Endomorphisms of graph algebras

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Abstract

We initiate a systematic investigation of endomorphisms of graph $C^*$-algebras $C^*(E)$, extending several known results on endomorphisms of the Cuntz algebras $O_n$. Most but not all of this study is focused on endomorphisms which permute the vertex projections and globally preserve the diagonal MASA $D_E$ of $C^*(E)$. Our results pertain both automorphisms and proper endomorphisms. Firstly, the Weyl group and the restricted Weyl group of a graph $C^*$-algebra are introduced and investigated. In particular, criteria of outerness for automorphisms in the restricted Weyl group are found. We also show that the restriction to the diagonal MASA of an automorphism which globally preserves both $D_E$ and the core AF-subalgebra eventually commutes with the corresponding one-sided shift. Secondly, we exhibit several properties of proper endomorphisms, investigate invertibility of localized endomorphisms both on $C^*(E)$ and in restriction to $D_E$, and develop a combinatorial approach to analysis of permutative endomorphisms.

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1. Introduction

The main aim of this article is to carry out systematic investigations of a certain natural class of endomorphisms and, in particular, automorphisms of graph $C^*$-algebras $C^*(E)$. Namely, we focus on those endomorphisms which globally preserve the diagonal MASA $D_E$. This leads to the concept of the Weyl group of a graph algebra, arising from the normalizer of a maximal abelian subgroup of the automorphism group of $C^*(E)$, consisting of those automorphisms which fix the diagonal $D_E$ point-wise. We investigate in depth an important subgroup of the Weyl group corresponding to those automorphisms which globally preserve the core AF-subalgebra $F_E$ as well – the restricted Weyl group of $C^*(E)$. We develop powerful novel techniques of both analytic and combinatorial nature for the study of automorphisms these groups comprise.

By analogy with the theory of semi-simple Lie groups, Cuntz introduced in [18] the Weyl group of the simple, purely infinite $C^*$-algebras $O_n$. Quite similarly with the classical theory, it arises as the quotient of the normalizer of a maximal abelian subgroup of the automorphism group of the algebra. So defined Weyl group is discrete albeit infinite, and the abelian subgroup in question is an inductive limit of higher dimensional tori. Cuntz posed a problem of determining the structure of the important subgroup of the Weyl group, corresponding to those automorphisms in the Weyl group which globally preserve the canonical UHF-subalgebra of $O_n$, [18]. After 30 years, this question has been finally answered in [9]. In the present paper, we take this programme one step further, expanding it from Cuntz algebras $O_n$ to a much wider class of graph $C^*$-algebras.

The theory of graph $C^*$-algebras began in earnest in the late nineties, [33,32], and since then it has developed into a fully fledged and very active area of research within operator algebras. For a good general introduction to graph $C^*$-algebras we refer the reader to [39]. In the case of finite graphs, the corresponding $C^*$-algebras essentially coincide with Cuntz–Krieger algebras, that were introduced much earlier, [20], in connection with topological Markov chains. The importance of graph algebras (or Cuntz–Krieger algebras) stems to a large extent from numerous applications they have found. Not trying to be exhaustive in any way, we only mention: their role in classification of purely infinite, simple $C^*$-algebras, [40,43], and related to that applications to the problem of semiprojectivity of Kirchberg algebras, [3,44,42]; their connection with objects of interest in noncommutative geometry and quantum group theory, [24,7,36]; their strong interplay with theory of symbolic dynamical systems, going back to the original paper of Cuntz and Krieger, [20,4]. It is also worth mentioning that graph $C^*$-algebras (similarly to Cuntz algebras) in many ways behave as purely combinatorial objects, and are thus strongly connected to their purely algebraic counterparts, the Leavitt path algebras, [1]. Therefore we believe that many of the results of the present paper are applicable to the latter algebras as well.

The analysis of endomorphisms of graph algebras, developed in the present article, owes a great deal to the close relationship between such algebras and the Cuntz algebras $O_n$. The $C^*$-algebras $O_n$ were first defined and investigated by Cuntz in his seminal paper [17], and they bear his name ever since. The Cuntz algebras have been extensively used in many a diverse contexts, including classification of $C^*$-algebras, quantum field theory, self-similar sets, wavelet theory, coding theory, spectral flow, subfactors and index theory, among others.
Systematic investigations of endomorphisms of $O_n$, $2 \leq n < \infty$, were initiated by Cuntz in [18]. A fundamental bijective correspondence $\lambda_u \leftrightarrow u$ between unital $*$-endomorphisms and unitaries in $O_n$ was established therein. Using this correspondence, Cuntz proved a number of interesting results, in particular with regard to those endomorphisms which globally preserve either the core UHF-subalgebra $F_n$ or the diagonal MASA $D_n$. Likewise, investigations of automorphisms of $O_n$ began almost immediately after the birth of the algebras in question, [18].

Endomorphisms of the Cuntz algebras have played a role in certain aspects of index theory, both from the $C^*$-algebraic and von Neumann algebraic point of view, e.g. see [27,29,13]. One of the most interesting applications of endomorphisms of $O_n$, found by Bratteli and Jørgensen in [5,6], is in the area of wavelets. In particular, permutative endomorphisms have been used in this context. These were further investigated by Kawamura, [31].

The present article builds directly on the progress made recently in the study of localized endomorphisms of $O_n$ by Conti, Szymański and their collaborators. In particular, much better understanding of those endomorphisms of $O_n$ which globally preserve the core UHF-subalgebra or the canonical diagonal MASA has been obtained in [14] and [23], respectively. In [45,15,12,8,10], a novel combinatorial approach to the study of permutative endomorphisms of $O_n$ has been introduced, and subsequently significant progress in the investigations of such endomorphisms and automorphisms has been obtained. In particular, a striking relationship between permutative automorphisms of $O_n$ and automorphisms of the full two-sided $n$-shift has been found in [9].

In the present article, we extend this analysis of endomorphisms of the Cuntz algebras to the much larger class of graph $C^*$-algebras. Most of our results (but not all) are concerned with algebras corresponding to finite graphs without sinks, which may be identified with Cuntz–Krieger algebras of finite 0–1 matrices, [20]. From the very beginning, the theory of such algebras has been closely related to dynamical systems. In particular, endomorphisms of Cuntz–Krieger algebras have been studied in the context of index theory, [28]. Quasi-free automorphisms of Cuntz–Krieger algebras (and even more generally, Cuntz–Pimsner algebras) have been studied in [30,46,21]. An interesting connection between automorphisms of Cuntz–Krieger algebras and Markov shifts has been investigated by Matsumoto, [34,35].

The present paper is organized as follows. In Section 2, we set up notation and present some preliminaries. In particular, we introduce a class of endomorphisms $\{\lambda_u: u \in U_E\}$ of a graph $C^*$-algebra $C^*(E)$ corresponding to unitaries commuting with the vertex projections (whose set we denote $U_E$).

In Section 3, an analogue of the Weyl group for graph $C^*$-algebras is introduced and investigated. Namely, let $D_E$ be the canonical abelian subalgebra of $C^*(E)$. Then, under a mild hypothesis, the group Aut$_{D_E}$($C^*(E)$) of those automorphisms of $C^*(E)$ which fix $D_E$ pointwise is a maximal abelian subgroup of Aut($C^*(E)$) (Propositions 3.2 and 3.3). The Weyl group $W_E$ of $C^*(E)$ is defined as the quotient of the normalizer of Aut$_{D_E}$($C^*(E)$) by itself. Then we exhibit several structural properties of the Weyl group, analogous to those discovered by Cuntz in the case of $O_n$, [18]. In particular, the Weyl group of $C^*(E)$ is countable and discrete (Proposition 3.5).

In Section 4, we investigate an important subgroup of the Weyl group corresponding to those automorphisms which globally preserve both the diagonal $D_E$ and the core AF-subalgebra $F_E$, the restricted Weyl group $RW_E$ of $C^*(E)$. We prove certain facts about the normalizer of $F_E$ (Theorem 4.2) and give a characterization of automorphisms globally preserving $F_E$ (Lemma 4.5). We obtain a very convenient criterion of outerness for a large class of such automorphisms (Corollary 4.7). Then we take a closer look at the action on the diagonal $D_E$ induced by the restriction of those automorphisms of the graph algebra which preserve both the diagonal
and the core AF-subalgebra. By earlier results of Cuntz and Krieger and Matsumoto, [20,34,35], it is known that all shift commuting automorphisms of the diagonal $D_E$ extend to automorphisms of $C^*(E)$. In the case of the Cuntz algebra, it was shown in [9] that all automorphisms of the diagonal which (along with their inverses) eventually commute with the shift can be extended to automorphisms of $O_n$. In the present paper, we show that the automorphisms of $D_E$ arising from restrictions of AF-subalgebra $F_E$ preserving automorphisms of $C^*(E)$ eventually commute with the shift (Theorem 4.13). This provides a solid starting point for a future complete determination of the restricted Weyl group $RW_E$ of a graph $C^*$-algebra $C^*(E)$.

In Section 5, we study localized endomorphisms of a graph algebra $C^*(E)$, that is endomorphisms $\lambda_u$ corresponding to unitaries $u$ from the algebraic part of the core AF-subalgebra which commute with the vertex projections. We obtain an algorithmic criterion of invertibility of such localized endomorphisms (Theorem 5.1), as well as a criterion of invertibility of the restriction of a localized endomorphism to the diagonal MASA (Theorem 5.3). These theorems extend the analogous results for Cuntz algebras obtained earlier in [15]. Localized endomorphisms constitute the best understood and most studied class of endomorphisms of the Cuntz algebras, and it is natural to begin systematic investigations of endomorphisms of graph $C^*$-algebras from them.

In Section 6, a special class of localized endomorphisms corresponding to permutation unitaries is investigated. Automorphisms of $C^*(E)$ of this type give rise to a large and interesting subgroup of the restricted Weyl group $RW_E$. For a permutative endomorphism $\lambda_u$ of a graph algebra $C^*(E)$, combinatorial criteria are given for $\lambda_u$ to be an automorphism of $C^*(E)$ or its restriction to be an automorphism of the diagonal $D_E$ (Lemma 6.1, Lemma 6.3 and Theorem 6.4). In this way, we obtain a far-reaching generalization of the techniques developed by Conti and Szymański in [15] for dealing with permutative endomorphisms of the Cuntz algebras. A few examples are worked out in detail, illustrating applications of the combinatorial machinery developed in the present paper.

2. Notation and preliminaries

2.1. Directed graphs and their $C^*$-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph, where $E^0$ and $E^1$ are (countable) sets of vertices and edges, respectively, and $r, s : E^1 \to E^0$ are range and source maps, respectively. The $C^*$-algebra $C^*(E)$ corresponding to a graph $E$ is by definition the universal $C^*$-algebra generated by mutually orthogonal projections $P_v, v \in E^0$, and partial isometries $S_e, e \in E^1$, subject to the following relations

\[(GA1)\quad S_e^*S_e = P_{r(e)} \quad \text{and} \quad S_f^*S_e = 0 \quad \text{if} \quad e \neq f \in E^1,\]

\[(GA2)\quad S_eS_e^* \leq P_{s(e)} \quad \text{for} \quad e \in E^1,\]

\[(GA3)\quad P_v = \sum_{s(e)=v} S_eS_e^* \quad \text{if} \quad v \in E^0 \text{ emits finitely many and at least one edge.}\]

It follows from the above definition that a graph $C^*$-algebra $C^*(E)$ is unital if and only if the underlying graph $E$ has finitely many vertices, in which case $1 = \sum_{v \in E^0} P_v$.

A path $\mu$ of length $|\mu| = k \geq 1$ is a sequence $\mu = (\mu_1, \ldots, \mu_k)$ of $k$ edges $\mu_j$ such that $r(\mu_j) = s(\mu_{j+1})$ for $j = 1, \ldots, k - 1$. We also view the vertices as paths of length 0. The set of all paths of length $k$ is denoted $E^k$. The range and source maps naturally extend from edges $E^1$ to paths $E^k$. A sink is a vertex $v$ which emits no edges, i.e. $s^{-1}(v) = \emptyset$. A source is a vertex $w$
which receives no edges, i.e. \( r^{-1}(v) = \emptyset \). By a loop we mean a path \( \mu \) of length \( |\mu| \geq 1 \) such that \( s(\mu) = r(\mu) \). We say that a loop \( \mu = (\mu_1, \ldots, \mu_k) \) has an exit if there is a \( j \) such that \( s(\mu_j) = r(\mu) \) emits at least two distinct edges.

As usual, for a path \( \mu = (\mu_1, \ldots, \mu_k) \) of length \( k \) we denote by \( S_\mu = S_{\mu_1} \cdots S_{\mu_k} \) the corresponding partial isometry in \( C^*(E) \). It is known that each \( S_\mu \) is non-zero, with the domain projection \( P_{r(\mu)} \). Then \( C^*(E) \) is the closed span of \( \{ S_\mu S_\nu^* : \mu, \nu \in E^* \} \), where \( E^* \) denotes the collection of all finite paths (including paths of length zero). Here we agree that \( S_\nu = P_v \) for \( v \in E^0 \) viewed as path of length 0. Also note that \( S_\mu S_\nu^* \) is non-zero if and only if \( r(\mu) = r(\nu) \). In that case, \( S_\mu S_\nu^* \) is a partial isometry with domain and range projections equal to \( S_v S_v^* \) and \( S_\mu S_\nu^* \), respectively.

The range projections \( P_\mu = S_\mu S_\mu^* \) of all partial isometries \( S_\mu \) mutually commute, and the abelian \( C^* \)-subalgebra of \( C^*(E) \) generated by all of them is called the diagonal subalgebra and denoted \( D_E \). If \( E \) does not contain sinks and all loops have exits then \( D_E \) is a MASA (maximal abelian subalgebra) in \( C^*(E) \) by \([25, \text{Theorem 5.2}]\). We set \( D_E^0 = \text{span} \{ P_v : v \in E^0 \} \) and, more generally, \( D_E^k = \text{span} \{ P_{\mu} : \mu \in E^k \} \) for \( k \geq 0 \).

There exists a strongly continuous action \( \gamma \) of the circle group \( U(1) \) on \( C^*(E) \), called the gauge action, such that \( \gamma_t(S_e) = t S_e \) and \( \gamma_t(P_v) = P_v \) for all \( e \in E^1, v \in E^0 \) and \( t \in U(1) \subseteq \mathbb{C} \). The fixed-point algebra \( C^*(E)^\gamma \) for the gauge action is an AF-algebra, denoted \( F_E \) and called the core AF-subalgebra of \( C^*(E) \). \( F_E \) is the closed span of \( \{ S_\mu S_\nu^* : \mu, \nu \in E^*, |\mu| = |\nu| \} \). For \( k \in \mathbb{N} = \{ 0, 1, 2, \ldots \} \) we denote by \( F_E^k \) the linear span of \( \{ S_\mu S_\nu^* : \mu, \nu \in E^*, |\mu| = |\nu| = k \} \). For an integer \( m \in \mathbb{Z} \) we denote by \( C^*(E)^{(m)} \) the spectral subspace of the gauge action corresponding to \( m \). That is, \( C^*(E)^{(m)} := \{ x \in C^*(E) : \gamma_z(x) = z^m x, \forall z \in U(1) \} \). In particular, \( C^*(E)^{(0)} = C^*(E)^\gamma \).

2.2. Endomorphisms determined by unitaries

We denote by \( \mathcal{U}_E \) the collection of all those unitaries in the multiplier algebra \( M(C^*(E)) \) which commute with all vertex projections \( P_v, v \in E^0 \). That is

\[
\mathcal{U}_E := \mathcal{U}(\{ D_E^0 \} \cap M(C^*(E))).
\] (1)

These unitaries will play a crucial role throughout this paper. Let \( u \in \mathcal{U}_E \). Then \( u S_e, e \in E^1 \), are partial isometries in \( C^*(E) \) which together with projections \( P_v, v \in E^0 \), satisfy (GA1)–(GA3). Thus, by universality in the definition of \( C^*(E) \), there exists a \(*\)-homomorphism \( \lambda_u : C^*(E) \to C^*(E) \) such that

\[
\lambda_u(S_e) = u S_e \quad \text{and} \quad \lambda_u(P_v) = P_v, \quad \text{for } e \in E^1, \ v \in E^0.
\] (2)

Clearly, \( \lambda_u(1) = 1 \) whenever \( C^*(E) \) is unital. In general, \( \lambda_u \) may be neither injective nor surjective. However, the following proposition is an immediate consequence of the gauge-invariant uniqueness theorem \([2, \text{Theorem 2.1}]\).

**Proposition 2.1.** If \( u \in \mathcal{U}_E \) belongs to the minimal unitization of the core AF-subalgebra \( F_E \), then endomorphism \( \lambda_u \) is automatically injective.

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4 The reader should be aware that in some papers (e.g. in \([18,45,15]\)) a different convention: \( \lambda_u(S_e) = u^* S_e \) is used.
Note that \( \{ \lambda_u : u \in \mathcal{U}_E \} \) is a semigroup with the following multiplication law:

\[
\lambda_u \circ \lambda_w = \lambda_{uw}, \quad u \ast w = \lambda_u(w)u.
\] (3)

We say \( \lambda_u \) is invertible if \( \lambda_u \) is an automorphism of \( C^*(E) \). For \( K \subset \mathcal{U}_E \) we denote \( \lambda(K)^{-1} := \{ \lambda_u \in \text{Aut}(C^*(E)) : u \in K \} \). It turns out that \( \lambda_u \) is invertible if and only if it is injective and \( u^* \) is in the range of its unique strictly continuous extension to \( M(C^*(E)) \), still denoted \( \lambda_u \). (Such an extension exists since \( \lambda_u \) fixes each vertex projection and thus it fixes an approximate unit comprised of finite sums of vertex projections.) Indeed, if \( \lambda_u \) is injective and there exists a \( w \in M(C^*(E)) \) such that \( \lambda_u(w) = u^* \) then \( w \) must belong to \( \mathcal{U}_E \) and \( \lambda_u \ast w = \lambda_{uw} = \text{id} \). Thus \( \lambda_u \) is surjective and hence an automorphism. In this case, we have \( \lambda_u^{-1} = \lambda_w \).

In the present paper, we mainly deal with finite graphs without sinks. If \( E \) is such a graph then the mapping \( u \mapsto \lambda_u \) establishes a bijective correspondence between \( \mathcal{U}_E \) and the semigroup of those unital \( * \)-homomorphisms from \( C^*(E) \) into itself which fix all the vertex projections \( P_v \), \( v \in E_0 \). Indeed, if \( \rho \) is such a homomorphism then \( u := \sum_{e \in E^1} \rho(S_e)S_e^* \) belongs to \( \mathcal{U}_E \) and \( \rho = \lambda_u \) (cf. [46]). If \( u \in \mathcal{F}_E \cap \mathcal{U}_E \) then \( \lambda_u \) is automatically invertible with inverse \( \lambda_{u^*} \) and the map

\[
\mathcal{F}_E \cap \mathcal{U}_E \ni u \mapsto \lambda_u \in \text{Aut}(C^*(E))
\] (4)

is a group homomorphism with range inside the subgroup of quasi-free automorphisms of \( C^*(E) \), see [21,46].

If \( \lambda_u \) is an endomorphism of \( C^*(E) \) corresponding to a unitary \( u \) in the linear span of \( \{ S_\mu S_\nu^* : \mu, \nu \in E^*, |\mu| = |\nu| \} \) and the identity (i.e. in the minimal unitization of the algebraic part of the core AF-subalgebra), then we call \( \lambda_u \) localized, cf. [13].

Let \( E \) be a finite graph without sinks, and let

\[
\varphi(x) = \sum_{e \in E^1} S_e x S_e^*
\] (5)

be the usual shift on \( C^*(E) \), [20]. It is a unital, completely positive map. One can easily verify that the shift is an injective \( * \)-homomorphism when restricted to the relative commutant of \( \{ P_v : v \in E_0 \} \). We will denote this relative commutant by \( B_E \), i.e.

\[
B_E := (D_E^0)' \cap C^*(E).
\] (6)

In particular, \( \mathcal{U}_E = \mathcal{U}(B_E) \). We have \( \varphi(B_E) \subseteq B_E \) and thus \( \varphi(\mathcal{U}_E) \subseteq \mathcal{U}_E \). It is also clear that \( \varphi(\mathcal{F}_E) \subseteq \mathcal{F}_E \) and \( \varphi(\mathcal{D}_E) \subseteq \mathcal{D}_E \). For \( k \geq 1 \) we denote

\[
u_k := u \varphi(u) \cdots \varphi^{k-1}(u),
\] (7)

and agree that \( u_k^* \) stands for \( (u_k)^* \). For each \( u \in \mathcal{U}_E \) and for any two paths \( \mu, \nu \in E^* \) we have

\[
\lambda_u(S_\mu S_\nu^*) = u_{|\mu|} S_\mu S_\nu^* u_{|\nu|}^*.
\] (8)
The above equality is established with help of the identity $\varphi(u)S_e = Seu\varphi(u)$, which holds for all $e \in E^1$ because the unitary $u$ commutes with all vertex projections $P_v$, $v \in E^0$, by hypothesis. Indeed,

$$\varphi(u)S_e = \left( \sum_{f \in E^1} S_f u S^*_f \right) S_e = S_e u S^*_e S_e = S_e P(e) = S_e P_r(e) = S_e u = S_e u.$$ 

More generally, if $x \in C^*(E)$ commutes with all vertex projections then

$$S_\alpha x = \varphi|\alpha|(x) S_\alpha$$

for each finite path $\alpha$. Furthermore, we have

$$\text{Ad}(u) = \lambda u \varphi(u)$$

for $u \in U_E$, (10)

3. The Weyl group

For algebras $A \subseteq B$ (with a common identity element) we denote by $N_B(A) = \{u \in U(B) : uAu^* = A\}$ the normalizer of $A$ in $B$, and by $A' \cap B = \{b \in B : (\forall a \in A) ab = ba\}$ the relative commutant of $A$ in $B$.

**Proposition 3.1.** Let $E$ be a finite graph without sinks, and let $u \in U_E$. Then the following hold:

1. If $uD_Eu^* \subseteq D_E$ then $\lambda_u(D_E) \subseteq D_E$.
2. If $\lambda_u(D_E) = D_E$ then $uD_Eu^* \subseteq D_E$.

**Proof.** Part (1) is established in the same way as the analogous statement about $O_n$ in [18].

For part (2), first note that $D_E$ is a $C^*$-algebra generated by $\{\varphi^k(S_e S^*_e) : e \in E^1, k = 0, 1, 2, \ldots\}$, where $\varphi^0 = \text{id}$. In particular, $D_E$ is generated by $D^1_E$ and $\varphi(D_E)$. Now, if $u \in U_E$ and $\lambda_u(D_E) = D_E$ then $uD^1_Eu^* = \lambda_u(D^1_E) \subseteq D_E$ and $u\varphi(D_E)u^* = u\varphi(\lambda_u(D_E))u^* = \lambda_u(\varphi(D_E)) \subseteq D_E$. Hence $uD_Eu^* \subseteq D_E$. \(\square\)

Further note that if $E$ is a finite graph without sinks in which every loop has an exit then $D_E$ is a MASA in $C^*(E)$ and in part (2) of Proposition 3.1 we can further conclude that $uD_Eu^* = D_E$, i.e. that $u \in N_{C^*(E)}(D_E)$. In that case, by virtue of [25, Theorem 10.1], every $u \in N_{C^*(E)}(D_E)$ can be uniquely written as $u = dw$, where $d \in U(D_E)$ and $w \in S_E$. Here $S_E$ denotes the group of all unitaries in $U(C^*(E))$ of the form $\sum S_\alpha S^*_\beta$ (finite sum). Therefore, one can show, as in [15, Section 2] that the normalizer of the diagonal in $C^*(E)$ is a semi-direct product

$$N_{C^*(E)}(D_E) = U(D_E) \rtimes S_E.$$ (11)

For $C^*$-algebras $A \subseteq B$ (with a common identity), we denote by $\text{Aut}(B, A)$ the collection of all those automorphisms $\alpha$ of $B$ such that $\alpha(A) = A$, and by $\text{Aut}_A(B)$ those automorphisms of $B$ which fix $A$ point-wise. Similarly, $\text{End}_A(B)$ denotes the collection of all unital $*$-homomorphisms of $B$ which fix $A$ point-wise.
Proposition 3.2. Let $E$ be a finite graph without sinks in which every loop has an exit. Then the mapping $u \mapsto \lambda_u$ establishes a group isomorphism
\[
\text{End}_{D_E}(C^*(E)) = \text{Aut}_{D_E}(C^*(E)) \cong \mathcal{U}(D_E).
\]

Proof. It follows immediately from formula (3) that the mapping $u \mapsto \lambda_u$ is a group homomorphism from $\mathcal{U}(D_E)$ into $\text{Aut}(C^*(E))$. If $\mu$ is a finite path then $\lambda_u(P_\mu) = \text{Ad}(u|_\mu)(P_\mu)$ by (8). Hence, if $u \in D_E$, then $\lambda_u$ fixes $D_E$ point-wise. Consequently, the mapping $u \mapsto \lambda_u$ establishes a group isomorphism into Aut$_{D_E}(C^*(E))$. The map is one-to-one, as already noted in Section 2. To see that the map is onto End$_{D_E}(C^*(E))$, recall (again from Section 2) that every endomorphism fixing the vertex projections is of the form $\lambda_w$ for some $w \in \mathcal{U}_E$. Since $\lambda_w$ fixes $D_E$ point-wise, proceeding by induction on $|\mu|$ one shows that $w$ commutes with all projections $P_\mu$. Thus $w \in \mathcal{U}(D_E)$, since $D_E$ is a MASA in $C^*(E)$ by [25, Theorem 5.2]. \hfill \Box

We note that under the hypothesis of Proposition 3.2 the fixed-point algebra for the action of Aut$_{D_E}(C^*(E))$ on $C^*(E)$ equals $D_E$. Indeed, each element in this fixed-point algebra is also fixed by all Ad$(u)$, $u \in \mathcal{U}(D_E)$, and thus belongs to $D_E$.

For the proof of the following Proposition 3.3, we note the following simple fact. If $E$ is a finite graph without sinks in which every loop has an exit then $D_E$ does not contain minimal projections. Indeed, if $p$ is a non-zero projection in $D_E$ then there exists a path $\alpha$ such that $P_\alpha \lesssim p$. Extend $\alpha$ to a path ending at a loop, and then further to a path $\beta$ which ends at a vertex emitting at least two edges, say $e$ and $f$. Then for the path $\mu = \beta e$ we have $P_\mu \lesssim P_e \lesssim p$.

Proposition 3.3. Let $E$ be a finite graph without sinks in which every loop has an exit. Then the normalizer of Aut$_{D_E}(C^*(E))$ in Aut$(C^*(E))$ coincides with Aut$(C^*(E), D_E)$. If, in addition, the center of $C^*(E)$ is trivial, then Aut$_{D_E}(C^*(E))$ is a maximal abelian subgroup of Aut$(C^*(E))$.

Proof. Let $\alpha \in \text{Aut}(C^*(E))$ be in the normalizer of Aut$_{D_E}(C^*(E))$. Thus for each $w \in \mathcal{U}(D_E)$ there is a $u \in \mathcal{U}(D_E)$ such that $\alpha \lambda_u = \lambda_u \alpha$. Then $\alpha(d) = \alpha \lambda_u(d) = \lambda_u \alpha(d)$ for all $d \in D_E$. Whence $\alpha(D_E) \subseteq D_E$. Replacing $\alpha$ with $\alpha^{-1}$ we get the reverse inclusion, and thus $\alpha(D_E) = D_E$. This proves the first part of the proposition.

Now let $\alpha \in \text{Aut}(C^*(E))$ commute with all elements of Aut$_{D_E}(C^*(E))$. Then, in particular, $\text{Ad}(u)\alpha(x) = \alpha \text{Ad}(u)(x)$ for all $u \in \mathcal{U}(D_E)$ and $x \in C^*(E)$. Thus $u^*\alpha(u)$ belongs to the center of $C^*(E)$ and hence it is a scalar. Therefore for each projection $p \in D_E$ we have $\alpha(2p - 1) = \pm(2p - 1)$, and hence $\alpha(p)$ equals either $p$ or $1 - p$. The case $\alpha(p) = 1 - p$ is impossible, since taking a projection $0 \not\lesssim q \lesssim p$ we would get $\alpha(q) \lesssim 1 - p$ and thus $\alpha(q) \not\in \{q, 1 - q\}$. Hence $\alpha$ fixes all projections, and thus all elements of $D_E$. The claim now follows from Proposition 3.2. \hfill \Box

The quotient of Aut$(C^*(E), D_E)$ by Aut$_{D_E}(C^*(E))$ will be called the Weyl group of $C^*(E)$ (cf. [18]), and denoted $\mathfrak{W}_E$. That is,
\[
\mathfrak{W}_E := \text{Aut}(C^*(E), D_E) / \text{Aut}_{D_E}(C^*(E)).
\]

Remark 3.4. Since triviality of the center of $C^*(E)$ plays a role in our considerations, it is worthwhile to mention the following. If $E$ is a finite graph without sinks in which every loop has an exit then the following conditions are equivalent.
\(\alpha(P(e,\mu))\)

(i) The center of \(C^*(E)\) is trivial.
(ii) \(C^*(E)\) is indecomposable. That is, there is no decomposition \(C^*(E) \cong A \oplus B\) into direct sum of two non-zero \(C^*\)-algebras.
(iii) There are no two non-empty, disjoint, hereditary and saturated subsets \(X\) and \(Y\) of \(E^0\) such that: for each vertex \(v \in E^0 \setminus (X \cup Y)\) there is no loop through \(v\), and there exist paths from \(v\) to both \(X\) and \(Y\).

Indeed, it is shown in [22] that conditions (ii) and (iii) above are equivalent. Obviously, (i) implies (ii). To see that (ii) implies (i), we first note that in the present case \(D_E\) is a MASA, [25], and thus it contains the center \(Z(C^*(E))\) of \(C^*(E)\). Thus, the (closed, two-sided) ideal of \(C^*(E)\) generated by any element of \(Z(C^*(E))\) is gauge-invariant. But for a finite graph \(E\), \(C^*(E)\) contains only finitely many gauge-invariant ideals, [26] (see also [2] for the complete description of gauge-invariant ideals for arbitrary graphs). Hence \(Z(C^*(E))\) is finite dimensional. Thus if \(Z(C^*(E))\) is non-trivial then it contains a non-trivial projection and consequently \(C^*(E)\) is decomposable.

Just as in the case of the Cuntz algebras, the Weyl group of a graph algebra turns out to be countable.

**Proposition 3.5.** Let \(E\) be a finite graph. Then the Weyl group \(\mathfrak{W}_E\) is countable.

**Proof.** For each coset in the quotient \(\text{Aut}(C^*(E), D_E)/\text{Aut}_{D_E}(C^*(E))\) choose a representative \(\alpha\) and define a mapping from \(\text{Aut}(C^*(E), D_E)/\text{Aut}_{D_E}(C^*(E))\) to \(\bigoplus_{|E^1|+|E^0|} C^*(E)\) by \(\alpha \mapsto \bigoplus_e \alpha(S_e) \bigoplus_v \alpha(P_v)\). This mapping is one-to-one and the target space is separable. Thus it suffices to show that its image is a discrete subset of \(\bigoplus_{|E^1|+|E^0|} C^*(E)\).

Let \(\alpha \in \text{Aut}(C^*(E), D_E)\) be such that \(||\alpha(x) - x|| < 1/2\) for all \(x \in \{S_e: e \in E^1\} \cup \{P_v: v \in E^0\}\). We claim that \(\alpha|_{D_E} = \text{id}\). To this end, we show by induction on \(|\mu|\) that \(\alpha(P_\mu) = P_\mu\) for each path \(\mu\). Indeed, if \(v \in E^0\) then \(||\alpha(P_v) - P_v|| < 1/2\) and thus \(\alpha(P_v) = P_v\) since \(\alpha(P_v) \in D_E\) and \(D_E\) is commutative. This establishes the base for induction. Now suppose that \(\alpha(P_\mu) = P_\mu\) for all paths \(\mu\) of length \(k\). Let \((e, \mu)\) be a path of length \(k + 1\). Then, by the inductive hypothesis, we have

\[
\|\alpha(P_{(e,\mu)}) - P_{(e,\mu)}\| = \|\alpha(S_e)P_\mu \alpha(S_e^*) - S_eP_\mu S_e^*\| \\
\leq \|\alpha(S_e)P_\mu \alpha(S_e^*) - \alpha(S_e)P_\mu S_e^*\| + \|\alpha(S_e)P_\mu S_e^* - S_eP_\mu S_e^*\| \\
< \frac{1}{2} + \frac{1}{2} = 1.
\]

Thus \(\alpha(P_{(e,\mu)}) = P_{(e,\mu)}\), since \(\alpha(P_{(e,\mu)})\) and \(P_{(e,\mu)}\) commute. This yields the inductive step.

Now suppose that \(\alpha, \beta \in \text{Aut}(C^*(E), D_E)\) are such that \(||\alpha(x) - \beta(x)|| < 1/2\) for all \(x \in \{S_e: e \in E^1\} \cup \{P_v: v \in E^0\}\). Then also \(||\beta^{-1}\alpha(x) - x|| < 1/2\) for all such \(x\), and hence \(\beta^{-1}\alpha \in \text{Aut}_{D_E}(C^*(E))\) by the preceding paragraph. This completes the proof. \(\square\)

For the remainder of this section, we assume that \(E\) is a finite graph without sinks in which every loop has an exit. Clearly, each automorphism \(\alpha\) of the graph \(E\) gives rise to an automorphism of the algebra \(C^*(E)\), still denoted \(\alpha\), such that \(\alpha(S_e) = S_{\alpha(e)}\), \(e \in E^1\), and \(\alpha(P_v) = P_{\alpha(v)}\).
v ∈ E^0. With a slight abuse of notation, we are identifying automorphisms of graph E with the corresponding automorphisms of C*-algebra C*(E), and denote this group Γ_E.

We denote by \( \mathcal{G}_E \) the subgroup of Aut(C*(E)) generated by automorphisms of the graph E and \( \lambda(S_E ∩ U_E)^{-1} \). That is,

\[
\mathcal{G}_E := \{ \lambda(S_E ∩ U_E)^{-1} ∪ Γ_E \} ⊆ \text{Aut}(C*(E)).
\] (13)

If \( u ∈ S_E ∩ U_E \) then \( \lambda_u(\mathcal{D}_E) ⊆ \mathcal{D}_E \). Consequently, if \( u ∈ S_E ∩ U_E \) and \( \lambda_u \) is invertible then \( \lambda_u \) belongs to Aut(C*(E), \( \mathcal{D}_E \)), since \( \mathcal{D}_E \) is a MASA in C*(E). Since each graph automorphism gives rise to an element of Aut(C*(E), \( \mathcal{D}_E \)) as well, we have \( \mathcal{G}_E ⊆ \text{Aut}(C*(E), \mathcal{D}_E) \). We would like to stress that the class of automorphisms comprising \( \mathcal{G}_E \) can be viewed as ‘purely combinatorial’ transformations, which facilitates their algorithmic analysis.

Now, let \( u ∈ S_E ∩ U_E \) be such that \( \lambda_u \) is invertible. Then, as noted in Section 2 above, \( \lambda_u^{-1} = \lambda_w \) for some \( w ∈ U_E \) and \( w ∈ N_{C^*(E)}(\mathcal{D}_E) \) by Proposition 3.1. Let \( w = dz \) with \( d ∈ \mathcal{U}(\mathcal{D}_E) \) and \( z ∈ S_E \). Then we have \( z ∈ S_E ∩ U_E \). Since \( \lambda_u dz = \lambda_u λ_d z = \text{id} = λ_1 \), we have \( \lambda_u(d)λ_u(z)u = u * dz = 1 \). Thus \( d = 1 \), by (11), and \( \lambda_u^{-1} = \lambda_z \). This shows that \( \lambda(S_E ∩ U_E)^{-1} \) is a group. If \( \alpha \) is an automorphism of \( E \) and \( u ∈ S_E ∩ U_E \) then \( αλ_u α^{-1} = \lambda_w \) for some \( w ∈ U_E \), since \( αλ_u α^{-1} \) fixes all the vertex projections. A short calculation shows that this \( w \) belongs to \( S_E \). It follows that \( \lambda(S_E ∩ U_E)^{-1} \) is a normal subgroup of \( \mathcal{G}_E \). Let \( Γ_E^0 \) be the normal subgroup of \( Γ_E \) consisting of those automorphisms which fix all vertex projections. It is easy to verify that \( Γ_E^0 = λ(\mathcal{P}_E^1 ∩ U_E) \), where we denote \( \mathcal{P}_E^1 = \mathcal{F}_E^1 ∩ S_E \). We have

\[
\mathcal{G}_E = λ(S_E ∩ U_E)^{-1} Γ_E \quad \text{and} \quad λ(S_E ∩ U_E)^{-1} ∩ Γ_E = Γ_E^0.
\] (14)

Proposition 3.6. Let \( E \) be a finite graph without sinks in which every loop has an exit. Then there is a natural embedding of \( \mathcal{G}_E \) into the Weyl group \( \mathbb{W}_E \) of C*(E).

Proof. Since \( \mathcal{G}_E ⊆ \text{Aut}(C*(E), \mathcal{D}_E) \), it suffices to show that \( \mathcal{G}_E ∩ \text{Aut}_{\mathcal{D}_E}(C^*(E)) = \{ \text{id} \} \). Indeed, if \( β ∈ \mathcal{G}_E \) then \( β = λ_u α \) for some \( u ∈ S_E ∩ U_E \) and \( α ∈ \text{Aut}(E) \). If, in addition, \( β ∈ \text{Aut}_{\mathcal{D}_E}(C^*(E)) \) then \( α ∈ Γ_E^0 \) and thus \( α = λ_w \) for some \( w ∈ \mathcal{P}_E^1 ∩ U_E \). Therefore \( β = λ_u λ_w = λ_{u+w} \) and \( u * w ∈ S_E \). By Proposition 3.2 we also have \( β = λ_d \) for some \( d ∈ \mathcal{U}(\mathcal{D}_E) \). Consequently \( u * w = d = 1 \), since \( S_E ∩ \mathcal{U}(\mathcal{D}_E) = \{ 1 \} \), and thus \( β = \text{id} \).

One of the more difficult issues arising in dealing with automorphisms of graph C*-algebras is deciding if the automorphism at hand is outer or inner. The following theorem provides a criterion useful for certain automorphisms belonging to the subgroup \( \mathcal{G}_E \) of the Weyl group \( \mathbb{W}_E \), see Corollary 4.7 below.

Theorem 3.7. Let \( E \) be a finite graph without sinks in which every loop has an exit. Then

\[
\mathcal{G}_E ∩ \text{Inn}(C^*(E)) ⊆ \{ \text{Ad}(w): w ∈ S_E \}.
\]

Proof. By (14), each element of \( \mathcal{G}_E \) is of the form \( λ_u α \), with \( u ∈ S_E ∩ U_E \) and \( α ∈ Γ_E \). If such an automorphism is inner and equals Ad(y) for some \( y ∈ \mathcal{U}(C^*(E)) \) then \( y ∈ N_{C^*(E)}(\mathcal{D}_E) \), and thus we have \( λ_u α = \text{Ad}(d) \text{Ad}(w) \) for some \( d ∈ \mathcal{U}(\mathcal{D}_E) \) and \( w ∈ S_E \), by (11). Hence \( ρ := λ_u α \text{Ad}(w^*) = λ_d φ(d^*) \). But then \( d φ(d^*) = \sum_{v ∈ E^1} ρ(S_v) S_v^* ∈ S_E \). Since \( S_E ∩ \mathcal{U}(\mathcal{D}_E) = \{ 1 \} \), we
have $d \varphi (d^*) = 1$. Therefore $d = \varphi (d)$ and this implies (via a straightforward calculation) that $d$ belongs to the center of $C^*(E)$. Hence $\text{Ad}(u) = \text{id}$ and thus $\lambda_u \alpha = \text{Ad}(w)$, as required.

4. The restricted Weyl group

In order to define and study the restricted Weyl group of a graph $C^*$-algebra, we need some preparation on endomorphisms which globally preserve its core AF-subalgebra. Recall that $\gamma : U(1) \to \text{Aut}(C^*(E))$ is the gauge action, such that $\gamma_t(S_e) = zS_e$ for all $e \in E^1$ and $z \in U(1)$. Then $\mathcal{F}_E$ is the fixed-point algebra for action $\gamma$.

The same argument as in Proposition 3.1 yields the following.

**Proposition 4.1.** Let $E$ be a finite graph without sinks, and let $u \in U_E$. Then the following hold:

1. If $u \mathcal{F}_E u^* \subseteq \mathcal{F}_E$ then $\lambda_u(\mathcal{F}_E) \subseteq \mathcal{F}_E$.
2. If $\lambda_u(\mathcal{F}_E) = \mathcal{F}_E$ then $u \mathcal{F}_E u^* \subseteq \mathcal{F}_E$.

It turns out that in many instances the normalizer of the core AF-subalgebra $\mathcal{F}_E$ of $C^*(E)$ is trivial. The following theorem is essentially due to Mikael Rørdam, [41].

**Theorem 4.2.** Let $E$ be a directed graph with finitely many vertices and no sources. Suppose further that the relative commutant of $\mathcal{F}_E$ in $C^*(E)$ is trivial. Then $u \in U(C^*(E))$ and $u \mathcal{F}_E u^* \subseteq \mathcal{F}_E$ imply that $u \in U(\mathcal{F}_E)$.

**Proof.** If $x \in \mathcal{F}_E$ then $uxu^* \in \mathcal{F}_E$ and thus $\gamma_t(x) \gamma_t(u)^* = \gamma_t(uxu^*) = uxu^*$ for each $t \in U(1)$. Consequently, $uxu^*$ belongs to $\mathcal{F}'_E \cap C^*(E) = \mathbb{C}$. Thus, for each $t \in U(1)$ there exists a $z(t) \in \mathbb{C}$ such that $\gamma_t(u) = z(t)u$. Clearly, $t \mapsto z(t)$ is a continuous character of the circle group $U(1)$. Hence there is an integer $m$ such that $z(t) = t^m$. If $m = 0$ then $u$ is invariant under the gauge action and we are done. Otherwise, by passing to $u^*$ if necessary, we may assume that $m > 0$.

For each vertex $v \in E^0$ choose one edge $e_v$ with range $v$ and let $T = \sum_{v \in E^0} S_{e_v}$. We have $\gamma_t(T) = tT$ for all $t \in U(1)$ and $T$ is an isometry, since $E$ has no sources. Furthermore, $TT^* \neq 1$, for otherwise each vertex would emit exactly one edge. As $E^0$ is finite and there are no sources, this would entail existence of a loop disjoint from the rest of the graph, and consequently $\mathcal{F}'_E \cap C^*(E)$ would contain the non-trivial center of $C^*(E)$, contrary to the assumptions. Now $T^m u^*$ is fixed by the gauge action and thus it is an isometry in $\mathcal{F}_E$. Since $\mathcal{F}_E$ is an AF-algebra, $T^m u^*$ must be unitary. But then $T^m$ and thus $T$ itself would be unitary, a contradiction. This completes the proof.

**Remark 4.3.** Several results of this paper depend on triviality of the center $\mathcal{Z}(C^*(E))$ of the graph algebra $C^*(E)$, the center $\mathcal{Z}(\mathcal{F}_E)$ of the core AF-subalgebra $\mathcal{F}_E$, or the relative commutant $\mathcal{F}'_E \cap C^*(E)$, respectively. The most interesting case to us is when $E$ is finite without sinks and where all loops have exits. Since in this case $D_E$ is maximal abelian in $C^*(E)$, [25], it follows immediately that

$$\mathcal{Z}(C^*(E)) \subseteq \mathcal{Z}(\mathcal{F}_E) = \mathcal{F}'_E \cap C^*(E).$$

The first inclusion above may be strict, even if $C^*(E)$ is simple (see [37] for examples and discussion). However, when $E$ is strongly connected (i.e. for every pair of vertices $v$, $w$ there is
a path from \(v\) to \(w\) and has period 1 (i.e. the greatest common divisor of the lengths of all loops is 1) then \(\mathcal{F}_E\) is simple and thus its center is trivial, [37, Theorem 6.11].

We begin our analysis of automorphisms of \(C^*(E)\) which globally preserve \(\mathcal{F}_E\) by identifying those which fix \(\mathcal{F}_E\) point-wise.

**Proposition 4.4.** Let \(E\) be a finite graph without sinks in which every loop has an exit. Suppose also that the center of \(\mathcal{F}_E\) is trivial. Then

\[
\text{End}_{\mathcal{F}_E}(C^*(E)) = \text{Aut}_{\mathcal{F}_E}(C^*(E)) = \{\gamma_z : z \in U(1)\}.
\]

**Proof.** Let \(\alpha \in \text{End}_{\mathcal{F}_E}(C^*(E))\). Then \(\alpha|_{\mathcal{D}_E} = \text{id}\) and thus \(\alpha = \lambda_u\) for some \(u \in U(\mathcal{D}_E)\), as noted in Section 2. Then an easy induction on \(k\) shows that \(u\) commutes with all \(\mathcal{F}_E^k\), and thus \(u \in \mathcal{F}_E' \cap \mathcal{D}_E\). But \(\mathcal{F}_E' \cap \mathcal{D}_E\) is the center of \(\mathcal{F}_E\), since \(\mathcal{D}_E\) is a MASA in \(\mathcal{F}_E\). Hence \(u\) is a scalar and \(\alpha = \lambda_u\) is a gauge automorphism. \(\square\)

In what follows, we will be mainly working with a finite graph \(E\) without sinks and sources. For each vertex \(v \in E^0\) we select exactly one edge \(e_v\) with \(r(e_v) = v\), and define

\[
T = \sum_{v \in E^0} S_{e_v}, \tag{15}
\]

as in the proof of Theorem 4.2. Then \(T\) is an isometry such that \(\gamma_z(T) = zT\) for all \(z \in U(1)\).

**Lemma 4.5.** Let \(E\) be a finite graph without sinks and sources in which every loop has an exit. Also, we assume that the center of \(\mathcal{F}_E\) is trivial. For an automorphism \(\alpha \in \text{Aut}(C^*(E))\) the following conditions are equivalent.

1. \(\alpha \gamma_z = \gamma_z \alpha\) for each \(z \in U(1)\).
2. For each \(e \in E^1\) there is a \(w \in \mathcal{F}_E\) such that \(\alpha(S_e) = wT\), where \(T\) is an isometry as in (15).
3. \(\alpha(\mathcal{F}_E) = \mathcal{F}_E\).

**Proof.** (1) \(\Rightarrow\) (3): If \(\alpha\) commutes with each \(\gamma_z\) then \(\alpha(\mathcal{F}_E) \subseteq \mathcal{F}_E\), since \(\mathcal{F}_E\) is the fixed-point algebra for the gauge action. Also, \(\alpha \gamma_z = \gamma_z \alpha\) implies \(\alpha^{-1} \gamma_z = \gamma_z \alpha^{-1}\), and thus \(\alpha^{-1}(\mathcal{F}_E) \subseteq \mathcal{F}_E\) as well.

(3) \(\Rightarrow\) (1): Suppose \(\alpha(\mathcal{F}_E) = \mathcal{F}_E\). Then for each \(x \in \mathcal{F}_E\) and \(z \in U(1)\) we have \(\alpha \gamma_z \alpha^{-1} \gamma_z^{-1}(x) = x\). Then by Proposition 4.4 there exists an \(\omega(z) \in U(1)\) such that \(\alpha \gamma_z \alpha^{-1} \gamma_z^{-1}(x) = \gamma_{\omega(z)}(x)\). The mapping \(z \mapsto \omega(z)\) is a continuous character of \(U(1)\). Indeed,

\[
\gamma_{\omega(y)} = \alpha \gamma_{\omega(y)} \alpha^{-1} \gamma_{\omega(y)}^{-1}
= (\alpha \gamma_z \alpha^{-1} \gamma_z^{-1}) \gamma_z (\alpha \gamma_y \alpha^{-1} \gamma_y^{-1}) \gamma_y \gamma_z^{-1}
= \gamma_{\omega(z)} \gamma_{\omega(y)} \gamma_y \gamma_z^{-1}
= \gamma_{\omega(z) \omega(y)}.
\]

Hence there exists an \(m \in \mathbb{Z}\) such that \(\omega(z) = z^m\) and, consequently, we have \(\alpha \gamma_z = \gamma_{z^{m+1}} \alpha\) for all \(z \in U(1)\). Therefore \(\gamma_z(\alpha^{-1}(S_e)) = z^{m+1} \alpha^{-1}(S_e)\) for all \(e \in E^1\). Since \(\alpha^{-1}\) is an automorphism,
this implies that $C^*(E)$ is generated by the spectral subspace $C^*(E)^{(m+1)}$ of the gauge action. Thus $m+1 = \pm 1$. However, the case $m+1 = -1$ is impossible. Indeed, let $T$ be an isometry as in (15). Since $1 = \sum_{e \in E} S_e S_e^*$, we have $\alpha^{-1}(T) = \sum_{e \in E} x_e S_e^*$, where $x_e = \alpha^{-1}(T) S_e$ are partial isometries in $\mathcal{F}_E$ such that $x_e^* x_e = S_e^* S_e = P_{r(e)}$. Thus

$$\sum_{e \in E} x_e^* x_e = \sum_{e \in E} P_{r(e)} > 1,$$

due to our assumptions on the graph $E$. On the other hand,

$$\sum_{e \in E} x_e^* x_e = \sum_{e \in E} \alpha^{-1}(T) S_e S_e^* \alpha^{-1}(T^*) \leq \alpha^{-1}(TT^*) \leq 1.$$

This yields a contradiction $\sum_{e \in E} x_e^* x_e > \sum_{e \in E} x_e x_e^*$, since $\mathcal{F}_E$ being an AF-algebra is stably finite.

(1) $\Rightarrow$ (2): We have $\alpha(S_e) = \alpha(S_e) T^* T = \sum_{e \in E} \alpha(S_e) S_e^* S_{e_\alpha}$. If $\alpha$ commutes with the gauge action then $\alpha(S_e) S_e^*$ belongs to $\mathcal{F}_E$, and the claim follows.

(2) $\Rightarrow$ (1): This is clear. \qed

Now, we turn to the definition of the restricted Weyl group of $C^*(E)$. If $E$ has no sinks and all loops have exits, then each $\alpha \in \text{Aut}_{D_E}(C^*(E))$ automatically belongs to $\text{Aut}(C^*(E), \mathcal{F}_E)$ by Proposition 3.2, above. Thus we may consider the quotient of

$$\text{Aut}(C^*(E), \mathcal{F}_E, D_E) := \text{Aut}(C^*(E), D_E) \cap \text{Aut}(C^*(E), \mathcal{F}_E)$$

by $\text{Aut}_{D_E}(C^*(E))$, which will be called the restricted Weyl group of $C^*(E)$ and denoted $\mathfrak{RW}_E$. That is,

$$\mathfrak{RW}_E := \text{Aut}(C^*(E), \mathcal{F}_E, D_E) / \text{Aut}_{D_E}(C^*(E)).$$  \hspace{1cm} (16)

We denote by $\mathcal{P}_E^k$ the collection of all unitaries in $\mathcal{U}(C^*(E))$ of the form $\sum S_{\alpha} S_{\beta}^*$ with $|\alpha| = |\beta| = k$. Clearly, each $\mathcal{P}_E^k$ is a finite subgroup of $\mathcal{U}(C^*(E))$, and we have $\mathcal{P}_E^k \subseteq \mathcal{P}_E^{k+1}$ for each $k$. We set $\mathcal{P}_E := \bigcup_{k=0}^{\infty} \mathcal{P}_E^k$. As shown in [38], every $u \in \mathcal{N}_{\mathcal{F}_E}(D_E)$ can be uniquely written as $u = dw$, where $d \in \mathcal{U}(D_E)$ and $w \in \mathcal{P}_E$. That is, the normalizer of the diagonal in $\mathcal{F}_E$ is a semi-direct product

$$\mathcal{N}_{\mathcal{F}_E}(D_E) = \mathcal{U}(D_E) \rtimes \mathcal{P}_E.$$  \hspace{1cm} (17)

We denote by $\mathcal{G}_E$ the subgroup of $\text{Aut}(C^*(E))$ generated by automorphisms of the graph $E$ and $\lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1}$. That is,

$$\mathcal{G}_E := \{ \lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1} \cup \Gamma_E \} \subseteq \text{Aut}(C^*(E)).$$  \hspace{1cm} (18)

By Proposition 3.6, there is a natural embedding of $\mathcal{G}_E$ into the Weyl group $\mathcal{W}_E$ of $C^*(E)$. Its restriction yields an embedding of $\mathcal{G}_E$ into the restricted Weyl group $\mathfrak{RW}_E$ of $C^*(E)$. Furthermore, $\lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1}$ is a normal subgroup of $\mathcal{G}_E$. Hence we have

$$\mathcal{G}_E = \lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1} \Gamma_E \quad \text{and} \quad \lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1} \cap \Gamma_E = \Gamma_E^0.$$  \hspace{1cm} (19)
Similarly to Theorem 3.7, in the restricted case we have the following.

**Proposition 4.6.** Let $E$ be a finite graph without sinks and sources in which every loop has an exit, and such that the relative commutant of $F_E$ in $C^*(E)$ is trivial. Then

$$G \mathcal{R}_E \cap \text{Inn}(C^*(E)) \subseteq \{ \text{Ad}(w) : w \in \mathcal{P}_E \}.$$  

**Proof.** By Theorem 3.7, every element of $G \mathcal{R}_E \cap \text{Inn}(C^*(E))$ is of the form $\text{Ad}(w)$ with $w \in S_E$. Since this $\text{Ad}(w)$ globally preserves $F_E$, by hypothesis, Theorem 4.2 implies that $w \in \mathcal{P}_E$. □

Since for each $w \in \mathcal{P}_E$ the corresponding inner automorphism $\text{Ad}(w)$ of $C^*(E)$ has finite order, Proposition 4.6 immediately yields the following.

**Corollary 4.7.** Let $E$ be a finite graph without sinks and sources in which every loop has an exit, and such that the relative commutant of $F_E$ in $C^*(E)$ is trivial. Then every element of infinite order in $G \mathcal{R}_E$ has infinite order in $\text{Out}(C^*(E))$ as well.

In general, it is a non-trivial matter to verify outerness of an automorphism of a graph algebra. Corollary 4.7 solves this problem for a significant class of automorphisms. In the case of Cuntz algebras, an analogous result was proved in [45, Theorem 6], and provided a convenient outerness criterion for permutative automorphisms – probably, the most studied class of automorphisms of $\mathcal{O}_n$.

Now, we turn to analysis of the action of the restricted Weyl group on the diagonal MASA. Let $E$ be a finite graph without sinks in which every loop has an exit. If $\alpha$ is an automorphism of $\mathcal{D}_E$ then, following [9], we say that $\alpha$ has property (P) if there exists a non-negative integer $m$ such that the endomorphism $\alpha \varphi^m$ commutes with the shift $\varphi$. In that case, we also say that $\alpha$ satisfies (P) with $m$. We define

$$\mathfrak{A}_E := \{ \alpha \in \text{Aut}(\mathcal{D}_E) : \text{ both } \alpha \text{ and } \alpha^{-1} \text{ have property (P)} \}. \quad (20)$$

Clearly, $\mathfrak{A}_E$ is a subgroup of $\text{Aut}(\mathcal{D}_E)$. In the case of the Cuntz algebras, it was shown in [9] that the restricted Weyl group is naturally isomorphic to $\mathfrak{A}_E$. Now, we investigate this problem for graph algebras.

**Lemma 4.8.** Let $E$ be a finite graph without sinks and sources in which every loop has an exit. Also, we assume that the center of $F_E$ is trivial. If $\alpha \in \text{Aut}(C^*(E), \mathcal{D}_E) \cap \text{Aut}(C^*(E), F_E)$ then for each $e \in E^1$ there exists a partial unitary $d \in D_E$ and a partial isometry $w \in F_E$ equal to a finite sum of words of the form $S_\mu S_\nu^*$ with $|\mu| = |\nu|$, and such that

$$\alpha(S_e) = dwT, \quad (21)$$

where $T$ is an isometry as in (15).

**Proof.** For each $e \in E^1$, $S_e$ is a partial isometry normalizing $\mathcal{D}_E$ (i.e., $S_e \mathcal{D}_ES_e^* \subseteq \mathcal{D}_E$ and $S_e^* \mathcal{D}_ES_e \subseteq \mathcal{D}_E$). Since $\alpha(\mathcal{D}_E) = \mathcal{D}_E$, it follows that $\alpha(S_e)$ normalizes $\mathcal{D}_E$ as well. By [25, Theorem 10.1], $\alpha(S_e)$ equals $dv$ for some partial unitary $d \in \mathcal{D}_E$ and $v$ a partial isometry which can
be written as the sum of a finite collection of words $S_\mu S_\nu^*$. Since $\alpha$ commutes with the gauge action by Lemma 4.5, we have $\gamma_\alpha(z) = z\nu$ for each $z \in U(1)$. Thus for each of the words in the decomposition of $v$ we have $|\mu| = |\nu| + 1$. Now let $T$ be as in (15). Then $\alpha(S_e) = dv = dwT$ is the desired decomposition, with $w = vT^*$. □

**Proposition 4.9.** Let $E$ be a finite graph without sinks and sources in which every loop has an exit. Also, we assume that the center of $\mathcal{F}_E$ is trivial. Let $\alpha \in \text{Aut}(\mathcal{C}^*(E), \mathcal{D}_E) \cap \text{Aut}(\mathcal{C}^*(E), \mathcal{F}_E)$ and let $T$ be an isometry as in (15). For each $e \in E^1$ let $d_e$ and $w_e$ be as in Lemma 4.8 so that $\alpha(S_e) = d_e w_e T$. Then for each path $\mu$ of length $r$ we have

$$\alpha(S_\mu) = D_\mu W_\mu T^r,$$

where $D_\mu \in \mathcal{D}_E$ and $W_\mu$, a sum of words in $\mathcal{F}_E$, are such that

$$D_\mu = (d_{\mu_1} w_{\mu_1} T)(d_{\mu_2} w_{\mu_2} T) \cdots (d_{\mu_r} w_{\mu_r} T)^* (w_{\mu_1} T)^* \cdots (w_{\mu_r} T)^*,$$

$$W_\mu = (w_{\mu_1} T)(w_{\mu_2} T) \cdots (w_{\mu_r} T) w_{\mu_r} T^{* (r - 1)}.$$

Furthermore, $D_\mu$ is a partial unitary and $W_\mu$ is a partial isometry. Thus we also have

$$\alpha(P_\mu) = D_\mu D_\mu^* W_\mu T^r T^{* r} W_\mu^*.$$

**Proof.** This is established by a somewhat tedious but not complicated inductive argument. We illustrate it with the case $r = 2$ only. Since $T^* T = 1$ and $w_{\mu_1}$ is a partial isometry, we have

$$\alpha(S_{\mu_1} S_{\mu_2}) = (d_{\mu_1} w_{\mu_1} T)(d_{\mu_2} w_{\mu_2} T) = d_{\mu_1} w_{\mu_1} (w_{\mu_1}^* w_{\mu_1})(T d_{\mu_2} T^*) T w_{\mu_2} T.$$

But both $w_{\mu_1}^* w_{\mu_1}$ and $T d_{\mu_2} T^*$ belong to $\mathcal{D}_E$ (since $w_{\mu_1}$ and $T$ are partial isometries normalizing $\mathcal{D}_E$). Thus the above expression equals

$$[(d_{\mu_1} w_{\mu_1} T)(d_{\mu_2} w_{\mu_2} T)^*][(w_{\mu_1} T) w_{\mu_2} T^*]T^2,$$

as required. Note that $D_\mu = (d_{\mu_1} w_{\mu_1} T)(d_{\mu_2} w_{\mu_2} T)^* = d_{\mu_1} (w_{\mu_1} T d_{\mu_2} T^* w_{\mu_1}^*)$ is a partial unitary in $\mathcal{D}_E$, and $W_\mu = (w_{\mu_1} T) w_{\mu_2} T^*$ is a sum of words in $\mathcal{F}_E$ and a partial isometry (as a product of partial isometries with mutually commuting domain and range projections). □

**Corollary 4.10.** Keeping the notation and hypothesis from Proposition 4.9, let

$$M = \min \{ k : \forall e \in E^1 \ d_e \in \mathcal{D}_E^k, \ w_e \in \mathcal{F}_E^k \}.$$

Then $D_\mu \in \mathcal{D}_E^{M + |\mu| - 1}$ and $W_\mu \in \mathcal{F}_E^{M + |\mu| - 1}$.

**Remark 4.11.** In fact, the conclusion of Proposition 4.9 remains valid with no hypothesis on the graph $E$ and for any endomorphism $\alpha$ of $\mathcal{C}^*(E)$, if we know that $\alpha(S_e) = d_e w_e T$ for each $e \in E^1$. 
Remark 4.12. Let $T = \sum_{v \in E^0} S_e v$ and $T' = \sum_{v \in E^0} S_{f_e} v^*$ be two isometries as in (15). Set $\tilde{U} = \sum_{v \in E^0} S_{f_e} S_e^*$. Then $\tilde{U}$ is a partial isometry in the finite dimensional $C^*$-algebra $F^1_E$. We may extend it to a unitary $U \in \mathcal{P}^1_E$, and then we have $T' = U T$. Thus if $\alpha \in \text{Aut}(C^*(E), F_E) \cap \text{Aut}(C^*(E), D_E)$ and $\alpha(S_e) = d_e w_e T = d'_e w'_e T'$, as in Lemma 4.8, then $d'_e = d_e$ and $w'_e U = w_e$. In particular, if we fix $T$, then for each $e \in E^1$ there is a unique such $w_e$ which satisfies $\alpha(S_e) = d_e w_e T$ and $w_e = \alpha(S_e S_e^*) w_e T T^*$.

Let $\alpha$ be an endomorphism of $C^*(E)$, where $E$ is a finite graph without sinks. Define a unital, completely positive map $\Phi_\alpha : C^*(E) \to C^*(E)$ by

$$\Phi_\alpha(x) = \sum_{e \in E^1} \alpha(S_e) x \alpha(S_e^*).$$

(22)

Then the following braiding relation holds

$$\alpha \varphi = \Phi_\alpha \alpha.$$  

(23)

If $\alpha = \lambda_u$ for some $u \in U_E$, then $\Phi_\alpha = \text{Ad}(u) \circ \varphi$.

Now, we are in the position to show how elements of the restricted Weyl group act on the diagonal, by automorphisms of $D_E$ (or, equivalently, homeomorphisms of its spectrum) which eventually commute with the shift.

Theorem 4.13. Let $E$ be a finite graph without sinks and sources in which every loop has an exit. Also, we assume that the center of $F_E$ is trivial. If $\alpha \in \text{Aut}(C^*(E), D_E) \cap \text{Aut}(C^*(E), F_E)$ then the restriction of $\alpha$ to $D_E$ belongs to $\mathfrak{A}_E$. This yields a group homomorphism

$$\text{Res} : \text{Aut}(C^*(E), D_E) \cap \text{Aut}(C^*(E), F_E) \to \mathfrak{A}_E$$

and a group embedding

$$\mathfrak{W}_E \hookrightarrow \mathfrak{A}_E.$$  

Proof. It suffices to show that there exists an $m$ such that $\alpha^{-1} \varphi^{m+1} = \varphi \alpha^{-1} \varphi^m$ on $D_E$. But $\alpha \varphi \alpha^{-1} \varphi^m = \Phi_\alpha \varphi^m$. Thus, we must show that $\Phi_\alpha \varphi^m = \varphi^{m+1}$ on $D_E$ for a sufficiently large $m$. To this end, for each $e \in E^1$ write $\alpha(S_e) = d_e w_e T$ as in Lemma 4.8. Let $m$ be so large that all $w_e$ belong to $F_{E_{m+1}}$. Then, for $x \in D_E$, using relation (9) we have $T \varphi^m(x) = \varphi^{m+1}(x) T$ and each $w_e$ commutes with $\varphi^{m+1}(x)$. Consequently, we have

$$\Phi_\alpha \varphi^m(x) = \sum_{e \in E^1} d_e w_e T \varphi^m(x) T^* w_e^* d_e^* = \varphi^{m+1}(x) \sum_{e \in E^1} d_e w_e T T^* w_e^* d_e^* = \varphi^{m+1}(x),$$

as required.

Finally, the kernel of the Res homomorphism coincides with $\text{Aut}_{D_E}(C^*(E))$. Thus, the restriction gives rise to a homomorphic embedding of the restricted Weyl group $\mathfrak{W}_E$ into $\mathfrak{A}_E$. □

In the case of $O_n$, it was shown in [9] that the restriction mapping from Theorem 4.13 is surjective. In this way, the restricted Weyl group of the Cuntz algebra has been identified with the
group of homeomorphisms which (along with their inverses) eventually commute with the full one-sided $n$-shift. The more general case of graph algebras is more complicated. We would like to pose it as an open problem to determine the exact class of automorphisms of $D_E$ (or, equivalently, homeomorphisms of the underlying space) which arise as restrictions of automorphisms of the graph algebra which preserve both the diagonal and the core AF-subalgebra.

5. The localized automorphisms

Throughout this section, we assume that $E$ is a finite graph without sinks. Recall that an endomorphism $\lambda_u$ of $C^*(E)$ is called localized if the corresponding unitary $u$ belongs to a finite dimensional algebra $F^k_{E} \cap U_E$ for some $k$. Our main aim in this section is to produce an invertibility criterion for localized endomorphisms, analogous to [15, Theorem 3.2].

Let $u$ be unitary in $F^k_{E} \cap U_E$, for some fixed $k \geq 1$. Using (7) and (8) we see that $\lambda_u(x) = \text{Ad}(u_r)(x)$ for all $x \in F^r_{E}$ and $r \geq 1$. Following [45], for each pair $e, f \in E^1$ we define a linear map $a_{e,f}^u : F^k_{E} \rightarrow F^k_{E}$ by

$$a_{e,f}^u(x) = S_{e}^* u^s x u S_{f}, \quad x \in F^k_{E}. \quad (24)$$

Denote $V_k := F^k_{E} / D^0_{E}$, the quotient vector space, and let $L(V_k)$ be the space of linear maps from $V_k$ to itself. Since $a_{e,f}^u(D^0_{E}) \subseteq D^0_{E}$, there is an induced map $\tilde{a}_{e,f}^u : V_k \rightarrow V_k$. Now we define $A_u$ as the subring of $L(V_k)$ generated by $\{ \tilde{a}_{e,f}^u : e, f \in E^1 \}$.

We denote by $H$ the linear span of the generators $S_e$'s. Let $u$ be as above. Following [13, p. 386], we define inductively

$$\Xi_0 = F^k_{E}, \quad \Xi_r = \lambda_u(H)^s \Xi_{r-1} \lambda_u(H), \quad r \geq 1. \quad (25)$$

Then $\{ \Xi_r \}_r$ is a non-increasing sequence of finite dimensional, self-adjoint subspaces of $F^k_{E}$ and thus it eventually stabilizes. If $\Xi_p = \Xi_{p+1}$ then we have $\Xi_u := \bigcap_{r=0}^{\infty} \Xi_r = \Xi_p$.

If $\alpha, \beta$ are paths of length $r$, then we denote by $T_{\alpha,\beta}$ the linear map from $F^k_{E}$ to itself defined by $T_{\alpha,\beta} = a_{\alpha_r,\beta_r}^u \cdots a_{\alpha_1,\beta_1}^u$. We have $T_{\alpha,\beta}(x) = S_\alpha^* \text{Ad}(u_r^s)(x) S_\beta$ for all $x \in F^k_{E}$. Note that $T_{\alpha,\beta} T_{\mu,\nu} = 0$ if either $\alpha \mu$ or $\beta \nu$ does not form a path. It easily follows from our definitions that the space $\Xi_r$ is linearly spanned by elements of the form $T_{\alpha,\beta}(x)$, for $\alpha, \beta \in E^r, x \in F^k_{E}$.

**Theorem 5.1.** Let $E$ be a finite graph without sinks, and let $u \in U_E$ be a unitary in $F^k_{E}$ for some $k \geq 1$. Then the following conditions are equivalent:

1. $\lambda_u$ is invertible with localized inverse;
2. the sequence of unitaries $\{ \text{Ad}(u_r^s)(u_r) \}_{m \geq 1}$ eventually stabilizes;
3. the ring $A_u$ is nilpotent;
4. $\Xi_u \subseteq D^0_{E}$.

**Proof.** (1) $\Rightarrow$ (3): If the inverse of $\lambda_u$ is localized then there exists an $l$ such that $\lambda_u^{-1}(F^k_{E}) \subseteq F^l_{E}$. Let $\alpha = (e_1, e_2, \ldots, e_l)$ and $\beta = (f_1, f_2, \ldots, f_l)$ be paths of length $l$, and consider an ele-
ment $T_{\alpha,\beta} = a_{e_1,f_1}^u a_{e_2,f_2}^u \cdots a_{e_l,f_l}^u$ of $A^l_u$. Let $b \in \mathcal{F}^{k-1}_E$ and let $x = \lambda^{-1}_u(b)$. Then $x \in \mathcal{F}_E^l$ and we have $b = \lambda_u(x) = \text{Ad}(u_l(x))$. Therefore

$$T_{\alpha,\beta}(b) = a_{e_1,f_1}^u a_{e_2,f_2}^u \cdots a_{e_l,f_l}^u(b) = \text{Ad}(u_l^*)(b)S_\beta = \text{Ad}(u_l^*)xS_\beta.$$ 

Since $x$ can be written as $\sum_{|\gamma|=|\rho|=l} \sum_{e,f} c_{\gamma,\rho}(x)S_\gamma S_\rho^*$ for some $c_{\gamma,\rho}(x) \in \mathbb{C}$, we have

$$T_{\alpha,\beta}(b) = S_\alpha^* \left( \sum_{|\gamma|=|\rho|=l} c_{\gamma,\rho}(x)S_\gamma S_\rho^* \right) S_\beta = \begin{cases} c_{\alpha,\beta}(x)P_{r(\alpha)}, & \text{if } r(\alpha) = r(\beta), \\ 0, & \text{if } r(\alpha) \neq r(\beta) \end{cases}$$

because $S_\alpha^*S_\gamma = P_{r(\alpha)}$ if $\alpha = \gamma$, and $S_\alpha^*S_\gamma = 0$ otherwise. This implies that $T_{\alpha,\beta}(b) \in \mathcal{D}_E^0$, and hence we see that $A^l_u = 0$.

$(3) \Rightarrow (4)$: Let $A^l_u = 0$ for some positive integer $l$. Then $T_{\alpha,\beta}(b) \in \mathcal{D}_E^0$ for all $b \in \mathcal{F}^{k-1}_E$ and all $\alpha, \beta$ such that $|\alpha| = |\beta| = l$. But this immediately yields $\mathcal{E}_l \subseteq \mathcal{D}_E^0$ and, consequently, $\mathcal{E}_u \subseteq \mathcal{D}_E^0$.

$(4) \Rightarrow (2)$: Let $\mathcal{E}_u \subseteq \mathcal{D}_E^0$, and let $l$ be a positive integer such that $\mathcal{E}_l = \mathcal{E}_u$. Let $b \in \mathcal{F}^{k-1}_E$ and let $\alpha, \beta \in \mathcal{E}_l$. Then $T_{\alpha,\beta}(b)$ belongs to $\mathcal{D}_E^0$ and thus it commutes with $\varphi^m(u)$ for all $m$, since $u$ commutes with the vertex projections. Consequently, for each $r \geq 1$ we have

$$\text{Ad}(u_{l+r}^*)(b) = \text{Ad}(\varphi^{l+1-r}(u^*) \cdots \varphi^l(u^*)) \left( \sum_{\alpha,\beta \in \mathcal{E}_l} S_\alpha T_{\alpha,\beta}(b)S_\beta^* \right)$$

$$= \sum_{\alpha,\beta \in \mathcal{E}_l} S_\alpha \text{Ad}(\varphi^{l}(u^*) \cdots u^*)(T_{\alpha,\beta}(b))S_\beta^*$$

$$= \sum_{\alpha,\beta \in \mathcal{E}_l} S_\alpha T_{\alpha,\beta}(b)S_\beta^*.$$ 

Thus for each $b \in \mathcal{F}^{k-1}_E$ the sequence $\text{Ad}(u_{m}^*)(b)$ stabilizes from $m = l + 1$. Write $u^* = \sum_{e,f \in \mathcal{E}_l} S_e b_{e,f} S_f^*$, for some $b_{e,f} \in \mathcal{F}^{k-1}_E$. Then for each $m$ we have

$$\text{Ad}(u_{m+1}^*)(u^*) = \sum_{e,f \in \mathcal{E}_l} \text{Ad}(\varphi^{m-1}(u^*) \cdots \varphi(u^*)u^*)(S_e b_{e,f} S_f^*)$$

$$= \sum_{e,f \in \mathcal{E}_l} S_e \text{Ad}(\varphi^{m-1}(u^*) \cdots \varphi(u^*)u^*)(b_{e,f})S_f^*$$

and, consequently, the sequence $\text{Ad}(u_{m}^*)(u^*)$ stabilizes from $m = l + 2$.

$(2) \Rightarrow (1)$: Suppose that the sequence $\text{Ad}(u_{m}^*)(u^*)$ eventually stabilizes. Hence $\text{Ad}(u_{r}^*)(u^*) = w$ for all sufficiently large $r$. It follows that $\lambda_{u}(w) = \text{Ad}(u_r)(w) = u^*$ for suitable $r$ depending on $w$. Thus $\lambda_{u}(w)u = u^*u = 1$ and, consequently, $\lambda_{u}$ is invertible with inverse $\lambda_{w}$. This completes the proof.

We also include a different and much more direct proof of implication $(1) \Rightarrow (2)$, that sheds additional light on the equivalent conditions of the theorem and is interesting in its own right.
Let $\lambda_u$ be invertible, and suppose that there exists an $l \in \mathbb{N}$ and a unitary $v \in \mathcal{U}_E$ in $\mathcal{F}_E^l$ such that $\lambda_u \lambda_v = \mathrm{id}$. Then we have $\lambda_u(v)u = 1$. Since $v \in \mathcal{F}_E^l$, $u^* = \lambda_u(v) = \text{Ad}(u_l)(v)$, and hence $\text{Ad}(u^*_l)(u^*) = v$. Now for $r \geq 1$ we have

$$\text{Ad}(u^*_l)^{r}(u^*) = \text{Ad}(\varphi^1(u^*) \cdots \varphi^r(u^*)u^*)(u^*)$$

$$= \varphi^1(u^*) \cdots \varphi^r(u^*) \text{Ad}(u^*_l)(u^*) \varphi^1(u) \cdots \varphi^r(u^*) \varphi^r(u)$$

$$= \varphi^1(u^*) \cdots \varphi^r(u) \varphi^1(u) \cdots \varphi^r(u^*)$$

$$= v$$

since $v$ commutes with $\varphi^m(u)$ for every $m \geq l$. Thus we can conclude that $\text{Ad}(u^*_m)(u^*)$ stabilizes at $v$ from $m = l$. $\square$

**Remark 5.2.** Let $u \in \mathcal{F}_E^1 \cap \mathcal{U}_E$, so that $\lambda_u$ is quasi-free. Since $\mathcal{F}_E^0 = D_E^0$, we have $V_1 = D_E^0 / D_E^0 = \{0\}$ and consequently each $\tilde{a}_{e,f}^u$ is a zero map. Therefore $A_u = \{0\}$ and Theorem 5.1 trivially implies that $\lambda_u$ is an automorphism of $C^*(E)$.

If $u \in \mathcal{U}_E$ normalizes $D_E$ then $\lambda_u(D_E) \subseteq D_E$. It may well happen that such a restriction is an automorphism of $D_E$ even though $\lambda_u$ is not invertible. For unitaries in the algebraic part of $\mathcal{F}_E$ this can be checked in a way similar to [15, Theorem 3.4]. Indeed, let $u \in \mathcal{U}_E \cap N_{\mathcal{F}_E^1}(D_E^1)$. Then it follows from (9) that $u \in N_{\mathcal{F}_E^1}(D_E)$. Furthermore, the subspace $D_E^{k-1}$ of $\mathcal{F}_E^k$ is invariant under the action of all maps $a_{e,f}^u$, $e, f \in E^1$. We denote by $b_{e,f}^u$ the restriction of $a_{e,f}^u$ to $D_E^{k-1}$, and by $\tilde{b}_{e,f}^u$ the map induced on $V_k^D := D_E^{k-1} / D_E^0$. Let $A_u^D$ be the subring of $\mathcal{L}(V_k^D)$ generated by $\{\tilde{b}_{e,f}^u : e, f \in E^1\}$. Also, we consider a nested sequence of subspaces $\mathcal{E}_r^D$ of $D_E^{k-1}$, defined inductively as

$$\mathcal{E}_0^D = D_E^{k-1}, \quad \mathcal{E}_r^D = \lambda_u(H)^r \mathcal{E}_{r-1}^D \lambda_u(H), \quad r \geq 1.$$  \hspace{1cm} (26)

Each $\mathcal{E}_r^D$ is finite dimensional and self-adjoint. We set $\mathcal{E}_u^D := \bigcap_r \mathcal{E}_r^D$.

**Theorem 5.3.** Let $E$ be a finite graph without sinks and let $u \in \mathcal{U}_E \cap N_{\mathcal{F}_E}(D_E^k)$, for some $k \geq 1$. Then the following conditions are equivalent:

1. $\lambda_u$ restricts to an automorphism of $D_E$;
2. the ring $A_u^D$ is nilpotent;
3. $\mathcal{E}_u^D \subseteq D_E^0$.

**Proof.** (1) $\Rightarrow$ (3): Since the algebraic part $\bigcup_{t=0}^{\infty} D_E^t$ of $D_E$ coincides with the linear span of all projections in $D_E$, every automorphism of $D_E$ restricts to an automorphism of $\bigcup_{t=0}^{\infty} D_E^t$. Thus, there exists an $l$ such that $(\lambda_u|_{D_E^l})^{-1}(D_E^{k-1}) \subseteq D_E^l$. Let $\alpha = (e_1, e_2, \ldots, e_l)$ and $\beta = (f_1, f_2, \ldots, f_l)$ be in $E^l$, and let $R_{\alpha, \beta} := b_{e_1,f_1}^u \cdots b_{e_l,f_l}^u b_{e_2,f_2}^u b_{e_1,f_1}^u$ be in $(A_u^D)^l$. Then the same argument as in the proof of implication (1) $\Rightarrow$ (3) in Theorem 5.1 yields that $R_{\alpha, \beta}(d) \in D_E^l$ for all $d \in D_E^{k-1}$. Thus $(A_u^D)^l = \{0\}$. But as in the proof of implication (3) $\Rightarrow$ (4) in Theorem 5.1, this implies that $\mathcal{E}_u^D \subseteq D_E^0$. 


(3) ⇒ (2): Let $\Sigma^D_u \subseteq \mathcal{D}^0_E$ and let $l$ be a positive integer such that $\Sigma^D_l = \Sigma^D_u$. Let $d \in \mathcal{D}^{k-1}_E$ and let $\alpha, \beta \in E^l$. Then $R_{\alpha, \beta}(d)$ belongs to $\mathcal{D}^0_E$, and this entails $(A^D_u)^l = \{0\}$, i.e. the ring $A^D_u$ is nilpotent.

(2) ⇒ (1): Suppose that $A^D_u$ is nilpotent. We show by induction on $r \geq k$ that all $\mathcal{D}^r_E$ are in the range of $\lambda_u$ restricted to $\bigcup_{t=0}^{\infty} \mathcal{D}^t_E$.

Firstly, let $r = k$ and $d \in \mathcal{D}^k_E$. Similarly to the argument in the implication (4) ⇒ (2) of the proof of Theorem 5.1 one shows that the sequence $\text{Ad}(\varphi^m(u^*) \cdots \varphi(u^*)u^*)(d)$ eventually stabilizes at some $f \in \bigcup_{t=0}^{\infty} \mathcal{D}^t_E$. It then follows that $d = \lambda_u(f)$.

For the inductive step, suppose that $r \geq k$ and $\mathcal{D}^r_E \subseteq \lambda_u(\bigcup_{t=0}^{\infty} \mathcal{D}^t_E)$. Since $\mathcal{D}^{r+1}_E$ is generated by $\mathcal{D}^r_E$ and $\varphi^r(\mathcal{D}^1_E)$, it suffices to show that $\varphi^r(y)$ belongs to $\lambda_u(\bigcup_{t=0}^{\infty} \mathcal{D}^t_E)$ for all $y \in \mathcal{D}^1_E$. However, $\varphi^r(y)$ commutes with $u$ and $\varphi^{r-1}(y) \in \mathcal{D}^r_E$ is in $\lambda_u(\bigcup_{t=0}^{\infty} \mathcal{D}^t_E)$ by the inductive hypothesis. Thus the sequence

$$\text{Ad}(\varphi^m(u^*) \cdots \varphi(u^*)u^*)(\varphi^r(y)) = \varphi(\text{Ad}(\varphi^{m-1}(u^*) \cdots \varphi(u^*)u^*)(\varphi^{r-1}(y)))$$

eventually stabilizes at $\lambda_u^{-1}(\varphi^r(y)) \in \bigcup_{t=0}^{\infty} \mathcal{D}^t_E$. □

It should be noted that, in the setting of Theorem 5.3, it may well happen that $\lambda_u | \mathcal{D}_E$ is an automorphism of $\mathcal{D}_E$ while $\lambda_u$ is a proper endomorphism of $\mathcal{C}^*_{\text{st}}(E)$. In that case there may not exist any unitary $w \in \mathcal{C}^*_{\text{st}}(E)$ such that $(\lambda_u | \mathcal{D}_E)^{-1} = \lambda_w | \mathcal{D}_E$, and thus $(\lambda_u | \mathcal{D}_E)^{-1}$ is not localized in our sense (as defined in the first paragraph of this section). See [15,12], [16, Theorem 3.8] and [11, Proposition 3.2] for examples and further discussion of this interesting point.

6. The permutative automorphisms

Throughout this section, we assume that $E$ is a finite graph without sinks. Our main goal in this section is to give a combinatorial criterion for invertibility of $\lambda_u$, $u \in \mathcal{P}_E$, analogous to [9, Corollary 4.12]. Both the statement of the criterion (see Theorem 6.4 below) and its proof are quite similar to those given in the case of the Cuntz algebras in [15], suitably generalized to the present case of graph $\mathcal{C}^*_{\text{st}}$-algebras. The key idea is to break the process into two steps, Condition (b) and Condition (d), of which the former detects those endomorphisms which restrict to automorphisms of the diagonal $\mathcal{D}_E$.

It is useful to look at collections of paths of a fixed length beginning or ending at the same vertex. Hence we introduce the following notation. For $v, w \in E^0$ and $k \in \mathbb{N}$, let $E^k_{v, w} := \{\alpha \in E^k : r(\alpha) = v\}$, $E^k_{*, v} := \{\alpha \in E^k : s(\alpha) = v\}$ and $E^k_{v, w} := \{\alpha \in E^k : r(\alpha) = v, s(\alpha) = w\}$. Then $E^k = \bigcup_{v \in E^0} E^k_{v, v} = \bigcup_{v \in E^0} E^k_{v, *}$ and $E^k_{v, w} = \bigcup_{v, w \in E^0} E^k_{v, w}$, disjoint unions. If $u \in \mathcal{P}_E^k$, $k > 0$, then there exist permutations $\sigma_v \in \text{Perm}(E^k_{v, *})$ such that

$$u = \sum_{v \in E^0} \sum_{\alpha \in E^k_{v, *}} S_{\sigma_v(\alpha)} S^*_{\alpha}. \tag{27}$$

A unitary $u \in \mathcal{P}_E^k$, $k > 0$, commutes with all the vertex projections if and only if there exist permutations $\sigma_{v, w} \in \text{Perm}(E^k_{v, w})$ such that

$$u = \sum_{v, w \in E^0} \sum_{\alpha \in E^k_{v, w}} S_{\sigma_{v, w}(\alpha)} S^*_{\alpha}. \tag{28}$$
If the unitary \( u \in \mathcal{P}_k^E \cap \mathcal{U}_E \) is understood, as in Eq. (28), then we will denote by \( \sigma = \bigcup_{v, w \in \mathcal{E}^0} \sigma_{v, u} \) the corresponding permutation of \( E^k \). In that case, we will also write \( \lambda_u = \lambda_\sigma \).

Now let \( u \in \mathcal{P}_k^E \cap \mathcal{U}_E \), \( e, f \in E^1 \), and consider the linear map \( a(u)_{e, f} \), as defined in (24). With respect to the basis \( \{ S_\mu S_\nu^*: \mu, \nu \in E^{k - 1} \} \) of \( \mathcal{F}_E^{k - 1} \) so ordered that the initial vectors span \( \mathcal{D}_E^{k - 1} \), the matrix of \( a(u)_{e, f} \) has the block form

\[
\begin{pmatrix}
 b_{e, f}^u & c_{e, f}^u \\
 0 & d_{e, f}^u
\end{pmatrix}
\]

similarly to [15, Section 4]. The first block corresponds to the subspace \( \mathcal{D}_E^{k - 1} \) of \( \mathcal{F}_E^{k - 1} \). Thus, the map \( \tilde{a}(u)_{e, f} \in \mathcal{L}(V_k) \) has a matrix

\[
\begin{pmatrix}
 b_{e, f}^u & 0 \\
 0 & d_{e, f}^u
\end{pmatrix}
\]

with the first block corresponding to the subspace \( V_k \) of \( V_k \). Note that the passage from the space \( \mathcal{F}_E^{k - 1} \) to its quotient \( V_k \) does not affect the matrix for \( d_{e, f}^u \) and thus there is no tilde over it in formula (30). It is an immediate corollary to Theorem 5.1 that an endomorphism \( \lambda_u \) of \( C^*(E) \) is invertible if and only if the following two conditions are satisfied.

Condition (b): the ring generated by \( \{ \tilde{b}_{e, f}^u : e, f \in E^1 \} \) is nilpotent.

Condition (d): the ring generated by \( \{ d_{e, f}^u : e, f \in E^1 \} \) is nilpotent.

The remainder of this section is devoted to the description of a convenient combinatorial interpretation of these two crucial conditions, similar to the one appearing in [15] and used in the analysis of permutative endomorphisms of the Cuntz algebras. By virtue of Theorem 5.3, Condition (b) alone is equivalent to the restriction of \( \lambda_u \) to the diagonal \( \mathcal{D}_E \) being an automorphism.

6.1. Condition (b)

We fix \( u \in \mathcal{P}_k^E \cap \mathcal{U}_E \) and denote by \( \sigma \) the corresponding permutation, as above. If \( e \neq f \) then \( b_{e, f}^u = 0 \). Thus, it suffices to consider the ring generated by maps \( b_{e, e}^u := b_{e, e}^u, e \in E^1 \). Since \( b_{e, e}^u(1) = P_{r(e)} \), the matrix of \( b_{e, e}^u \) has exactly one 1 in the row corresponding to each \( \alpha \in E_{s, r(e)}^{k - 1} \), and 0’s elsewhere. Consequently, each \( b_{e, e}^u \) may be identified with a mapping

\[
f_{e}^u : E_{s, r(e)}^{k - 1} \rightarrow E_{s, s(e)}^{k - 1}, \quad f_{e}^u(\alpha) = \beta,
\]

whenever \( b_{e, e}^u \) has 1 in the \( \alpha \rightarrow \beta \) entry. If the unitary \( u \) is given by a permutation \( \sigma \) then

\[
f_{e}^u(\alpha) = \beta \quad \Leftrightarrow \quad \exists g \in E^1 \text{ s.t. } \sigma(e, \alpha) = (\beta, g).
\]

The product \( b_{e, e}^u b_{g, g}^u \) corresponds to the composition \( f_{g}^u \circ f_{e}^u \) (in reversed order of \( e \) and \( g \)). Now Condition (b) may be phrased in terms of mappings \( \{ f_{e}^u \} \) rather than \( \{ b_{e, e}^u \} \), as follows:

There exists an \( m \) such that for all \( e_1, \ldots, e_m \in E^1 \) if \( T = f_{e_m}^u \circ \cdots \circ f_{e_1}^u \) then for all \( v \in E^0 \) and \( \alpha \in E^{k - 1} \) either \( E_{s, v}^{k - 1} \cap T^{-1}(\alpha) = \emptyset \) or \( E_{s, v}^{k - 1} \subseteq T^{-1}(\alpha) \).
Taking into account (31) above, we arrive at the following:

**Condition** (b): There exists an integer \( m \in \mathbb{Z} \) such that for all \( e_1, \ldots, e_m \in E^1 \) either: (i) \( f^u_{e_1} \circ \cdots \circ f^u_{e_m} \) has the empty domain, or (ii) its domain equals \( E^{k-1}_{s,e(e)} \) and its range consists of exactly one element.

In the remainder of this section, notation \((\alpha, \beta)\) indicates either a single path in \( E^* \) or an ordered pair in the cartesian product \( E^* \times E^* \). This will be clear from context.

**Lemma 6.1.** Let \( E \) be a finite graph without sinks in which every loop has an exit. Let \( u \in P_c^1 \cap \mathcal{U}_E \). Then Condition (b) holds for \( u \) (and hence \( \lambda_u \mid \mathcal{D}_E \) is an automorphism of \( \mathcal{D}_E \)) if and only if there exists a partial order \( \leq \) on \( \bigcup_{v \in E^0} E^{k-1}_{s,v} \times E^{k-1}_{s,v} \) such that:

1. if \( v \in E^0 \setminus r(E^1) \) then each element of \( E^{k-1}_{s,v} \times E^{k-1}_{s,v} \) is minimal, each diagonal element \((\alpha, \alpha)\) is minimal, and there are no other minimal elements;
2. if \( e \in E^1 \) and \( \alpha \neq \beta \in E^{k-1}_{s,e(e)} \) then \((f^u_\alpha(\alpha), f^u_\beta(\beta)) \leq (\alpha, \beta)\).

**Proof.** At first suppose that Condition (b) holds for \( u \). Define a relation \( \leq \) as follows. For any \( \alpha \in E^{k-1} \) set \((\alpha, \alpha) \leq (\alpha, \alpha)\). If \( \gamma \neq \delta \) then \((\alpha, \beta) \leq (\gamma, \delta)\) if and only if there exists a sequence \( e_1, \ldots, e_d \in E^1 \), possibly empty, such that \( \alpha = (f^u_{e_1} \circ \cdots \circ f^u_{e_d})(\gamma) \) and \( \beta = (f^u_{e_1} \circ \cdots \circ f^u_{e_d})(\delta) \). Reflexivity and transitivity of \( \leq \) are obvious. To see that \( \leq \) is also antisymmetric, suppose that \((\alpha, \beta) \leq (\gamma, \delta)\) and \((\gamma, \delta) \leq (\alpha, \beta)\). If \((\alpha, \beta) \neq (\gamma, \delta)\) then, by definition of \( \leq \), \( \alpha \neq \beta \), \( \gamma \neq \delta \) and there exist edges \( e_1, \ldots, e_d, g_1, \ldots, g_h \) such that \((\alpha, \beta) = (f^u_{e_1} \circ \cdots \circ f^u_{e_d})(\gamma, \delta)\) and \((\gamma, \delta) = (f^u_{g_1} \circ \cdots \circ f^u_{g_h})(\alpha, \beta)\). That is, \( f^u_{e_1} \circ \cdots \circ f^u_{e_d} \circ f^u_{g_1} \circ \cdots \circ f^u_{g_h} \) has two distinct fixed points, a contradiction with Condition (b). Thus \((\alpha, \beta) \leq (\gamma, \delta)\) and \( \leq \) is also antisymmetric. Hence \( \leq \) is a partial order satisfying condition (2) above. By the very definition of \( \leq \), if \( v \in E^0 \setminus r(E^1) \) then each element of \( E^{k-1}_{s,v} \times E^{k-1}_{s,v} \) is minimal, and likewise each diagonal element \((\alpha, \alpha)\) is minimal. If any other element were minimal for \( \leq \) then there would exist \( \alpha \neq \beta \) and \( e \in E^1 \) such that \( f^u_\alpha(\alpha) = \alpha \) and \( f^u_\beta(\beta) = \beta \). Thus \( f^u_e \) would have two distinct fixed points, contradicting Condition (b). Thus condition (1) holds true as well.

Conversely, if a partial order \( \leq \) with the required properties exists, then counting shows that each sufficiently long composition product of mappings \( \{f^u_e\} \) either has the empty domain or its range consists of a single element (and the domain is as required, due to (31)). This completes the proof. \( \square \)

**Remark 6.2.** By the diagram of \( f^u_e \) we mean a directed graph with vertices corresponding to the union of the domain and the range of the map \( f^u_e \) and with an edge from vertex \( \alpha \) to \( \beta \) if and only if \( f^u_e(\alpha) = \beta \). Combining the diagrams of all \( f^u_e, e \in E^1 \), we obtain a directed graph whose edges are labeled by \( \{f^u_e\} \) or simply by the edges of \( E \), see Example 6.5 below. Our Condition (b) is equivalent to existence of a positive integer \( m \) such that all words of length \( m \) are synchronizing for this labeled graph.\(^5\) Also note that if the conditions of Lemma 6.1 are satisfied and \( e \in E^1 \) is such that \( s(e) = r(e) \), then the diagram of \( f^u_e \) is a rooted tree with the root being the unique fixed point, cf. [15, Section 4.1].

\(^5\) We are grateful to Rune Johansen for pointing this out.
6.2. Condition (d)

Again, we fix a \( u \in \mathcal{P}_E^k \cap \mathcal{U}_E \) and denote by \( \sigma \) the corresponding permutation. It is easy to verify that for each row of the matrix \( d_{e,g}^u \) either may have 1 in one place and 0’s elsewhere or may consist of all zeros. This matrix has 1 in \((\alpha, \beta)\) row and \((\gamma, \delta)\) column if and only if there exists an \( h \in E^1 \) such that \( S_\alpha S_\beta^* S_\gamma^* S_h^* S_\delta S_g \). In turn, this takes place if and only if
\[
\sigma(e, \alpha) = (\gamma, h) \quad \text{and} \quad \sigma(g, \beta) = (\delta, h).
\]
(33)

For each \( e, g \in E^1 \) we now define a mapping \( f_{e,g}^u \), as follows. The domain \( D(f_{e,g}^u) \) consists of all \((\alpha, \beta) \in E_{*,r(e)} \times E_{*,r(g)}^{k-1}\) for which the \((\alpha, \beta)\) row of \( d_{e,g}^u \) is non-zero, and the corresponding value is \( f_{e,g}^u(\alpha, \beta) = (\gamma, \delta) \in E^{k-1} \times E^{k-1}\) for \((\gamma, \delta)\) satisfying (33). Note that, by the very definition of \( d_{e,g}^u \), in such a case we must necessarily have \( \alpha \neq \beta \) and \( \gamma \neq \delta \). We denote
\[
\Psi_u := E^{k-1} \times E^{k-1} \setminus \{(\alpha, \alpha) : \alpha \in E^{k-1}\}.
\]
We also denote by \( \Delta_u \) the subset of \( \Psi_u \) consisting of all those \((\alpha, \beta)\) for which there exist \( e, g \in E^1 \) such that \((\alpha, \beta)\) belongs to the domain \( D(f_{e,g}^u) \).

It is a simple matter to verify that in terms of mappings \( \{f_{e,g}^u\} \) Condition (d) may be rephrased as follows (cf. [15, Section 4.3]).

**Condition (d):** There exists an \( m \) such that for all \((e_1, g_1), \ldots, (e_m, g_m) \in \Psi_u\) the domain of the map \( f_{e_1, g_1}^u \circ \cdots \circ f_{e_m, g_m}^u \) is empty.

The proof of the following lemma is essentially the same as that of [15, Lemma 4.10] and thus it is omitted.

**Lemma 6.3.** Let \( E \) be a finite graph without sinks in which every loop has an exit. Let \( u \in \mathcal{P}_E^k \cap \mathcal{U}_E \). Then Condition (d) holds for \( u \) if and only if there exists a partial order \( \leq \) on \( \Psi_u \) such that:

1. the set of minimal elements coincides with \( \Psi_u \setminus \Delta_u \);
2. if \( e, g \in E^1 \) and \((\alpha, \beta) \in D(f_{e,g}^u)\) then \( f_{e,g}^u(\alpha, \beta) \leq (\alpha, \beta) \).

Combining Lemma 6.1 with Lemma 6.3 we obtain a combinatorial criterion of invertibility of permutative endomorphisms, similar to [15, Corollary 4.12].

**Theorem 6.4.** Let \( E \) be a finite graph without sinks in which every loop has an exit, and let \( u \in \mathcal{P}_E^k \cap \mathcal{U}_E \). Then the endomorphism \( \lambda_u \) is invertible if and only if conditions of Lemma 6.1 and Lemma 6.3 hold for \( u \).

6.3. Examples

We give two examples with small graphs illustrating the combinatorial machinery developed in the preceding section. In Example 6.5, we exhibit a proper permutative endomorphism of \( C^*(E) \) which restricts to an automorphism of the diagonal MASA \( \mathcal{D}_E \). On the other hand, in Example 6.6 we provide an order 2 permutative automorphism of a Kirchberg algebra \( C^*(E) \) with \( K_0(C^*(E)) \cong \mathbb{Z} \cong K_1(C^*(E)) \), which is neither quasi-free nor comes from a graph automorphism.
Example 6.5. Consider the following graph $E$ (the Fibonacci graph):

At level $k = 2$, there are 2 permutations in $\mathcal{P}_E^2 \cap \mathcal{U}_E \cong \mathbb{Z}_2$. Denoting edge $e_j$ simply by $j$, the non-trivial transposition is $(1, 32)$. The corresponding map $f_1 : \{1, 3\} \to \{1, 3\}$ is such that $f_1(1) = 3$ and $f_1(3) = 1$. Thus Condition (b) does not hold and, consequently, the corresponding endomorphism is surjective neither on $C^*(E)$ nor on $D_E$.

At level $k = 3$, there are 24 permutations in $\mathcal{P}_E^3 \cap \mathcal{U}_E \cong S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. These are generated by $\sigma = (132, 321)$, $\tau = (111, 132, 321)$, $\upsilon = (113, 323)$ and $\omega = (211, 232)$. Of these 24 permutations, only $\tau \upsilon$ and $\text{id}$ satisfy Condition (b). However, $\tau \upsilon$ does not satisfy Condition (d). Thus, $\lambda_{\tau \upsilon}$ is a proper endomorphism of $C^*(E)$ (i.e., it is not surjective) which restricts to an automorphism of $D_E$.

The maps $f_1 : \{(11), (13), (32)\} \to \{(11), (13), (32)\}$, $f_2 : \{(11), (13), (32)\} \to \{(21), (23)\}$, and $f_3 : \{(21), (23)\} \to \{(11), (13), (32)\}$ corresponding to permutation $\tau \upsilon$ and involved in verification of Condition (b) are illustrated by the following labeled graph.

Note that the diagram of the map $f_1$, corresponding to an edge whose source and range coincide, gives rise to a rooted tree. This is the left hand side of the diagram above, with the root (the unique fixed point for $f_1$) indicated by a star.

Example 6.6. Consider the following graph $E$:

At level $k = 2$, there are 8 permutations in $\mathcal{P}_E^2 \cap \mathcal{U}_E \cong \mathbb{Z}_2^3$. Denoting edge $e_j$ simply by $j$, these are generated by transpositions $\sigma = (25, 63)$, $\tau = (11, 52)$ and $\upsilon = (36, 44)$. Of these 8
permutations, only $\sigma$ and id satisfy Condition (b). Since $\sigma$ satisfies Condition (d) as well, $\lambda_\sigma$ is an automorphism of $C^*(E)$. We have

\[
\begin{align*}
\lambda_\sigma(S_2) &= S_6S_3S_5^* + S_2S_1S_1^*, \\
\lambda_\sigma(S_6) &= S_2S_5S_3^* + S_6S_4S_4^*, \\
\lambda_\sigma(S_j) &= S_j, \quad j = 1, 3, 4, 5,
\end{align*}
\]

and it follows immediately that $\lambda_\sigma^2 = \text{id}$.

We note that in the present case $K_0(C^*(E)) \cong \mathbb{Z} \cong K_1(C^*(E))$ by [19] and [33, Section 4], see also [39]. Thus $C^*(E)$ is not isomorphic to a Cuntz algebra, and hence this example is not covered in any way by the results of [15].

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References