[\hat{u}^{\,n+1}]_h - \hat{u}_h^{n+1}\|_\infty \leqslant C\,\Delta N^{-1+q}\log N, \tag{30}$$

which is the bound that we use to prove the uniform convergence of totally discrete method.

This uniform convergence of the totally discrete scheme is obtained as follows. Splitting the global error in the form

$$\|[u(t_n)]_h - u_h^n\|_\infty \leqslant \|[u(t_n)]_h - [\hat{u}^{\,n}]_h\|_\infty + \|[\hat{u}^{\,n}]_h - \hat{u}_h^n\|_\infty + \|\hat{u}_h^n - u_h^n\|_\infty,$$

bound (30) of remark, joined to the stability result (9) and the consistency one (12) for the time discretization, prove

$$\|[u(t_n)]_h - u_h^n\|_\infty \leqslant C(\Delta t^2 + \Delta t\, N^{-1+q} \log N) + \|[u(t_{n-1})]_h - u_h^{n-1}\|_\infty.$$

From this recurrence relation, we deduce the main result of the paper.

Theorem 7. *Let u be the solution of* (1) *and $\{u_h^n\}_n$ the solution of* (24)–(28). *Under the hypotheses of Lemma* 1 *and Theorem* 5, *if $N^{-q} \leqslant C\,\Delta t$, $0 < q < 1$, there exists a constant C such that*

$$\|[u(t_n)]_h - u_h^n\|_\infty \leqslant C(\Delta t + N^{-1+q} \log N). \tag{31}$$

5. Numerical results

In this section, we show the numerical results obtained in the integration of some problems of type (1). In all cases we begin with $N = 16$ and $\Delta t = 0.1$ and we multiply N by two and divide Δt also by two.

In the first example, we take $a(x) = 1$, $b(x) = 0$, $T = 1$ and we fit $u_0(x)$ and $f(x,t)$ to get that the exact solution is given by

$$u(x,t) = \mathrm{e}^{-t}(C_1 + C_2 x - \mathrm{e}^{-(1-x)/\varepsilon}),$$

where $C_1 = \mathrm{e}^{-1/\varepsilon}$ and $C_2 = 1 - \mathrm{e}^{-1/\varepsilon}$. As we know the exact solution, we compute exactly the pointwise errors by

$$\mathrm{e}_\varepsilon^{N,\Delta t}(i,n) = |u(x_i,t_n) - u^N(x_i,t_n)|,$$

where the superscript indicates the number of mesh points used in the spatial direction, $t_n = n\,\Delta t$ and Δt is the time step. For each ε, the maximum nodal error is given by

$$E_{\varepsilon,N,\Delta t} = \max_{i,n} \mathrm{e}_\varepsilon^{N,\Delta t}(i,n),$$

and for each N and Δt, the ε-uniform maximum nodal error is defined by $E_{N,\Delta t} = \max_\varepsilon E_{\varepsilon,N,\Delta t}$. Proceeding as in [14], the numerical rate of convergence is given by

$$p_{\varepsilon,N} = \frac{\log(E_{\varepsilon,N,\Delta t}/E_{\varepsilon,2N,\Delta t/2})}{\log 2},$$

and the numerical ε-uniform rate of convergence is

$$p_N = \frac{\log(E_{N,\Delta t}/E_{2N,\Delta t/2})}{\log 2}.$$

We show the results in Table 1.

In the second problem we take $a(x) = 2 - x^2$, $b(x) = x$, $u_0 = 0$, $f = 10t^2\,\mathrm{e}^{-t}x(1-x)$ and $T = 3$. Now, the exact solution is not known and we estimate the pointwise error by

$$\tilde{e}_\varepsilon^{N,\Delta t}(i,n) = |u^N(x_i,t_n) - u^{2N}(x_i,t_n)|,$$

Table 1
Maximum nodal errors and numerical order of convergence

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{0}	1.3076E-3 0.882	7.9078E-4 0.940	3.6986E-4 0.969	1.8894E-4 0.984	9.5517E-5 0.992	4.8028E-5
2^{-2}	1.7398E-2 0.845	9.6845E-3 0.924	5.1056E-3 0.961	2.6223E-3 0.981	1.3289E-3 0.990	6.6891E-4
2^{-4}	4.0133E-2 0.651	2.5552E-2 0.688	1.5865E-2 0.731	9.5603E-3 0.772	5.5999E-3 0.806	3.2019E-3
2^{-6}	5.9664E-2 0.675	3.7372E-2 0.778	2.1792E-2 0.816	1.2381E-2 0.829	6.9704E-3 0.836	3.9052E-3
2^{-8}	6.8794E-2 0.599	4.5409E-2 0.744	2.7112E-2 0.828	1.5275E-2 0.873	8.3396E-3 0.889	4.5045E-3
2^{-10}	7.1475E-2 0.564	4.8331E-2 0.706	2.9637E-2 0.795	1.7085E-2 0.852	9.4669E-3 0.891	5.1050E-3
2^{-12}	7.2175E-2 0.555	4.9125E-2 0.693	3.0386E-2 0.778	1.7720E-2 0.829	9.9766E-3 0.866	5.4749E-3
2^{-14}	7.2351E-2 0.553	4.9327E-2 0.691	3.0557E-2 0.772	1.7892E-2 0.821	1.0130E-2 0.853	5.6076E-3
2^{-16}	7.2396E-2 0.552	4.9378E-2 0.689	3.0630E-2 0.772	1.7937E-2 0.819	1.0170E-2 0.850	5.6439E-3
2^{-18}	7.2407E-2 0.552	4.9391E-2 0.689	3.0643E-2 0.772	1.7948E-2 0.818	1.0180E-2 0.848	5.6536E-3
2^{-20}	7.2410E-2 0.553	4.9394E-2 0.689	3.0646E-2 0.772	1.7951E-2 0.818	1.0183E-2 0.848	5.6561E-3
2^{-22}	7.2410E-2 0.552	4.9395E-2 0.689	3.0646E-2 0.772	1.7952E-2 0.818	1.0183E-2 0.848	5.6567E-3
2^{-24}	7.2411E-2 0.553	4.9395E-2 0.689	3.0647E-2 0.772	1.7952E-2 0.818	1.0184E-2 0.848	5.6568E-3
2^{-26}	7.2411E-2 0.553	4.9395E-2 0.689	3.0647E-2 0.772	1.7952E-2 0.818	1.0184E-2 0.848	5.6568E-3
$E_{N,\Delta t}$ p_N	7.2411E-2 0.553	4.9395E-2 0.689	3.0647E-2 0.772	1.7952E-2 0.818	1.0184E-2 0.848	5.6568E-3

Table 2
Maximum nodal errors and numerical order of convergence

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{0}	2.3143E-3 0.838	1.2944E-3 0.924	6.8229E-4 0.962	3.5016E-4 0.981	1.7736E-4 0.991	8.92588E-5
2^{-2}	1.1249E-2 0.832	6.3202E-3 0.914	3.3531E-3 0.958	1.7260E-3 0.979	8.7574E-4 0.989	4.4114E-4
2^{-4}	1.6783E-2 1.050	8.1043E-3 0.881	4.4011E-3 0.786	2.5525E-3 0.778	1.4883E-3 0.780	8.6694E-4
2^{-6}	3.090E-2 1.021	1.5221E-2 1.045	7.3746E-3 0.989	3.7162E-3 0.944	1.9318E-3 0.917	1.0230E-3
2^{-8}	3.5742E-2 0.885	1.9347E-2 0.955	9.9801E-3 0.979	5.0641E-3 0.980	2.5678E-3 0.971	1.3097E-3
2^{-10}	3.6717E-2 0.843	2.0475E-2 0.916	1.0851E-2 0.934	5.6789E-3 0.947	2.9448E-3 0.956	1.5184E-3
2^{-12}	3.6931E-2 0.833	2.0732E-2 0.906	1.1062E-2 0.924	5.8286E-3 0.933	3.0536E-3 0.931	1.6016E-3
2^{-14}	3.6982E-2 0.831	2.0794E-2 0.904	1.1111E-2 0.923	5.8618E-3 0.930	3.0767E-3 0.926	1.6192E-3
2^{-16}	3.6995E-2 0.830	2.0810E-2 0.904	1.1123E-2 0.922	5.8697E-3 0.930	3.0818E-3 0.925	1.6226E-3
2^{-18}	3.6998E-2 0.830	2.0814E-2 0.904	1.1126E-2 0.922	5.8716E-3 0.929	3.0830E-3 0.925	1.6233E-3
2^{-20}	3.6999E-2 0.830	2.0815E-2 0.904	1.1127E-2 0.922	5.8721E-3 0.929	3.0833E-3 0.925	1.6235E-3
2^{-22}	3.6999E-2 0.830	2.0815E-2 0.904	1.1127E-2 0.922	5.8722E-3 0.929	3.0834E-3 0.925	1.6235E-3
2^{-20}	3.6999E-2 0.830	2.0815E-2 0.904	1.1127E-2 0.922	5.8722E-3 0.929	3.0834E-3 0.925	1.6235E-3
2^{-20}	3.6999E-2 0.830	2.0815E-2 0.904	1.1127E-2 0.922	5.8722E-3 0.929	3.0834E-3 0.925	1.6235E-3
$\tilde{E}_{N,\Delta t}$ p_N	3.6999E-2 0.830	2.0815E-2 0.904	1.1127E-2 0.922	5.8722E-3 0.929	3.0834E-3 0.925	1.6235E-3

Table 3
Maximum nodal errors and numerical order of convergence

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{0}	5.8409E-3 0.652	3.7160E-3 0.798	2.1373E-3 0.889	1.1540E-3 0.941	6.0090E-4 0.970	3.0678E-4
2^{-2}	1.1388E-2 0.764	6.7069E-3 0.865	3.6824E-3 0.928	1.9358E-3 0.962	9.9350E-4 0.981	5.0343E-4
2^{-4}	2.1411E-2 0.816	1.2161E-2 0.854	6.7273E-3 0.883	3.6487E-3 0.915	1.9345E-3 0.926	1.0184E-3
2^{-6}	3.2531E-2 0.642	2.0842E-2 0.772	1.2206E-2 0.847	6.7845E-3 0.899	3.6374E-3 0.938	1.8987E-3
2^{-8}	3.5325E-2 0.543	2.4241E-2 0.653	1.5419E-2 0.773	9.0213E-3 0.861	4.9681E-3 0.925	2.6160E-3
2^{-10}	3.5716E-2 0.493	2.5384E-2 0.639	1.6300E-2 0.736	9.7878E-3 0.835	5.4853E-3 0.918	2.9025E-3
2^{-12}	3.5824E-2 0.482	2.5647E-2 0.636	1.6499E-2 0.726	9.9728E-3 0.827	5.6225E-3 0.909	2.9936E-3
2^{-14}	3.5857E-2 0.480	2.5711E-2 0.636	1.6546E-2 0.724	1.0016E-2 0.825	5.6528E-3 0.908	2.0133E-3
2^{-16}	3.5865E-2 0.479	2.5727E-2 0.636	1.6558E-2 0.724	1.0026E-2 0.825	5.6598E-3 0.907	2.0174E-3
2^{-18}	3.5867E-2 0.479	2.5730E-2 0.636	1.6560E-2 0.724	1.0028E-2 0.825	5.6615E-3 0.907	2.0183E-3
2^{-20}	3.5868E-2 0.479	2.5731E-2 0.636	1.6561E-2 0.724	1.0029E-2 0.825	5.6619E-3 0.907	2.0186E-3
2^{-22}	3.5868E-2 0.479	2.5732E-2 0.636	1.6561E-2 0.724	1.0029E-2 0.825	5.6620E-3 0.907	2.0186E-3
2^{-24}	3.5868E-2 0.479	2.5732E-2 0.636	1.6561E-2 0.724	1.0029E-2 0.825	5.6621E-3 0.907	2.0186E-3
2^{-26}	3.5868E-2 0.479	2.5732E-2 0.636	1.6561E-2 0.724	1.0029E-2 0.825	5.6621E-3 0.907	2.0186E-3
$\tilde{E}_{N,\Delta t}$ p_N	3.5868E-2 0.479	2.5732E-2 0.636	1.6561E-2 0.724	1.0029E-2 0.825	5.6621E-3 0.907	2.0186E-3

where $u^{2N}(x_i, t_n)$ is the solution on the corresponding Shishkin mesh with $2N$ points and half step time. Based on this error, we define, as in the first example, $\tilde{E}_{\varepsilon,N,\Delta t}$ and $\tilde{E}_{N,\Delta t}$. Note that since the meshes are not uniform, we use a linear interpolation to obtain u^{2N} on the coarse mesh. The results are given in Table 2.

Finally, in the third example we take $a(x)=2-x^2$, $b(x)=x^2+1+\cos(\pi x)$, $u_0=0$, $f=\sin(\pi x)$ and $T=1$. Again the exact solution is not known. The data of this problem do not satisfy the compatibility requirements imposed in our analysis; in this case, neither $\partial^2 u/\partial t^2$ nor $L^2_{x,\varepsilon}u$ are continuous functions at the points $(x,t)=(0,0)$ and $(1,0)$. Nevertheless, a similar behavior of numerical solutions is observed. The results are given in Table 3.

We wish to point out also that the restriction $N^{-q} \leqslant C\Delta t$, which we have imposed in the uniform convergence analysis, with $0<q<1$, seems not necessary to achieve the uniform behavior of the errors and the numerical order of uniform convergence.

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