JOURNAL OF ALGEBRA 122, 106–111 (1989)

On Projective Summands of Induced Modules

GEOFFREY R. ROBINSON

Department of Mathematics, UMIST, P.O. Box 88, Manchester M60 1QD, England

Communicated by Michel Broué

Received September 1987

INTRODUCTION

Let G be a finite group, p be a prime, k be an algebraically closed field of characteristic p, and H be a subgroup of G. Using Green correspondence, it is often possible to obtain information about non-projective summands of modules of the form $\mathrm{Ind}_H^G(M)$ (M a kH-module) in terms of projective summands of induced modules for various sections of G.

The main theme of this paper is the study of projective summands of modules of the form $\mathrm{Ind}_H^G(M)$, especially when M is a simple kH-module. A crucial tool in our investigation is the Reynolds ideal of kG (which is the k-span of the p'-section sums of kG, or equivalently, is equal to $Z(kG) \cap \mathrm{Ann}(J(kG))$), which we denote by $\mathrm{Rey}(kG)$.

The first and last sections of the paper discuss the way in which $\mathrm{Rey}(kG)$ controls projective summands of arbitrary kG-modules. These sections were inspired to some extent by a paper of Landrock [3] dealing with projective summands of $\mathrm{Ind}_H^G(k)$ when $H \in \mathrm{Syl}_p(G)$.

The second section of the paper includes a useful reciprocity theorem, proved using the Reynolds ideal. In the third section, we show that if M is a simple kH-module, then projective summands of $\mathrm{Ind}_H^G(M)$ (if there are any) have relatively large dimension. Alternatively, Propositions 4 and 5 may be viewed as putting upper bounds on the size of certain Cartan invariants.

In the fourth section of the paper, we apply our earlier results to obtain information about modules of the form $\mathrm{Ind}_H^G(S)$ when S is a 1-dimensional kH-module. Again, Propositions 6 and 7 may be regarded as placing restrictions on the size of Cartan invariants in terms of group-theoretic structure.

0021-8693/89 $3.00

1. The Reynolds Ideal and Projective Summands

A Basis for the Reynolds Ideal

Let $\{S_i\colon 1 \leqslant i \leqslant t\}$ be a full set of representatives for the distinct isomorphism types of simple kG-modules. Let β_i be the (k-valued) character of S_i for $1 \leqslant i \leqslant t$. Let P_i be the projective cover of S_i for $1 \leqslant i \leqslant t$.

Let $f_i = \sum_{g \in G} \beta_i(g^{-1})\, g$. Then as $\{\beta_i\colon 1 \leqslant i \leqslant t\}$ is linearly independent over k, and since $\mathrm{Rey}(kG)$ has dimension t, it follows that $\{f_i\colon 1 \leqslant i \leqslant t\}$ is a basis for $\mathrm{Rey}(kG)$ (notice that each β_i is constant on p'-sections).

Proposition 1. *Let M be a kG-module. Then $f_i M \neq 0$ if and only if the projective cover of S_i is a summand of M. Furthermore,* $\dim_k(f_i M) = \dim_k(S_i) \times$ (*the multiplicity of P_i as a summand of M*).

Proof. We may suppose that M is indecomposable, and we do so. We note that $J(kG) f_i M = 0$, so that $f_i M$ is a semi-simple kG-module. Suppose that $f_i M \neq 0$. Choose $m \in M$ with $f_i m \neq 0$. Define the morphism of left kG-modules $\phi\colon kG \to M$ via $\phi(a) = am$ for $a \in kG$. Then $f_i kG \not\subset \ker \phi$.

Let e be a primitive idempotent of kG. Then $f_i e = \sum_{g \in G} \beta_i(eg^{-1})\, g$. Thus $f_i e \neq 0$ if and only if e does not annihilate S_i, which happens if and only if $kGe \cong P_i$. Then $f_i kGe \cong S_i$ if $kGe \cong P_i$, and $f_i kGe = 0$ otherwise. Also $f_i kG \cong \bigoplus_{\dim_k(S_i)\text{ times}} S_i$.

Since $f_i kG \not\subset \ker \phi$, there must be a primitive idempotent $e \in kG$ with $f_i kGe \not\subset \ker \phi$. Then $f_i kGe = \mathrm{Soc}(kGe) \cong S_i$, $kGe \cong P_i$. Since $\mathrm{soc}(kGe)$ is simple, $kGe \cap \ker \phi = \{0\}$.

Thus kGe is isomorphic to a submodule of M, so is isomorphic to a summand of M (as kGe is injective). Since M is indecomposable, the result now follows.

Our next result uses the Reynolds ideal to prove a fact about the kG-module which can be shown to be true for a wider class of algebras (via different methods).

Proposition 2. *Let M be a kG-module, and let $E = \mathrm{End}_{kG}(M)$. Then $J(E)M$ has no projective submodule.*

Proof. Suppose that X is a projective indecomposable submodule of $J(E)M$. Then X is a summand of M, say $X = iM$ for some primitive idempotent i of E. For some value of j we have $f_j X = \mathrm{soc}(X)$. Now $X = iJ(E)M$. Thus for some primitive idempotent i' of E we must have $f_j iJ(E)\, i'M \neq 0$. In particular $f_j i'M \neq 0$. Since i' is primitive, $i'M$ is indecomposable. Since $f_j i'M \neq 0$ we must have $f_j i'M = \mathrm{soc}(i'M)$ and $i'M \cong X$. Thus $i' = uiu^{-1}$ for some unit, u say, of E. Hence $iJ(E)i'M = iJ(E)iM$. Now $f_j X$ is iEi-invariant, and if follows that there is an algebra epimorphism from iEi onto $\mathrm{End}_{kG}(f_j X)$ (note that every kG-endomorphism of X which annihilates soc(X) is nilpotent). Since $\mathrm{End}_{kG}(f_j X)$ is a simple algebra, it follows that $iJ(E)i$ annihilates $f_j X$. Hence f_j annihilates $iJ(E)iM$ (so also $iJ(E)i'M$), a contradiction.

2. A Reciprocity Theorem

Let H be a subgroup of G, $\{\tilde{S}_j\colon 1 \leqslant j \leqslant r\}$ be a full set of isomorphism types of simple kH-modules, and $\tilde{P}_j$ be the projective cover of $\tilde{S}_j$.

Theorem 3. *For all i, j, the multiplicity of P_i is a summand of* $\mathrm{Ind}_H^G(\tilde{S}_j)$ *is the multiplicity of $\tilde{P}_j$ as a summand of* $\mathrm{Res}_H^G(S_i)$.

Proof. Let $\{\tilde{f}_j\colon 1 \leqslant j \leqslant r\}$ be the elements of $\mathrm{Rey}(kH)$ defined as the f_i's were for kG.

We note that for any j, $kG\tilde{f}_j$ is isomorphic to $\bigoplus_{\dim_k(\tilde{S}_j)\text{ copies}} \mathrm{Ind}_H^G(\tilde{S}_j)$. From the results of Section 1, it follows that $\dim_k(f_i kG\tilde{f}_j)$ is $\dim_k(S_i)$ times the multiplicity of P_i as a summand of $kG\tilde{f}_j$.

Thus we can conclude that $\dim_k(f_i kG\tilde{f}_j) = \dim_k(S_i) \times \dim_k(\tilde{S}_j) \times$ (multiplicity of P_i "in" $\mathrm{Ind}_H^G(\tilde{S}_j)$).

On the other hand, we see that $\dim_k(f_i kG\tilde{f}_j) = \dim_k(\tilde{f}_j^0 kG f_i^0)$ (where $(\sum \alpha_g g)^0 = \sum \alpha_g g^{-1}$). Thus, again from Section 1, we see that $\dim_k(\tilde{f}_j^0 kG f_i^0)$ is $\dim_k(S_j^*)$ times the multiplicity of $\tilde{P}_j^*$ as a summand of $\mathrm{Res}_H^G(kGf_i^0)$. Since kGf_i^0 is isomorphic to $\bigoplus_{\dim_k(S_i^*)\text{ copies}} S_i^*$, the result now follows easily (where M^* denotes the dual of a module M).[1]

3. On the Dimension of Projective Summands

Proposition 4. *For $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant r$, let m_{ij} denote the multiplicity of P_i as a summand of* $\mathrm{Ind}_H^G(\tilde{S}_j)$. *Then* $\dim_k(P_i) \geqslant \sum_{j=1}^r \sum_{l=1}^t c_{il} m_{lj} \dim_k(\tilde{P}_j)$ (*for any given i*) (*where c_{il} is the Cartan invariant* $\dim_k(\mathrm{Hom}_{kG}(P_i, P_l))$).

Furthermore, if H contains a Sylow p-subgroup of G, then equality holds if and only if P_i is simple.

Proof. We note that S_l occurs with multiplicity c_{il} as a composition factor of P_i. Since $\mathrm{Res}_H^G(S_l)$ contains $\tilde{P}_j$ as a summand with multiplicity m_{lj} (by the results of Section 2), it follows that $\mathrm{Res}_H^G(P_i)$ contains $\tilde{P}_j$ as a summand with multiplicity at least $\sum_{l=1}^t c_{il} m_{lj}$ (since $\tilde{P}_j$ is a projective kH-module). The inequality is thus established.

Suppose that $p \nmid [G:H]$, but that equality holds. Then there is certainly

[1] *Note added in proof.* The reciprocity formula above can also be proved using projective homomorphisms, and we have learned that various people seem to be aware of the result, among them D. J. Benson and P. J. Webb. However, there seems to be no proof in the literature, and we feel that the proof given here is of some interest.

some simple kG-module S_l such that $c_{li}m_{lj} \neq 0$ for some j. Furthermore, we must have $\mathrm{Res}^G_H(S_l) \cong \bigoplus_{j=1}^r \bigoplus_{m_{lj}\text{ copies}} \tilde{P}_j$. Thus $\mathrm{Res}^G_H(S_l)$ is projective. Since $p \nmid [G:H]$, it follows that S_l is projective. Since S_l is a composition factor of P_i, we have $i = l$, and $S_i = P_i$.

Conversely, if P_i is simple, then $\mathrm{Res}^G_H(P_i) \cong \bigoplus_{j=1}^r \bigoplus_{m_{ij}\text{ copies}} \tilde{P}_j$ and equality does indeed hold.

PROPOSITION 5. *For* $1 \leqslant j \leqslant r$, *we have* $[G:H]\dim_k(\tilde{P}_j) \geqslant \sum_{i=1}^t \sum_{l=1}^r \tilde{c}_{jl} m_{il} \dim_k(P_i)$. *Furthermore, equality holds if and only if* $\tilde{P}_j$ *is simple* ($\tilde{c}_{jl}$ *denotes the Cartan invariant* $\dim_k(\mathrm{Hom}_{kH}(\tilde{P}_j, \tilde{P}_l))$).

Proof. For $1 \leqslant l \leqslant r$, $\tilde{S}_l$ occurs $\tilde{c}_{lj}$ times as a composition factor of $\tilde{P}_j$, and P_i occurs m_{il} times as a summand of $\mathrm{Ind}^G_H(\tilde{S}_l)$. Thus P_i occurs at least $\sum_{l=1}^r \tilde{c}_{jl} m_{il}$ times as a summand of $\mathrm{Ind}^G_H(\tilde{P}_j)$. The inequality follows.

If equality holds, then there is some l with $\tilde{c}_{lj} \neq 0$ and $m_{il} \neq 0$ for some i. Then $\mathrm{Ind}^G_H(\tilde{S}_l)$ must be isomorphic to $\bigoplus_{i=1}^t \bigoplus_{(m_{il}\text{ copies})} P_i$. Thus $\mathrm{Ind}^G_H(\tilde{S}_l)$ is projective, so that $\tilde{S}_l$ must be projective (for $\tilde{S}_l$ is a summand of $\mathrm{Res}^G_H(\mathrm{Ind}^G_H(\tilde{S}_l))$). Since $\tilde{S}_l$ is a composition factor of $\tilde{P}_j$, we must have $\tilde{S}_l \cong \tilde{P}_j$. Conversely, if $\tilde{P}_j \cong \tilde{S}_l$, then $\mathrm{Ind}^G_H(\tilde{P}_j) \cong \bigoplus_{i=1}^t \bigoplus_{m_{ij}\text{ copies}} P_i$, and equality does hold.

4. APPLICATIONS

Let H, L be subgroups of G (possibly $H = L$). Let $\{\tilde{T}_i : 1 \leqslant i \leqslant s\}$ be a full set of isomorphism types of simple kL-modules (and let us keep the notation adopted earlier for the simple kH-modules, and the multiplicities m_{ij}).

Let n_{ij} denote the multiplicity of P_i as a summand of $\mathrm{Ind}^G_L(\tilde{T}_j)$.

PROPOSITION 6. *Suppose that* $\tilde{S}_j$ *and* $\tilde{T}_q$ *are one-dimensional. Then there are at least* $\sum_{i,l=1}^t c_{il} m_{lj} n_{iq}$ (L, H)*-double coset representatives,* w, *for which* $p \nmid |L^w \cap H|$ *and* $\mathrm{Res}^{L^w}_{L^w \cap H}(w^{-1} \otimes \tilde{T}_q) \cong \mathrm{Res}^H_{L^w \cap H}(\tilde{S}_j)$.

Proof. From the proof of Proposition 4, $\tilde{P}_j$ occurs at least $\sum_{l=1}^t c_{il} m_{lj}$ times as a summand of $\mathrm{Res}^G_H(P_i)$ (for each i). Since P_i occurs n_{iq} times as a summand of $\mathrm{Ind}^G_L(\tilde{T}_q)$ it follows that $\tilde{P}_j$ occurs at least $\sum_{i,l=1}^t c_{il} m_{lj} n_{iq}$ times as a summand of $\mathrm{Res}^G_H(\mathrm{Ind}^G_L(\tilde{T}_q))$.

On the other hand, we know that

$$\mathrm{Res}^G_H(\mathrm{Ind}^G_L(\tilde{T}_q)) \cong \bigoplus_{w \in L\backslash G/H} \mathrm{Ind}^H_{L^w \cap H}(\mathrm{Res}^{L^w}_{L^w \cap H}(w^{-1} \otimes \tilde{T}_q)).$$

Applying Theorem 3 within H, we see that for any $w \in L\backslash G/H$ the multiplicity of $\tilde{P}_j$ as a summand of $\mathrm{Ind}^H_{L^w \cap H}(\mathrm{Res}^{L^w}_{L^w \cap H}(w^{-1} \otimes \tilde{T}_q))$ is the same as the multiplicity of the projective cover of $\mathrm{Res}^{L^w}_{L^w \cap H}(w^{-1} \otimes \tilde{T}_q)$ as a summand of $\mathrm{Res}^{H}_{L^w \cap H}(\tilde{S}_j)$.

Since $\tilde{S}_j$ is one-dimensional, this last multiplicity is 1 if $p \nmid |L^w \cap H|$ and $\mathrm{Res}^{L^w}_{L^w \cap H}(w^{-1} \otimes \tilde{T}_q) \cong \mathrm{Res}^{H}_{L^w \cap H}(\tilde{S}_j)$, and is 0 otherwise. This suffices to complete the proof of Proposition 6.

Remark. Of course, Proposition 6 may be applied with $\tilde{S}_j$ and $\tilde{T}_q$ both being trivial.

PROPOSITION 7. *Suppose that $H \in \mathrm{Syl}_p(G)$. Then the number of (H, H) double coset representatives, w, for which $p \nmid |H^w \cap H|$ is at least $(\sum_{\{i: P_i \text{ is not simple}\}} m_{i1}) + \sum_{i,l=1}^{t} c_{il} m_{i1} m_{l1}$.*

Proof. Since $\mathrm{Res}^G_H(P_i)$ is projective, it follows that $\mathrm{Res}^G_H(P_i)$ contains $\tilde{P}_1$ ($\cong kH$) as a summand with multiplicity at least $1 + \sum_{l=1}^{t} c_{il} m_{l1}$ whenever P_i is not simple (from the proof of Proposition 4). It follows, then, that $\mathrm{Res}^G_H(\mathrm{Ind}^G_H(\tilde{S}_1))$ contains $\tilde{P}_1$ as a summand with multiplicity at least $(\sum_{\{i: P_i \text{ is not simple}\}} m_{i1}) + \sum_{i,l=1}^{t} m_{i1} m_{l1} c_{il}$. The result now follows as in Proposition 6.

EXAMPLE. Suppose that there is only one (H, H)-double coset representative, w_0 say, for which $p \nmid |H^{w_0} \cap H|$. Then $\sum_{i,l=1}^{t} c_{il} m_{i1} m_{l1} \leqslant 1$ from Proposition 6 (where we have labelled so that $\tilde{S}_1$ is the trivial kH-module). Hence there can be at most one value of i for which P_i occurs as a summand of $\mathrm{Ind}^G_H(\tilde{S}_1)$, and if there is such a value of i, then $m_{i1} = c_{ii} = 1$. The fact that $c_{ii} = 1$ forces P_i to be simple.

(This situation can occur—for example, if G has a split (B, N)-pair in characteristic p, and $H = B$.) Similar types of assertions can be made about modules of the form $\mathrm{Ind}^G_H(\tilde{S}_j)$ when $\tilde{S}_j$ is any one-dimensional module in the above situation.

5. FURTHER REMARKS ON PROJECTIVE SUMMANDS

Let A be a symmetric k-algebra (i.e., there is a symmetric k-linear form t: $A \to k$ such that $\ker t$ contains no non-zero right ideal of A).

We may define $\mathrm{Rey}(A)$ to be $Z(A) \cap \mathrm{Ann}(J(A))$. Let $K(A)$ denote the k-span of $\{ab - ba \colon a, b \in A\}$.

The following result (and its proof) is just a restatement of Satz D of Külshammer [2].

PROPOSITION 8. $\mathrm{Rey}(A) = [K(A) + J(A)]^{\perp}$.

Proof. Note that $x \in J(A)^{\perp}$ iff for every $j \in J(A)$ and every $a \in A$ we have $t(xja)=0$ iff $x \in \mathrm{Ann}(J(A))$ (since $\ker t$ contains no non-zero right ideal of A). Also $x \in K(A)^{\perp}$ iff for all $a, b \in A$ we have $t(x(ab-ba))=0$. This happens if and only if for all $a, b \in A$ we have $t(bxa)$ $(=t(xab))=$ $t(xba)$ $(=t(bax))$. Thus for a fixed a we need $t(bxa)=t(bax)$ for all $b \in A$ so $t(b(xa-ax))=0$ for all $b \in A$, so $xa-ax=0$ (as $\ker t$ contains no left ideal of A). Thus the above equality holds if and only if $x \in Z(A)$.

PROPOSITION 9. *Let P be a projective kG-module. Then $E=\mathrm{End}_{kG}(P)$ is a symmetric algebra.*

Proof. Since P is projective, there is some $\alpha \in \mathrm{End}_k(P)$ with $\mathrm{Tr}_1^G(\alpha)=1$. We define $t\colon E \to k$ as follows: given $x \in E$, then $x=\mathrm{Tr}_1^G(x_0)$ for some $x_0 \in \mathrm{End}_k(P)$, so set $t(x)=\mathrm{trace}(x_0)$. As in [1], t is symmetric, and is well defined. We claim that $\ker t$ contains no non-zero right ideal of $\mathrm{End}_{kG}(P)$. For given $0 \neq x \in E$, there is some $y_0 \in \mathrm{End}_k(P)$ with $\mathrm{trace}(xy_0) \neq 0$. Thus $t(x\,\mathrm{Tr}_1^G(y_0)) \neq 0$, as $\mathrm{Tr}_1^G(xy_0)=x\,\mathrm{Tr}_1^G(y_0)$.

Now let M be any kG-module, and write $M=P \oplus X$, where P is projective and X has no projective summand. There is an algebra morphism from $Z(kG)$ into $\mathrm{End}_{kG}(M)$. Furthermore, $\mathrm{Rey}(kG)$ annihilates X by Proposition 1, and $\mathrm{Rey}(kG)$ annihilates $J(\mathrm{End}_{kG}(P))$ by Propositions 1 and 2. Thus $\mathrm{Rey}(kG)$ maps into $Z(\mathrm{End}_{kG}(P)) \cap \mathrm{Ann}(J(\mathrm{End}_{kG}(P)))=$ $\mathrm{Rey}(\mathrm{End}_{kG}(P))$. Let $E=\mathrm{End}_{kG}(P)$. Then $\dim_k(\mathrm{Rey}(E))=\dim_k(E)-$ $\dim_k(K(E)+J(E))$, which is the number of matrix algebra summands of $E/J(E)$, which is the number of conjugacy classes of primitive idempotents of E (under the units of E), which is the number of isomorphism types of indecomposable summands of P.

On the other hand, if $f_{i_1} \cdots f_{i_s}$ are basis elements of $\mathrm{Rey}(kG)$ which do not annihilate P, their images in $\mathrm{Rey}(E)$ are linearly independent (for f_j annihilates any summand of P whose socle is not isomorphic to S_j). Thus the dimension of the image of $\mathrm{Rey}(kG)$ is the number of isomorphism types of summands of P, which is $\dim_k(\mathrm{Rey}(E))$. Hence we have (with the notation above):

THEOREM 10. *Under the natural algebra morphism from $Z(kG)$ into $\mathrm{End}_{kG}(M)$, the image of* $\mathrm{Rey}(kG)$ *is* $\mathrm{Rey}(\mathrm{End}_{kG}(P))$, *and the dimension of the image of* $\mathrm{Rey}(kG)$ *is the number of isomorphism types of indecomposable projective summands of M.*

REFERENCES

1. M. BROUÉ AND G. R. ROBINSON, Bilinear forms on G-algebras, *J. Algebra* **104** (1986), 377–396.
2. B. KÜLSHAMMER, Bemerkungen über die Gruppenalgebra als symmetrische Algebra, *J. Algebra* **72** (1981), 1–7.
3. P. LANDROCK, The modular Hecke algebra of a Sylow p-subgroup, *J. Algebra* **104** (1986), 141–145.