Limit results for the empirical process of squared residuals in GARCH models

István Berkes\textsuperscript{a,1,2}, Lajos Horváth\textsuperscript{b,*,2}

\textsuperscript{a}A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, P.O. Box 127, H-1364 Budapest, Hungary

\textsuperscript{b}Department of Mathematics, University of Utah, 155 South 1440 East, Salt Lake City, UT 84112-0090, USA

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Abstract

We study the asymptotic behavior of the empirical distribution function and the empirical process of squared residuals. We prove the Glivenko–Cantelli theorem for the empirical distribution function. We show that the two-parameter empirical process converges to a Gaussian process.

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1. Introduction

Empirical processes play a prominent role in statistics. Several statistical procedures involve functionals of empirical processes. For a review of empirical processes of independent, identically distributed random variables we refer to Shorack and Wellner (1986). In time series, some of the variables of interest cannot be observed directly and residuals are used in statistical analysis. Inference based on residuals is a fundamental tool in linear time-series models; see Brockwell and Davis (1991). Berkes and Horváth (2002) provides a survey of the asymptotic theory of residuals in non-linear time-series models. Li and Mak (1994) and Horváth and Kokoszka (2001) obtain multivariate central limit theorems for squared residual autocorrelations of ARCH...

sequences. Their results were generalized for GARCH\((p,q)\) sequences by Berkes et al. (2001a). Tjøstheim (1999) considers nonparametric tests based on squared residuals. The weak convergence of the empirical process of squared residuals of ARCH\((p)\) sequences is proven by Horváth et al. (2001). In this paper, we study the asymptotic behavior of the empirical process of squared residuals of GARCH\((p,q)\) sequences.

The GARCH\((p,q)\) process \(\{y_k, -\infty < k < \infty\}\) is defined by the equations

\[
y_k = \sigma_k \varepsilon_k
\]

and

\[
\sigma_k^2 = \omega + \sum_{1 \leq i \leq p} \alpha_i y_{k-i}^2 + \sum_{1 \leq j \leq q} \beta_j \sigma_{k-j}^2,
\]

where

\[
\omega > 0, \quad \alpha_i \geq 0, \quad 1 \leq i \leq p, \quad \beta_j \geq 0, \quad 1 \leq j \leq q
\]

are constants. We also assume that

\[
\{\varepsilon_i, -\infty < i < \infty\}
\]

are independent, identically distributed random variables.

The parameter of the process is the vector \(\theta = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)\). In case of GARCH\((1,1)\), Nelson (1990) proved that \((1.1)\) and \((1.2)\) have a unique stationary solution if and only if \(E \log(\beta_1 + \alpha_1 \varepsilon_0^2) < 0\). The general case was solved by Bougerol and Picard (1992a, b). Let

\[
\tau_n = (\beta_1 + \alpha_1 \varepsilon_n^2, \beta_2, \ldots, \beta_{q-1}) \in \mathbb{R}^{q-1},
\]

\[
\xi_n = (\varepsilon_n^2, 0, \ldots, 0) \in \mathbb{R}^{q-1}
\]

and

\[
\alpha = (\alpha_2, \ldots, \alpha_{p-1}) \in \mathbb{R}^{p-2}.
\]

(Clearly, by including extra terms with zero coefficients in \((1.2)\), we can achieve that \(\min(p, q) \geq 2\).) Define the \((p+q-1) \times (p+q-1)\) matrix \(A_n\), written in block form, by

\[
A_n = \begin{bmatrix}
\tau_n & \beta_q & \alpha & \alpha_p \\
I_{q-1} & 0 & 0 & 0 \\
\xi_n & 0 & 0 & 0 \\
0 & 0 & I_{p-2} & 0
\end{bmatrix},
\]

where \(I_{q-1}\) and \(I_{p-2}\) are the identity matrices of size \(q-1\) and \(p-2\), respectively.

The norm of any \(d \times d\) matrix \(M\) is defined by

\[
\|M\| = \sup\{\|Mx\|_d / \|x\|_d : x \in \mathbb{R}^d, x \neq 0\},
\]
where \( \| \cdot \|_d \) is the usual (Euclidian) norm in \( \mathbb{R}^d \). The top Lyapunov exponent \( \gamma_L \) associated with the sequence \( \{ A_n, -\infty < n < \infty \} \) is
\[
\gamma_L = \inf_{0 \leq n < \infty} \frac{1}{n + 1} E \log \| A_0 A_1 \ldots A_n \|,
\]
assuming that
\[
E(\log \| A_0 \|) < \infty. \tag{1.5}
\]
Bougerol and Picard (1992a, b) showed that if (1.5) holds, then (1.1) and (1.2) have a unique stationary solution if and only if
\[
\gamma_L < 0. \tag{1.6}
\]
For a generalization of Bougerol and Picard (1992a, b) we refer to Kazakevičius and Leipus (2002).

Assumptions (1.1)–(1.6) are a minimal set of conditions for the existence and stationarity of the GARCH\((p,q)\) sequence, so we assume throughout this paper that (1.1)–(1.6) are satisfied.

Assuming that \( y_1, y_2, \ldots, y_n \) have been observed, we wish to estimate the distribution function of \( \varepsilon_0^2 \). Since \( \varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_n^2 \) are not observed, we use the squared residuals. The definition of the squared residuals is based on the observation that \( \sigma_i^2 \) can be written as a linear function of \( y_{i-1}^2, y_{i-2}^2, \ldots \). Following Berkes et al. (2001b) we start with the recursive definition of a sequence of functions. Let \( u = (x, s_1, \ldots, s_p, t_1, \ldots, t_q) \). If \( q \geq p \), then let
\[
c_0(u) = x / (1 - (t_1 + \cdots + t_q)),
\]
\[
c_1(u) = s_1,
\]
\[
c_2(u) = s_2 + t_1 c_1(u),
\]
\[
\vdots
\]
\[
c_p(u) = s_p + t_1 c_{p-1}(u) + \cdots + t_{p-1} c_1(u),
\]
\[
c_{p+1}(u) = t_1 c_p(u) + \cdots + t_p c_1(u)
\]
\[
\vdots
\]
\[
c_q(u) = t_1 c_{q-1}(u) + \cdots + t_{q-1} c_1(u)
\]
(if \( p = q \), then we stop at \( c_{p+1}(u) = c_{q+1}(u) \)) and if \( q < p \), the equations above are replaced with
\[
c_0(u) = x / (1 - (t_1 + \cdots + t_q)),
\]
\[
c_1(u) = s_1,
\]
\[ c_2(u) = s_2 + t_1 c_1(u) \]

\[ \vdots \]

\[ c_{q+1}(u) = s_{q+1} + t_1 c_q(u) + \ldots + t_q c_1(u) \]

\[ \vdots \]

\[ c_p(u) = s_p + t_1 c_{p-1}(u) + \ldots + t_q c_{p-q}(u). \]

If \( i > R = \max(p, q) \), then

\[ c_i(u) = t_1 c_{i-1}(u) + t_2 c_{i-2}(u) + \ldots + t_q c_{i-q}(u). \quad (1.7) \]

Berkes et al. (2001b) (cf. also Nelson and Cao, 1992) showed that

\[ w_k(u) = c_0(u) + \sum_{1 \leq i < \infty} c_i(u) y_{k-i}^2 \]

exists with probability one for \( u \in U \), where

\[ U = \{ u : t_1 + t_2 + \ldots + t_q \leq q_0 \text{ and } u < \min(x, s_1, s_2, \ldots, s_p, t_1, t_2, \ldots, t_q) \leq \max(x, s_1, s_2, \ldots, s_p, t_1, t_2, \ldots, t_q) \leq \tilde{u} \} \]

with \( 0 < u < \tilde{u}, 0 < q_0 < 1 \) and \( q u < q_0 \). Under these assumptions \( U \) is a compact set and all elements of \( U \) can be parameters of GARCH(\( p, q \)) processes. They also showed that for \( \theta \in U \)

\[ \sigma_k^2 = w_k(\theta), \quad -\infty < k < \infty. \quad (1.8) \]

We cannot compute \( w_k(u) \) from the data, so we shall use

\[ \hat{w}_k(u) = c_0(u) + \sum_{1 \leq i \leq k-1} c_i(u) y_{k-i}^2, \quad 2 \leq k \leq n. \]

Relation (1.8) suggests that \( \hat{w}_k(\hat{\theta}_n) \) can be used as an estimator for \( \sigma_k^2 \), where \( \hat{\theta}_n \) is an estimator for \( \theta \). The squared residuals are

\[ \hat{\varepsilon}_k^2 = \frac{y_k^2}{\hat{w}_k(\hat{\theta}_n)}, \quad 2 \leq k \leq n. \]

We note that \( \hat{w}_k(\hat{\theta}_n) \geq \varepsilon > 0 \). In this paper we investigate the asymptotic properties of the empirical distribution function

\[ \hat{F}_k(t) = \frac{1}{k} \sum_{2 \leq i \leq k} I\{ \hat{\varepsilon}_i^2 \leq t \}, \quad 0 \leq t < \infty, \quad 2 \leq k \leq n \]
and the corresponding (sequential) empirical process of the squared residuals
\[ \hat{\varepsilon}_n(t,s) = n^{1/2}(\hat{F}_n(t) - F(t)), \quad 0 \leq t < \infty, \quad 0 \leq s \leq 1, \]
where
\[ F(t) = P\{ \varepsilon_0^2 \leq t \}, \quad 0 \leq t < \infty. \]
In the proofs we compare \( \hat{\varepsilon}_n(t,s) \) to \( \varepsilon_n(t,s) \), the empirical process of \( \varepsilon_2, \ldots, \varepsilon_n \), where
\[ \varepsilon_n(t,s) = n^{1/2}(F_{[ns]}(t) - F(t)), \quad 0 \leq t < \infty, \quad 0 \leq s \leq 1 \]
and
\[ F_k(t) = \frac{1}{k} \sum_{2 \leq i \leq k} I\{ \varepsilon_i^2 \leq t \}, \quad 0 \leq t < \infty. \]

2. Empirical process of squared residuals

We assume that all coordinates of \( \theta \) are strictly positive, i.e.
\[ \omega > 0, \quad \varepsilon_i > 0, \quad 1 \leq i \leq p \quad \text{and} \quad \beta_j > 0, \quad 1 \leq j \leq q. \tag{2.1} \]
We note that (1.6) implies
\[ \beta_1 + \beta_2 + \cdots + \beta_q < 1 \tag{2.2} \]
(cf. Bougerol and Picard, 1992b). We also assume that
\[ E|\varepsilon_0^2|^{\gamma} < \infty \quad \text{with some} \quad \gamma > 0 \tag{2.3} \]
and
\[ \lim_{t \to 0} t^{-\mu}P\{ \varepsilon_0^2 \leq t \} = 0 \quad \text{with some} \quad \mu > 0. \tag{2.4} \]
The next result shows that if \( \hat{\theta}_n \) is a strongly consistent estimator for \( \theta \), i.e.
\[ \hat{\theta}_n \to \theta \quad \text{a.s.,} \tag{2.5} \]
then \( \hat{F}_n \) satisfies the Glivenko–Cantelli theorem.

**Theorem 2.1.** If \( F \) is continuous and (2.1), (2.3)–(2.5) hold, then
\[ \lim_{n \to \infty} \sup_{0 \leq t < \infty} |\hat{F}_n(t) - F(t)| = 0 \quad \text{a.s.} \]
The proof of Theorem 2.1 is given in Section 5.

The Glivenko–Cantelli theorem stated in Theorem 2.1 holds under the condition that \( F \) is continuous. The weak convergence of \( \hat{\varepsilon}_n(t,s) \) will require the existence of a smooth density, \( f = F' \). Namely, we assume that
\[ L = \sup_{0 < s < \infty} sf(s) < \infty, \tag{2.6} \]
\[
\lim_{s \to 0} sf(s) = 0, \quad (2.7)
\]
\[
\lim_{s \to \infty} sf(s) = 0 \quad (2.8)
\]
and
\[
f \text{ is continuous on } (0, \infty). \quad (2.9)
\]
Lemma 4.2 will imply that
\[
d_0 = (d_1, \ldots, d_{p+q+1}) = E\frac{w'_0(\theta)}{w_0(\theta)} \quad (2.10)
\]
exists.
We assume that the estimator \( \hat{\theta}_n \) admits the representation
\[
\hat{\theta}_n - \theta = \frac{1}{n} \sum_{1 \leq i \leq n} (g_1(\varepsilon_i)\mathcal{L}_1(\varepsilon_{i-1}, \varepsilon_{i-2}, \ldots), g_2(\varepsilon_i)\mathcal{L}_2(\varepsilon_{i-1}, \varepsilon_{i-2}, \ldots), \ldots, g_{p+q+1}(\varepsilon_i)\mathcal{L}_{p+q+1}(\varepsilon_{i-1}, \varepsilon_{i-2}, \ldots)) + o_P(n^{-1/2}). \quad (2.11)
\]
We will show in Section 3 that the most often used estimators of \( \theta \) satisfy (2.11). The random variables in (2.11) are also assumed to have at least two finite moments:
\[
Eg_i(\varepsilon_0) = 0 \quad \text{and} \quad Eg_i^2(\varepsilon_0) < \infty, \quad 1 \leq i \leq p + q + 1 \quad (2.12)
\]
and
\[
E\mathcal{L}_i^2(\varepsilon_{-1}, \varepsilon_{-2}, \ldots) < \infty, \quad 1 \leq i \leq p + q + 1. \quad (2.13)
\]
Let
\[
\tilde{q}_{ij} = Eg_i(\varepsilon_0)g_j(\varepsilon_0), \quad \tilde{t}_i = E\mathcal{L}_i(\varepsilon_{-1}, \varepsilon_{-2}, \ldots),
\]
\[
\tilde{t}_{ij} = E\mathcal{L}_i(\varepsilon_{-1}, \varepsilon_{-2}, \ldots)\mathcal{L}_j(\varepsilon_{-1}, \varepsilon_{-2}, \ldots), \quad 1 \leq i, j \leq p + q + 1,
\]
\[
q_i(t) = E\{I\{\varepsilon_0^2 \leq t\}g_i(\varepsilon_0)\}, \quad 1 \leq i \leq p + q + 1
\]
and
\[
r(t, t', s, s') = (s \wedge s')(F(t \wedge t') - F(t)F(t')) + tf(t)ss' \sum_{1 \leq i \leq p+q+1} q_i(t')\tilde{t}_id_i
\]
\[
+ t'f(t')ss' \sum_{1 \leq i \leq p+q+1} q_i(t)\tilde{t}_id_i + tf(t)t'f(t')ss' \sum_{1 \leq i, j \leq p+q+1} \tilde{q}_{ij}\tilde{t}_{ij}d_id_j,
\]
where \( t \wedge t' = \min(t, t') \).

**Theorem 2.2.** If (2.1), (2.3), (2.4), (2.6)–(2.9) and (2.11)–(2.13) hold, then
\[
\hat{\theta}_n(t, s) \to \Gamma(t, s),
\]
where the convergence is in the Skorokhod space \( \mathcal{D}([0, \infty) \times [0, 1]) \) and \( \Gamma \) is a Gaussian process with \( E\Gamma(t, s) = 0 \) and \( E\Gamma(t, s)\Gamma(t', s') = r(t, t', s, s') \).
Before presenting the detailed proofs, we outline the basic idea. First, we show that
\[
\sup_{0 \leq s \leq 1} \sup_{0 < t < \infty} n^{1/2} \left( s \hat{F}_{[n]}(t) - \frac{1}{n} \sum_{2 \leq i \leq [ns]} F(t \hat{\theta}_i / \sigma_i^2) \right) - \tau_n(t, s) = o_p(1) \tag{2.14}
\]
and then
\[
\sup_{0 \leq s \leq 1} \sup_{0 < t < \infty} n^{-1/2} \sum_{2 \leq i \leq [ns]} (F(t \hat{\theta}_i / \sigma_i^2) - F(t)) - stf(t)n^{1/2}(\hat{\theta}_n - \theta)d_0^T = o_p(1). \tag{2.15}
\]
By (2.11) we can replace \( \hat{\theta}_n - \theta \) with partial sums and get
\[
\sup_{0 \leq s \leq 1} \sup_{0 < t < \infty} n^{-1/2} \sum_{2 \leq i \leq [ns]} (F(t \hat{\theta}_i / \sigma_i^2) - F(t)) - stf(t)n^{-1/2}(\hat{\theta}_n - \theta)d_0^T = o_p(1). \tag{2.16}
\]
On account of (2.14) and (2.16) (cf. Lemma 6.6) it will be sufficient to prove that
\[
\left\{ \left( \tau_n(t, s), n^{-1/2} \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq p+q+1} g_j(\epsilon_i)\epsilon_j(\epsilon_{i-1}, \epsilon_{i-2}, \ldots) \right) \right\},
\]
converges weakly to a vector valued Gaussian process \( \{(K(F(t), s), \tau), 0 \leq t < \infty, 0 \leq s \leq 1\} \) with
\[
EK(F(t), s) = 0, \quad E\tau = 0, \tag{2.17}
\]
\[
EK(F(t), s)K(F(t'), s') = (s \wedge s')(F(t \wedge t') - F(t)F(t')), \tag{2.18}
\]
\[
E\tau^2 = \sum_{1 \leq i, j \leq p+q+1} \bar{q}_{ij} \bar{\epsilon}_{ij} d_i d_j \tag{2.19}
\]
and
\[
E(\tau K(F(t), s)) = s \sum_{1 \leq i \leq p+q+1} q_i(t) \bar{\epsilon}_i d_i. \tag{2.20}
\]
Here $K(x,s)$ is a Kiefer process, i.e. a Gaussian process with $EK(x,s) = 0$ and $EK(x,s) K(y,t) = (x \wedge y - xy)(s \wedge t)$. Also, the limit process $\Gamma(t,s)$ in Theorem 2.2 can be represented as

$$\{\Gamma(t,s), 0 \leq t < \infty, 0 \leq s \leq 1\} \overset{D}{=} \{K(F(t),s) + stf(t)\tau, 0 \leq t < \infty, 0 \leq s \leq 1\}.$$

**Remark 2.1.** We note that $\{\Gamma(t,s) - s\Gamma(t,1), 0 \leq t < \infty, 0 \leq s \leq 1\} \overset{D}{=} \{K^*(F(t),s), 0 \leq t < \infty, 0 \leq s \leq 1\}$, where $K^*$ is a tied-down Kiefer process. The process $K^*$ has been extensively studied and, for example, the distributions of its supremum and square-integral are extensively tabulated in Picard (1985) and Blum et al. (1961).

**Remark 2.2.** Boldin (1998, 2000, 2002) and Viazilov (2001) study the non-squared residuals of the simpler ARCH($p$) and GARCH(1,1) models. However, the results and the assumptions are comparable to Theorems 2.1 and 2.2.

Next, we discuss some examples when condition (2.11) is satisfied. Section 4 contains some preliminary results on GARCH sequences. The proofs of Theorems 2.1 and 2.2 are presented in Sections 5 and 6.

### 3. Asymptotic linearity of estimators

The process $\{y_i, -\infty < i < \infty\}$ uniquely determines $\theta$ if (cf. Berkes et al., 2001b):

- the polynomials $\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_p x^p$ and $1 - \beta_1 x - \beta_2 x^2 - \cdots - \beta_q x^q$ are coprimes over the set of polynomials with real coefficients

$$\text{(3.1)}$$

and

$$\epsilon_0^2 \text{ is a nondegenerate random variable.} \quad \text{(3.2)}$$

In the examples we will discuss, $\hat{\theta}_n$ is given by

$$\hat{\theta}_n = \arg \max_{\theta \in U} \hat{L}_n(\theta),$$

where $\hat{L}_n(\theta)$ is a suitably chosen random function, $U$ is the set from Section 1 satisfying

$$\theta \in U. \quad \text{(3.3)}$$

#### 3.1. The quasi-maximum likelihood estimator

Let

$$\hat{L}_n(\theta) = \sum_{2 \leq k \leq n} \log \left\{ \frac{1}{\hat{w}_k^{1/2}(\theta)} \exp \left( -\frac{1}{2} \frac{y_k^2}{\hat{w}_k(\theta)} \right) \right\}. $$
If $E \varepsilon_0^2 = 1$ and $E |\varepsilon_0^2|^\kappa < \infty$ with some $\kappa > 1$, then (2.5) holds. If $E \varepsilon_0^4 < \infty$, then

$$\hat{\theta}_n - \theta = \frac{1}{n} \sum_{1 \leq k \leq n} (\varepsilon_k^2 - 1) \frac{w_k'(\theta)}{w_k(\theta)} A^{-1} + o_p(n^{-1/2}),$$

(3.4)

where

$$A = E \left\{ \left( \frac{w_0'(\theta)}{w_0(\theta)} \right)^T \frac{w_0'(\theta)}{w_0(\theta)} \right\}. \quad (3.5)$$

The strong consistency of the quasi-maximum likelihood estimator is due to Berkes et al. (2001b). The asymptotic linearity in (3.4) under the present condition is in Berkes et al. (2001b) and Berkes and Horváth (2001). For earlier results we refer to Lee and Hansen (1994) and Lumsdaine (1996).

### 3.2. Exponential-based estimator

Let

$$\hat{L}_n(u) = \sum_{2 \leq k \leq n} \log \left\{ \frac{1}{2w_k^{1/2}(u)} \exp \left( - \frac{|y_k|}{w_k^{1/2}(u)} \right) \right\}.$$ 

If $E|\varepsilon_0| = 1$ and $E|\varepsilon_0|^\kappa < \infty$ with some $\kappa > 1$, then (2.5) holds. If we also assume that $E \varepsilon_0^2 < \infty$, then

$$\hat{\theta}_n - \theta = \frac{1}{n} \sum_{1 \leq k \leq n} 2(|\varepsilon_k| - 1) \frac{w_k'(\theta)}{w_k(\theta)} A^{-1} + o_p(n^{-1/2}),$$

(3.6)

where $A$ is defined in (3.5). Berkes and Horváth (2001) obtained the consistency as well as (3.6)

### 3.3. The maximum likelihood estimator

Let $\mathcal{F}_{k-1}$ be the $\sigma$-algebra generated by $\varepsilon_{j}, -\infty < j \leq k - 1$. Conditionally on $\mathcal{F}_{k-1}$, the density of $y_k$ is $h(\cdot/\sigma_k)/\sigma_k$, where $h$ is the density of $\varepsilon_0$. So

$$\hat{L}_n(u) = \sum_{2 \leq k \leq n} \log \left\{ \frac{1}{w_k^{1/2}(u)} h \left( \frac{y_k}{w_k^{1/2}(u)} \right) \right\}$$

is the analogue of the log likelihood function. If the distributions, determined by the scale family of densities $th(yt)$, $t > 0$, are distinct, and further regularity conditions (cf. Lehmann, 1991, Section 6.2) are satisfied, then (2.5) holds. Berkes and Horváth (2001) showed that under some regularity conditions

$$\hat{\theta}_n - \theta = \frac{1}{n} \sum_{1 \leq k \leq n} \left( 1 + \varepsilon_k \frac{h'(\varepsilon_k)}{h(\varepsilon_k)} \right) 2 \left\{ E \left( \left( \frac{h''(\varepsilon_k)}{h(\varepsilon_k)} \right) \right)^{-1} \frac{w_k'(\theta)}{w_k(\theta)} A^{-1} + o_p(n^{-1/2}) \right\}$$

where $A$ is defined in (3.5).
4. Preliminary results

We say that \( \Delta \mathbf{u} = (\Delta x, \Delta s_1, \ldots, \Delta s_p, \Delta t_1, \ldots, \Delta t_q) \geq 0 \), if \( \Delta x \geq 0, \Delta s_i \geq 0, 1 \leq i \leq p \) and \( \Delta t_j \geq 0, 1 \leq j \leq q \). For any \( x^{(1)} < x^{(2)}, s_i^{(1)} < s_i^{(2)}, 1 \leq i \leq p, \ t_j^{(1)} < t_j^{(2)}, 1 \leq j \leq q \), let
\[
A = \{ \mathbf{u} = (x, s_1, \ldots, s_p, t_1, \ldots, t_q): x^{(1)} \leq x \leq x^{(2)}, s_i^{(1)} \leq s_i \leq s_i^{(2)}, 1 \leq i \leq p, t_j^{(1)} \leq t_j \leq t_j^{(2)}, 1 \leq j \leq q \}
\]
and \( \mathbf{u}^{(1)} = (x^{(1)}, s_1^{(1)}, \ldots, s_p^{(1)}, t_1^{(1)}, \ldots, t_q^{(1)}) \) and \( \mathbf{u}^{(2)} = (x^{(2)}, s_1^{(2)}, \ldots, s_p^{(2)}, t_1^{(2)}, \ldots, t_q^{(2)}) \).

Lemma 4.1. If \( \mathbf{u} \in U, \mathbf{u} + \Delta \mathbf{u} \in U \) and \( \Delta \mathbf{u} \geq 0 \), then
\[
\hat{w}_i(\mathbf{u}) \leq \hat{w}_i(\mathbf{u} + \Delta \mathbf{u}), \quad -\infty < i < \infty. \tag{4.1}
\]

If \( A \subseteq U \), then
\[
\inf_{\mathbf{u} \in A} \hat{w}_i(\mathbf{u}) = \hat{w}_i(\mathbf{u}^{(1)}), \quad -\infty < i < \infty \tag{4.2}
\]
and
\[
\sup_{\mathbf{u} \in A} \hat{w}_i(\mathbf{u}) = \hat{w}_i(\mathbf{u}^{(2)}), \quad -\infty < i < \infty. \tag{4.3}
\]

Proof. We claim that if \( \mathbf{u} \in U, \mathbf{u} + \Delta \mathbf{u} \in U \) and \( \Delta \mathbf{u} \geq 0 \), then
\[
c_i(\mathbf{u}) \leq c_i(\mathbf{u} + \Delta \mathbf{u}), \quad 0 \leq i < \infty. \tag{4.4}
\]
Since \( c_0(\mathbf{u}) = x/(1 - (t_1 + \cdots + t_q)) \), (4.4) holds if \( i = 0 \). If \( i \geq 1 \), then \( c_i(\mathbf{u}) \) is a polynomial of its coordinates with nonnegative coefficients and therefore (4.4) is proven.

The result in (4.1) is an immediate consequence of the definition of \( \hat{w}_i \) and (4.4). Clearly, (4.2) and (4.3) follow from (4.1).

Let \( |\cdot| \) denote the maximum norm of vectors and matrices. □

Lemma 4.2. If (2.1), (2.3), (2.4) hold and \( \mathbf{\theta} \in U \), then for any \( -\infty < \kappa < \infty \) there is \( \eta > 0 \) such that
\[
E \left( \sup_{\mathbf{u} \in U} \left\{ \frac{w_0(\mathbf{u})}{w_0(\mathbf{\theta})} : |\mathbf{u} - \mathbf{\theta}| \leq \eta \right\} \right)^\kappa < \infty. \tag{4.5}
\]

Also, for any \( \kappa > 0 \),
\[
E \left( \sup_{\mathbf{u} \in U} \left| \frac{w_0''(\mathbf{u})}{w_0(\mathbf{u})} \right| \right)^\kappa < \infty \tag{4.6}
\]
and
\[
E \left( \sup_{\mathbf{u} \in U} \left| \frac{w_0'''(\mathbf{u})}{w_0(\mathbf{u})} \right| \right)^\kappa < \infty. \tag{4.7}
\]
Proof. The result in (4.5) is a special case of Lemma 3.7 in Berkes and Horváth (2001). Berkes et al. (2001b) proved (4.6) and (4.7) (cf. also Berkes and Horváth, 2001, Lemma 3.6).

Lemma 4.3. If (2.1) and (2.3) hold, then
\[
\sup_{u \in U} |w_k(u) - \hat{w}_k(u)| \leq C q_{k, \zeta}, \quad 0 \leq k < \infty
\]  
with some \( C > 0 \), where \( q_* = q_0^{1/q} \) and
\[
\zeta = \sum_{0 \leq i < \infty} q_*^i y_{-i}^2.
\]
Also, there is \( \kappa > 0 \) such that
\[
E\zeta^\kappa < \infty.
\]  
(4.8)

Proof. The statement in (4.8) is a part of the proof of Lemma 5.9 in Berkes et al. (2001b), while (4.9) is obtained in the proof of Lemma 4.1 in Berkes and Horváth (2001).

5. Proof of Theorem 2.1

The proof is based on the following technical result. We assume that \( U \) is so large that \( \theta \in U \).

Lemma 5.1. If (2.1), (2.3) and (2.4) hold, then for any \( \xi > 0 \) there is \( \tau > 0 \) such that
\[
\limsup_{n \to \infty} \sup_{|u - \theta| \leq \tau} \left| \frac{1}{n} \sum_{2 \leq k \leq n} I\{ \varepsilon_k^2 \leq t \hat{w}_k(u)/\sigma_k^2 \} - F(t) \right| 
\leq 2(F(t(1 + \xi)) - F(t(1 - \xi))) \quad \text{a.s.}
\]  
(5.1)

for any \( 0 \leq t < \infty \).

Proof. By (4.8) we have
\[
\frac{1}{n} \sum_{2 \leq k \leq n} I\{ \varepsilon_k^2 \leq t(w_k(u) - C q_k^* \zeta)/\sigma_k^2 \} \leq \frac{1}{n} \sum_{2 \leq k \leq n} I\{ \varepsilon_k^2 \leq t \hat{w}_k(u)/\sigma_k^2 \} 
\leq \frac{1}{n} \sum_{2 \leq k \leq n} I\{ \varepsilon_k^2 \leq t(w_k(u) + C q_k^* \zeta)/\sigma_k^2 \}.
\]  
(5.2)

There is a random variable \( n_0 \) such that \( C q_k^* \zeta/\sigma_k^2 \leq C q_k^* \zeta/\omega \leq \zeta \) if \( k > n_0 \), so we can write
\[
\frac{1}{n} \sum_{2 \leq k \leq n} I\{ \varepsilon_k^2 \leq t(w_k(u) + C q_k^* \zeta)/\sigma_k^2 \} \leq \frac{n_0}{n} + \frac{1}{n} \sum_{2 \leq k \leq n} I\{ \varepsilon_k^2 \leq t(w_k(u)/\sigma_k^2 + \xi) \}.
\]  
(5.3)
Using Lemma 4.1 we get for any \( u \) satisfying \( |\theta - u| \leq \tau \)
\[
\frac{1}{n} \sum_{2 \leq k \leq n} I\{\varepsilon_k^2 \leq t(w_k(u)/\sigma_k^2 + \xi)\} \leq \frac{1}{n} \sum_{2 \leq k \leq n} I\{\varepsilon_k^2 \leq t(w_k(u^{(2)}/\sigma_k^2 + \xi)\},
\]
(5.4)
assuming that \( \tau \) is small enough, where \( u^{(2)} = (\omega + \tau, \alpha_1 + \tau, \ldots, \alpha_p + \tau, \beta_1 + \tau, \ldots, \beta_q + \tau) \). The ergodic theorem yields that
\[
\frac{1}{n} \sum_{2 \leq k \leq n} I\{\varepsilon_k^2 \leq t(w_0(u^{(2)}/\sigma_0^2 + \xi)\}
\rightarrow EI\{\varepsilon_0^2 \leq t(w_0(u^{(2)}/\sigma_0^2 + \xi)\} \quad \text{a.s.}
\]
(5.5)
By the independence of \( \varepsilon_0 \) and \( w_0(u^{(2)})/\sigma_0^2 \) we have
\[
EI\{\varepsilon_0^2 \leq t(w_0(u^{(2)}/\sigma_0^2 + \xi)\} = EF(t(w_0(u^{(2)}/\sigma_0^2 + \xi)\).
\]
The continuity of \( w_0(u) \) implies
\[
\lim_{\tau \to 0} \frac{w_0(u^{(2)})}{\sigma_0^2} = 1 \quad \text{a.s.}
\]
(5.6)
The dominated convergence theorem and (5.6) yield
\[
\lim_{\tau \to 0} EF(t(w_0(u^{(2)})/\sigma_0^2 + \xi)\) = F(t(1 + \xi)\).
\]
(5.7)
Putting together (5.2)–(5.6) we conclude that there is \( \tau = \tau(\xi) \) such that
\[
\limsup_{n \to \infty} \sup_{|u - \theta| \leq \tau} \frac{1}{n} \sum_{2 \leq k \leq n} I\{\varepsilon_k^2 \leq t\hat{w}_k(u)/\sigma_k^2\} \leq F(t) + 2(F(t(1 + \xi)) - F(t)) \quad \text{a.s.}
\]
(5.8)
Similar arguments show that for any \( \xi > 0 \) there is \( \tau = \tau(\xi) \) such that
\[
\liminf_{n \to \infty} \inf_{|u - \theta| \leq \tau} \frac{1}{n} \sum_{2 \leq k \leq n} I\{\varepsilon_k^2 \leq t\hat{w}_k(u)/\sigma_k^2\} \geq F(t) - 2(F(t) - F(t(1 - \xi))) \quad \text{a.s.}
\]
completing the proof of Lemma 5.1. \( \square \)

**Proof of Theorem 2.1.** First we show that for any \( 0 \leq t < \infty \)
\[
\lim_{n \to \infty} |\hat{F}_n(t) - F(t)| = 0 \quad \text{a.s.}
\]
(5.8)
Let \( \Delta > 0 \). Using Lemma 5.1, there are \( \tau > 0 \) and a random variable \( n_1 \) such that
\[
\sup_{|u - \theta| \leq \tau} \left| \frac{1}{n} \sum_{2 \leq k \leq n} I\{\varepsilon_k^2 \leq t\hat{w}_k(u)/\sigma_k^2\} - F(t) \right| \leq \Delta \quad \text{if} \quad n \geq n_1.
\]
(5.9)
By (2.5) there is a random variable \( n_2 \) such that
\[
|\hat{\theta}_n - \theta| \leq \tau \quad \text{if} \quad n \geq n_2.
\]
(5.10)
Now (5.8) follows from (5.9) and (5.10).
The uniform convergence in Theorem 2.1 follows from (5.8), the continuity of $F$ and Pólya’s lemma (cf. Roussas, 1997).

6. Proof of Theorem 2.2

We start with some technical lemmas. Let
\[ \hat{\xi}_i(t, u) = \hat{\xi}_{i, 1}(t, u) + \hat{\xi}_{i, 2}(t, u), \]
where
\[ \hat{\xi}_{i, 1}(t, u) = I\{ \epsilon_i^2 \leq t \hat{\eta}_i(u) \} - F(t \hat{\eta}_i(u)) - (I\{ \epsilon_i^2 \leq \eta_i(u) \} - F(\eta_i(u))) \]
and
\[ \hat{\xi}_{i, 2}(t, u) = I\{ \epsilon_i^2 \leq \eta_i(u) \} - F(\eta_i(u)) - (I\{ \epsilon_i^2 \leq t \} - F(t)). \]

Let
\[ S_k(t, u) = \sum_{1 \leq i \leq k} \xi_i(t, u). \]

Lemma 6.1. If (2.1), (2.3), (2.4) and (2.6) hold, then for any $A > 0$ there is a constant $C(A)$ such that for any $0 < t < \infty$ and $u$ satisfying $|u| \leq A$
\[ P \left\{ \max_{2 \leq k \leq n} |S_k(t, u)| \geq xn^{1/2} \right\} \leq \frac{C(A)}{x^4 n} \]
for all $x > 0$.

Proof. Let $\mathcal{F}_i$ be the $\sigma$-algebra generated by $\hat{\xi}_i, \hat{\xi}_{i-1}, \ldots$. It is easy to see that \{\{S_k(t, u), \mathcal{F}_k\} is a martingale for any $0 < t < \infty$ and $u \in \mathbb{R}^{p+q+1}$. Also
\[ E(\hat{\xi}_i^2(t, u)|\mathcal{F}_{i-1}) \leq 2(E(\hat{\xi}_{i, 1}^2(t, u)|\mathcal{F}_{i-1}) + 2E(\hat{\xi}_{i, 2}^2(t, u)|\mathcal{F}_{i-1})) \]
\[ \leq 2|F(t \hat{\eta}_i(u)) - F(\eta_i(u))| + 2|F(\eta_i(u)) - F(t)| \]
and therefore by Theorem 2.11 in Hall and Heyde (1980) we have
\[ E \max_{2 \leq k \leq n} |S_k(t, u)|^4 \leq C_1 \left\{ E \left( \sum_{2 \leq k \leq n} |F(t \hat{\eta}_k(u)) - F(\eta_k(u))| \right)^2 \right\} \]
\[ + E \left( \sum_{2 \leq k \leq n} |F(\eta_k(u)) - F(t)| \right)^2 + 1 \]
with some absolute constant $C_1 > 0$. 
By the Cauchy inequality we have
\[
E \left( \sum_{2 \leq k \leq n} |F(t\hat{\eta}_k(u)) - F(\eta_k(u))|^2 \right) \\
= \sum_{2 \leq k, i \leq n} E\{ |F(t\hat{\eta}_k(u)) - F(\eta_k(u))||F(t\hat{\eta}_i(u)) - F(\eta_i(u))| \} \\
\leq \sum_{2 \leq k, i \leq n} \{E|F(t\hat{\eta}_k(u)) - F(\eta_k(u))|^2\}^{1/2}\{E|F(t\hat{\eta}_i(u)) - F(\eta_i(u))|^2\}^{1/2}. \tag{6.2}
\]

We recall that \( \zeta \) and \( \varrho_\ast \) are defined in Lemma 4.3. We write
\[
E(F(t\hat{\eta}_k(u)) - F(\eta_k(u)))^2 \\
= E\{(F(t\hat{\eta}_k(u)) - F(\eta_k(u)))^2I\{C\varrho_\ast^k \leq \varrho_\ast^{k/2}\} \} \\
+ E\{(F(t\hat{\eta}_k(u)) - F(\eta_k(u)))^2I\{C\varrho_\ast^k > \varrho_\ast^{k/2}\} \} \\
= a_{k,1} + a_{k,2}. \tag{6.3}
\]

Using (4.9) we get that \( E(\log \max(\zeta,1))^4 < \infty \) and therefore the Markov inequality yields
\[
P\{C\varrho_\ast^k / \omega > \varrho_\ast^{k/2}\} \leq C_2k^{-4} \tag{6.4}
\]
with some constant \( C_2 \) resulting in
\[
a_{k,2} \leq C_2k^{-4}. \tag{6.5}
\]

The mean value theorem implies
\[
F(t\hat{\eta}_k(u)) - F(\eta_k(u)) = f(\eta_k^\ast)(t\hat{\eta}_k(u) - \eta_k(u)), \tag{6.6}
\]
where \( \eta_k^\ast \) is between \( t\hat{\eta}_k(u) \) and \( \eta_k(u) \). By (4.8) we have
\[
|\hat{\eta}_k(u) - \eta_k(u)|I\{C\varrho_\ast^k / \omega \leq \varrho_\ast^{k/2}\} \leq \varrho_\ast^{k/2}. \tag{6.7}
\]

Using the mean value theorem we get
\[
\sup_{|u| \leq A} |\eta_k(u) - 1| = \sup_{|u| \leq A} \left| \frac{w_k(\theta + n^{-1/2}u) - w_k(\theta)}{w_k(\theta)} \right| \\
\leq An^{-1/2}Z_{k,c}, \tag{6.8}
\]
if \( n \geq (A/c)^2 \), where
\[
Z_{k,c} = \sup_{v \in U} \frac{|w'_k(v)|}{w_k(v)} \sup_{|z-\theta| \leq c} \frac{w_k(z)}{w_k(\theta)}. \tag{6.9}
\]
Choosing \( c \) small enough, Lemma 4.2 yields that
\[
EZ_{k,c}^8 = C_3 < \infty. \tag{6.10}
\]
By (6.10) and the Markov inequality we have
\[ P\{An^{-1/2}Z_{k,c} \geq (1 - q_v^{1/2})/2\} \leq C_4 n^{-4} \]
and therefore
\[ E\{ |F(t\hat{n}_k(u)) - F(n_k(u))|^2 I\{An^{-1/2}Z_{k,c} \geq (1 - q_v^{1/2})/2\}\} \leq C_4 n^{-4}. \tag{6.11} \]
On the intersection of the events \( \{An^{-1/2}Z_{k,c} \leq (1 - q_v^{1/2})/2\} \) and \( \{C g_{k,-}^r(\omega) \leq g_v^{k/2} \leq q_v^{1/2}\} \) we have
\[ 1 - (1 - q_v^{1/2})/2 \leq \eta_k(u) \leq 1 + (1 - q_v^{1/2})/2 \]
as well as
\[ 1 - (1 - q_v^{1/2})/2 - q_v^{1/2} \leq \hat{n}_k(u) \leq 1 + (1 - q_v^{1/2})/2 + q_v^{1/2}, \]
so on this event, \( t/\eta_k^* \leq C_5 \), where \( \eta_k^* \) is defined in (6.6). By (6.6), (6.7) and (2.6) we have
\[ E\{ |F(t\hat{n}_k(u)) - F(n_k(u))|^2 I\{C g_{k,-}^r(\omega) \leq g_v^{k/2} \leq q_v^{1/2}\} I\{An^{-1/2}Z_{k,c} \leq (1 - q_v^{1/2})/2\}\} \leq \frac{(LC_5}{\omega})^2 q_v^{k}. \tag{6.12} \]
Putting together (6.11) and (6.12) we conclude that
\[ a_{k,1} \leq C_6 (n^{-4} + q_v^k), \tag{6.13} \]
so by (6.2), (6.3) and (6.5) we have
\[ E \left( \sum_{2 \leq k \leq n} |F(t\hat{n}_k(u)) - F(n_k(u))|^2 \right)^{1/2} \leq C_7 \sum_{2 \leq k, i \leq n} (k^{-4} + n^{-4} + q_v^k)^{1/2} (i^{-4} + n^{-4} + q_v^i)^{1/2} \leq C_8 \tag{6.14} \]
with some constants \( C_7 \) and \( C_8 \).
Similarly to (6.2), we have
\[ E \left( \sum_{2 \leq k \leq n} |F(n_k(u)) - F(t)|^2 \right)^{1/2} \leq \sum_{2 \leq k, i \leq n} \{E(F(n_k(u)) - F(t))^2\}^{1/2} \{E(F(n_i(u)) - F(t))^2\}^{1/2}. \tag{6.15} \]
Using (6.8) and (6.10), there is a constant \( C_9 \) such that
\[ P\{|\eta_k(u) - 1| \geq 1/2\} \leq P \left\{ Z_{k,c} \geq \frac{n^{1/2}}{2A} \right\} \leq C_9 n^{-4} \tag{6.16} \]
and therefore

\[ E \left\{ (F(t\eta_k(u)) - F(t))^2 I \left\{ |\eta_k(u) - 1| \geq \frac{1}{2} \right\} \right\} \leq C_9 n^{-4}. \]  

(6.17)

An application of the mean value theorem yields that there is \( \eta_k^* \) between \( t \) and \( t\eta_k(u) \) such that

\[ |F(t\eta_k(u)) - F(t)| = f(\eta_k^*) t |\eta_k(u) - 1|. \]  

(6.18)

Clearly, \( \eta_k(u) \geq 1/2 \) implies that \( t/\eta_k^* \leq 2 \), so by (6.8), (6.16), (6.18) and (2.6) we have

\[
|F(t\eta_k(u)) - F(t)| I\{|\eta_k(u) - 1| \leq 1/2\} \\
= \eta_k^* f(\eta_k^*) t |\eta_k(u) - 1| I\{|\eta_k(u) - 1| \leq 1/2\} \\
\leq 2A L n^{-1/2} Z_{k,c}.
\]  

(6.19)

Hence (6.10) implies

\[ E \left\{ (F(t\eta_k(u)) - F(t))^2 I\{|\eta_k(u) - 1| \leq 1/2\} \right\} \leq C_{10}/n \]  

(6.20)

with some \( C_{10} \). Putting together (6.15), (6.17) and (6.20) we conclude

\[ E \left( \sum_{2 \leq k \leq n} |F(t\eta_k(u)) - F(t)|^2 \right) \leq C_{11} n \]  

(6.21)

with some \( C_{11} \).

By (6.1), (6.14) and (6.21) we have that

\[ E \max_{2 \leq k \leq n} |S_k(t,u)|^4 \leq C_{12} n \]

and therefore the Markov inequality implies Lemma 6.1. \( \square \)

Lemma 6.1 yields that

\[ \max_{2 \leq k \leq n} |S_k(t,u)| = o_p(n^{1/2}) \]  

(6.22)

for any \( 0 \leq t < \infty \) and \( |u| \leq A \). The next lemma shows that (6.22) is uniform in \( t \) and Lemma 6.3 gives that (6.22) is uniform in \( t \) and \( u \).

**Lemma 6.2.** If (2.1), (2.3), (2.4), (2.6) and (2.7) hold, then for any \( A > 0 \)

\[ \sup_{0 \leq t < \infty} \max_{2 \leq k \leq n} |S_k(t,u)| = o_p(n^{1/2}), \]

if \( |u| \leq A \).
Proof. We divide \([0, \infty)\) into intervals with the points \(0 = t_1 < t_2 < \cdots < t_N < t_{N+1} = \infty\) to be defined later. Observe that

\[
\sup_{t_j \leq t \leq t_{j+1}} |S_k(t, u) - S_k(t_{j+1}, u)| = \left| \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq \hat{t}_i(u)\} + F(t) + F(t_{j+1} \hat{t}_i(u)) + I\{\varepsilon_i^2 \leq t_{j+1}\}) \right|
\]

\[
- \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_{j+1} \hat{t}_i(u)\} + F(t_{j+1}) + F(t_{j+1} \hat{t}_i(u)) + I\{\varepsilon_i^2 \leq t_{j+1}\})
\]

\[
\leq \max \left\{ \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_{j+1}\} - I\{\varepsilon_i^2 \leq t_j\} + F(t_{j+1} \hat{t}_i(u)) - F(t_{j} \hat{t}_i(u))) \right. 
\]

\[
+ \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_{j+1} \hat{t}_i(u)\} - I\{\varepsilon_i^2 \leq t_j \hat{t}_i(u)\}) + F(t_{j+1}(u)) - F(t_j(u)) \right\}. 
\]

Further elementary arguments give

\[
\sup_{t_j \leq t \leq t_{j+1}} |S_k(t, u) - S_k(t_{j+1}, u)| \leq |S_k(t_j, u)| + |S_k(t_{j+1}, u)|
\]

\[
+ \left| \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_{j+1}\} - F(t_{j+1})) - \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_j\} - F(t_j)) \right|
\]

\[
+ \sum_{2 \leq i \leq k} (F(t_{j+1} \hat{t}_i(u)) - F(t_{j} \hat{t}_i(u))) + k(F(t_{j+1}) - F(t_j)). 
\]

(6.23)

Hence

\[
\sup_{0 \leq t < \infty} \max_{2 \leq k \leq n} |S_k(t, u)| \leq 2A_{n,1} + A_{n,2} + 2A_{n,3} + A_{n,4} + A_{n,5}, 
\]

(6.24)

where

\[
A_{n,1} = \max_{1 \leq j \leq N+1} \max_{2 \leq k \leq n} |S_k(t_j, u)|,
\]

\[
A_{n,2} = \max_{0 \leq j \leq N} \max_{2 \leq k \leq n} \left| \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_{j+1}\} - F(t_{j+1})) - \sum_{2 \leq i \leq k} (I\{\varepsilon_i^2 \leq t_j\} - F(t_j)) \right|,
\]
\[ A_{n,3} = \max_{2 \leq j \leq N} \sum_{2 \leq i \leq n} |F(t_j \tilde{\eta}_i(u)) - F(t_j \eta_i(u))|, \]
\[ A_{n,4} = \max_{1 \leq j \leq N} \sum_{2 \leq i \leq n} (F(t_{j+1} \eta_i(u)) - F(t_j \eta_i(u))) \]

and
\[ A_{n,5} = n \max_{1 \leq j \leq N} (F(t_{j+1}) - F(t_j)). \]

Let \( A > 0 \) and choose \( 0 < T < \sup\{t: F(t) < 1\} \) such that
\[ \sup_{0 \leq t \leq T} tf(t) \leq A. \] (6.25)

Define \( 0 = t_1 < t_2 < \cdots < t_{N_1+1} \) by
\[ F(t_{j+1}) - F(t_j) = n^{-1/2} \Delta \quad \text{if} \quad 1 \leq j < N_1 \] (6.26)

and
\[ F(t_{N_1+1}) - F(t_{N_1}) \leq n^{-1/2} \Delta, \quad t_{N_1} \leq T/2 < t_{N_1+1} < 3T/4. \]

Let \( K \) be an integer satisfying \( K \geq 2/(\gamma(1 - \nu)) \) with \( 0 < \nu < 1 \) and \( \gamma \) be from (2.3). Now we define \( t_j, j > N_1 + 1: \)
\[ t_{N_1+i} = t_{N_1+1} + (i - 1)n^{-1/2-\nu}, \quad 1 \leq i \leq N_2 = [n^{3/4}], \]
\[ t_{N_1+N_2+\cdots+N_{j}+i} = t_{N_1+\cdots+N_j} + in^{(j-3)/4-\nu}, \quad 1 \leq i \leq N_{j+1} = [n^{3/4}], \]

\( 2 \leq j \leq K \) and \( N = N_1 + \cdots + N_{K+1} \). It is clear that \( N \leq C_{13}n^{3/4} \) with some \( C_{13} \). Next we get some properties of \( t_j \). By (2.6) we have for any \( 0 < \nu < u < \infty \) that
\[ F(u) - F(v) \leq L(u - v)/v. \] (6.27)

So using the definitions of \( t_j, j \geq N_1 \) we get that
\[ F(t_{j+1}) - F(t_j) \leq \frac{L}{t_j} (t_{j+1} - t_j) \leq L(t_{j+1} - t_j)/t_{N_1+\cdots+N_j}, \leq C_{14}n^{-1/2-\nu}, \] (6.28)

if \( N_1 + \cdots + N_j \leq j < N_1 + \cdots + N_{j+1} \) and therefore
\[ \max_{N_1 \leq j \leq N-1} (F(t_{j+1}) - F(t_j)) \leq C_{14}n^{-1/2-\nu}. \] (6.29)

Also \( t_N \geq n^{K(1-\nu)} \). Condition (2.3) gives that
\[ \lim_{x \to \infty} x^\nu (1 - F(x)) = 0 \]
and therefore
\[ 1 - F(t_N/2) \leq C_{15} n^{-K(1-\nu)} \leq C_{15} n^{-2}. \] (6.30)

It follows from (6.26), (6.29) and (6.30) that
\[ \max_{0 \leq j \leq N} (F(t_{j+1}) - F(t_j)) = O(n^{-1/2}). \] (6.31)

Using Lemma 6.1 we get for any \( x > 0 \) that
\[ P\{n^{-1/2} A_{n,1} \geq x\} \leq C_{13} n^{3/4} \frac{C(A)}{nx^4}, \]
resulting in
\[ n^{-1/2} A_{n,1} = o_P(1). \] (6.32)

The upper bound for the increments of the empirical process (cf. Csörgő and Révész, 1981) and (6.31) yield
\[ n^{-1/2} A_{n,2} = o_P(1). \] (6.33)

By (6.4) and the Borel–Cantelli lemma there is a random variable \( k_0 \) such that
\[ \frac{C}{C'} \leq q_0^{k_0} \leq q_0^{k/2} \quad \text{if} \quad k \geq k_0. \] (6.34)

Now
\[ A_{n,3} \leq k_0 + \max_{2 \leq j \leq N} \sum_{k_0 \leq i \leq n} |F(t_j \hat{\eta}_i(u)) - F(t_j \eta_i(u))|. \] (6.35)

Using (6.10) we obtain that
\[ \max_{2 \leq j \leq N} \sup_{|u| \leq A} \sum_{2 \leq i \leq n} |F(t_j \hat{\eta}_i(u)) - F(t_j \eta_i(u))|I\{A_{n}^{-1/2} Z_{i,c} \geq 1/4\} = O_P(n^{-3}), \] (6.36)

where \( Z_{i,c} \) is defined in (6.9). The mean value theorem, (2.6), (6.7), (6.8) and (6.34) imply that
\[ \max_{2 \leq j \leq N} \sup_{|u| \leq A} \sum_{k_0 \leq i \leq n} |F(t_j \hat{\eta}_i(u)) - F(t_j \eta_i(u))|I\{A_{n}^{-1/2} Z_{i,c} \leq 1/4\} \]
\[ = \max_{2 \leq j \leq N} \sum_{k_0 \leq i \leq n} f(\eta_{j,i}) |t_j \hat{\eta}_i(u) - \eta_i(u)| I\{A_{n}^{-1/2} Z_{i,c} \leq 1/4\} \]
\[ \leq L \sum_{k_0 \leq i \leq n} (t_j/\eta_{j,i}) q_0^{1/2} \leq 2L \frac{q_0^{1/2}}{1 - q_0^{1/2}} \] (6.37)
since $t_j/n_{j,i}^* \leq 2$ when $An^{-1/2}Z_{i,c} \leq 1/4$ holds. Combining (6.35)–(6.37) we conclude

$$A_{n,3} = O_P(1).$$  \hfill (6.38)

Next, we write

$$\max_{1 \leq j \leq N_1} \sum_{2 \leq i \leq n} (F(t_{j+1} \eta_i(u)) - F(t_j \eta_i(u)))$$

$$\leq 2 \max_{2 \leq j \leq N_1 + 1} \sum_{2 \leq i \leq n} |F(t_j \eta_i(u)) - F(t_j)| + n \max_{1 \leq j \leq N_1} (F(t_{j+1}) - F(t_j))$$

$$= A_{n,4}^{(1)} + A_{n,4}^{(2)}.$$  \hfill (6.39)

By (6.26) we have that

$$A_{n,4}^{(2)} = b''n + 1 = 2.$$  \hfill (6.40)

Applying (6.10) we obtain that

$$\max_{2 \leq j \leq N_1 + 1} \sum_{2 \leq i \leq n} |F(t_j \eta_i(u)) - F(t_j)|I\{An^{-1/2}Z_{i,c} \geq 1/2\} = O_P(n^{-3}).$$

The mean value theorem and (2.6), (6.8), (6.25) imply

$$\max_{2 \leq j \leq N_1 + 1} \sum_{2 \leq i \leq n} |F(t_j \eta_i(u)) - F(t_j)|I\{An^{-1/2}Z_{i,c} \leq 1/2\}$$

$$\leq \max_{2 \leq j \leq N_1 + 1} \sum_{2 \leq i \leq n} f(\eta_{j,i}^* \eta_{j,i}^* \eta_{j,i}^*(t_j/\eta_{j,i}^*))An^{-1/2}Z_{i,c}I\{An^{-1/2}Z_{i,c} \leq 1/2\}$$

$$\leq 2A \Delta n^{-1/2} \sum_{2 \leq i \leq n} Z_{i,c},$$  \hfill (6.41)

since $t_j/n_{j,i}^* \leq 2$ if $An^{-1/2}Z_{i,c} \leq 1/2$. It follows from (6.9) and (6.10) that $Z_{k,c}, 1 \leq k < \infty$ is a stationary and ergodic sequence with finite mean, so the ergodic theorem yields that

$$\frac{1}{n} \sum_{2 \leq i \leq n} Z_{i,c} = O(1) \quad \text{a.s.}$$  \hfill (6.42)

By (6.39)–(6.42) we have that

$$\max_{1 \leq j \leq N_1} \sum_{2 \leq i \leq n} (F(t_{j+1} \eta_i(u)) - F(t_j \eta_i(u))) = \Delta n^{1/2}O(1) \quad \text{a.s.,}$$  \hfill (6.43)

where O(1) does not depend on $\Delta$. We apply (6.27) and get that

$$\max_{N_1 < j \leq N-1} \sum_{2 \leq i \leq n} (F(t_{j+1} \eta_i(u)) - F(t_j \eta_i(u)))$$

$$\leq n \max_{N_1 < j \leq N-1} (t_{j+1} - t_j)/t_j$$

$$\leq Cn^{1/2-v}$$  \hfill (6.44)
on account of (6.28). Also, by (6.30), (6.16) and the Borel–Cantelli lemma
\[
\sum_{2 \leq i \leq n} (1 - F(t_N \eta_i(u))) \\
= \sum_{2 \leq i \leq n} (1 - F(t_N \eta_i(u)))I \{An^{-1/2}Z_{i,c} \leq 1/4\} \\
+ \sum_{2 \leq i \leq n} (1 - F(t_N \eta_i(u)))I \{An^{-1/2}Z_{i,c} > 1/4\} \\
\leq C_1 n^{-1} + O(n^{-2}) \text{ a.s.} (6.45)
\]
The bounds in (6.39)–(6.45) give that
\[
A_{n,4} = \Delta n^{1/2}O(1) + O(n^{1/2 - v}) \text{ a.s.} (6.46)
\]
for any \( \Delta > 0 \), where \( O(1) \) does not depend on \( \Delta \).
We use (6.26), (6.29) and (6.30) to see that
\[
A_{n,5} \leq C_{16} \Delta n^{1/2} \text{ for any } \Delta > 0 (6.47)
\]
with some constant \( C_{16} \).
Lemma 6.2 follows from (6.24), (6.32), (6.33), (6.38), (6.40) and (6.47).

**Lemma 6.3.** If (2.1), (2.3), (2.4), (2.6) and (2.7) hold, then for any \( \Delta > 0 \)
\[
\sup_{|u| \leq \Delta} \sup_{0 \leq t < \infty} \max_{2 \leq k \leq n} |S_k(t, u)| = o_P(n^{1/2}).
\]

**Proof.** Let \( N \geq 1 \) be an integer. The \( p + q + 1 \) dimensional cube \([-\Delta, \Delta]^{p+q+1}\) is divided into \( M = (2N)^{p+q+1}\) cubes with side length \( \Delta/N \). In case of a cube \( A(\ell) \), \( u_*(\ell) \), and \( u^*(\ell) \) denote the lower left and upper right vertex of \( A(\ell) \). (“Lower left” vertex means that all coordinates of \( u_*(\ell) \) are less than or equal to the corresponding coordinates of any elements of \( A(\ell) \). The “upper right” vertex is defined similarly.) Then
\[
\sup_{|u| \leq \Delta} |S_k(t, u)| \\
\leq \max_{1 \leq \ell \leq M} |S_k(t, u^*(\ell))| + \max_{1 \leq \ell \leq M} \sup_{u \in A(\ell)} |S_k(t, u) - S_k(t, u^*(\ell))|. (6.48)
\]
Using Lemma 4.1 we get that
\[
\sup_{u \in A(\ell)} |S_k(t, u) - S_k(t, u^*(\ell))| \\
\leq |S_k(t, u_*(\ell))| + |S_k(t, u^*(\ell))| + 2 \sum_{2 \leq i \leq n} (F(t \tilde{\eta}_i(u^*(\ell))) - F(t \tilde{\eta}_i(u_*(\ell)))). (6.49)
\]
Lemma 6.2 implies that
\[
\max_{1 \leq k \leq M} \sup_{0 \leq t < \infty} \max_{2 \leq i \leq n} \{|S_k(t, u_i(\ell))| + |S_k(t, u_i^*(\ell))|\} = o_p(n^{1/2}). \tag{6.50}
\]

Next we write
\[
\sum_{2 \leq i \leq n} (F(t\hat{\eta}_i(u^*(\ell))) - F(t\hat{\eta}_i(u_i(\ell))))
\]
\[
\leq \sum_{2 \leq i \leq n} |F(t\hat{\eta}_i(u^*(\ell))) - F(t\eta_i(u^*(\ell)))|
+ \sum_{2 \leq i \leq n} |F(t\hat{\eta}_i(u_i(\ell))) - F(t\eta_i(u_i(\ell)))|
+ \sum_{2 \leq i \leq n} (F(t\eta_i(u^*(\ell))) - F(t\eta_i(u_i(\ell)))) \tag{6.51}
\]

Using (6.27) and (4.8) we get for any \(u\) satisfying \(|u| \leq A\) and any \(0 \leq t < \infty\),
\[
\sum_{2 \leq i \leq n} |F(t\hat{\eta}_i(u)) - F(t\eta_i(u))|
\]
\[
\leq L \sum_{2 \leq i \leq n} \left| \frac{\hat{\eta}_i(u) - \eta_i(u)}{\hat{\eta}_i(u)} \right|
= L \sum_{2 \leq i \leq n} \left| \frac{\hat{\omega}_i(\theta + n^{-1/2}u)/\sigma_i^2}{\hat{\omega}_i(\theta + n^{-1/2}u)/\sigma_i^2} \right|
= L \sum_{2 \leq i \leq n} \left| \frac{\hat{\omega}_i(\theta + n^{-1/2}u) - \omega_i(\theta + n^{-1/2}u)}{\hat{\omega}_i(\theta + n^{-1/2}u)} \right|
\leq \frac{LC}{u} \sum_{2 \leq i < \infty} q_i^*, \tag{6.52}
\]
since \(\theta + n^{-1/2}u \in U\) if \(n\) is large and therefore \(\hat{\omega}_i(\theta + n^{-1/2}u) \geq c_0(\theta + n^{-1/2}u) \geq u\). It is clear from (6.52) that
\[
\max_{1 \leq k \leq M} \sup_{0 \leq t < \infty} \left\{ \sum_{2 \leq i \leq n} |F(t\hat{\eta}_i(u_i(\ell))) - F(t\eta_i(u_i(\ell)))|
+ \sum_{2 \leq i \leq n} |F(t\hat{\eta}_i(u^*(\ell))) - F(t\eta_i(u^*(\ell)))| \right\}
= O_p(1). \tag{6.53}
\]
The mean value theorem gives
\[ 0 \leq \frac{w_i(\theta + n^{-1/2}u_*(\ell)) - w_i(\theta + n^{-1/2}u_*(\ell))}{w_i(\theta + n^{-1/2}u_*(\ell))} \leq Z_{i,e}^* n^{-1/2} |u_*(\ell) - u_*(\ell)| \leq Z_{i,e}^* n^{-1/2} \frac{A}{N}(p + q + 1), \tag{6.54} \]

where
\[ Z_{i,c}^* = \sup_{v \in U} \frac{|w_i'(v)|}{w_i(v)} \sup_{|z - \theta| \leq c} \frac{w_i(z)}{w_i(\theta)} \sup_{|u - \theta| \leq c} \frac{w_i(\theta)}{w_i(u)}. \tag{6.55} \]

Choosing \( c > 0 \) small enough, Lemma 4.2 implies that
\[ EZ_{i,c}^* < \infty. \tag{6.56} \]

Next we use (6.27) and (6.54) and obtain that
\[ \sum_{2 \leq i \leq n} (F(t_{i,j}(u_*(\ell))) - F(t_{i,j}(u_*(\ell)))) \leq L \sum_{2 \leq i \leq n} \frac{\eta_i(u_*(\ell)) - \eta_i(u_*(\ell))}{\eta_i(u_*(\ell))} \]
\[ = L \sum_{2 \leq i \leq n} \frac{w_i(\theta + n^{-1/2}u_*(\ell)) - w_i(\theta + n^{-1/2}u_*(\ell))}{w_i(\theta + n^{-1/2}u_*(\ell))} \]
\[ \leq \frac{A}{N}(p + q + 1) \frac{L}{n^{1/2}} \sum_{2 \leq i \leq n} Z_{i,c}^* \tag{6.57} \]

assuming that \( n \geq (A/c)^{1/2} \). It is clear from (6.55) and (6.56) that \( \{Z_{i,c}^*, i \geq 2\} \) is a stationary and ergodic sequence with finite mean. The ergodic theorem gives
\[ \frac{1}{n} \sum_{2 \leq i \leq n} Z_{i,c}^* = O(1) \quad \text{a.s.} \tag{6.58} \]

so by (6.57) we have
\[ \max_{1 \leq \ell < M} \sup_{0 \leq t < \infty} \sum_{2 \leq i \leq n} (F(t_{i,j}(u_*(\ell))) - F(t_{i,j}(u_*(\ell)))) = \frac{1}{N} n^{1/2} O(1) \quad \text{a.s.} \tag{6.59} \]

where \( O(1) \) does not depend on \( N \). Lemma 6.3 now follows from (6.48)–(6.59). \( \square \)

**Lemma 6.4.** If (2.1), (2.3), (2.4), (2.6), (2.7) hold and
\[ |\hat{\theta}_n - \theta| = O_p(n^{-1/2}), \tag{6.60} \]
then
\[
\sup_{0 \leq t < \infty} \left| n^{1/2} \left( s \hat{F}_{[ns]}(t) - \frac{1}{n} \sum_{2 \leq i \leq [ns]} F(t \hat{\theta}_n(\hat{\sigma}_i^2)) - z_n(t,s) \right) \right| = o_p(1).
\]
(6.61)

**Proof.** By (6.60) for any \( A > 0 \) there are \( A_0 \) and \( n_0 \) such that
\[
P\left( |\hat{\theta}_n - \theta| \geq n^{-1/2} A \right) \leq A \quad \text{if} \quad n \geq n_0.
\]
(6.62)

Hence (6.61) follows from Lemma 6.3. \( \square \)

**Lemma 6.5.** If (2.1), (2.3), (2.4), (2.6)–(2.9) and (6.60) hold, then
\[
\sup_{0 \leq t < \infty} \left| n^{-1/2} \sum_{2 \leq i \leq [ns]} \left( F(t \hat{\theta}_i(\hat{\sigma}_i^2)) - F(t) \right) - stf(t)n^{1/2}(\hat{\theta}_n - \theta) d_0^T \right| = o_p(1),
\]
where \( d_0 \) is defined in (2.10).

**Proof.** It is enough to show that for any \( A > 0 \)
\[
\sup_{0 \leq t < \infty} \sup_{|u| \leq A} \left| n^{-1/2} \sum_{2 \leq i \leq [ns]} (F(t \hat{\eta}_i(u)) - F(t)) - stf(t)n^{1/2}(\hat{\theta}_n - \theta) d_0^T \right| = o_p(1).
\]
(6.63)

It follows from (6.52) that
\[
\sup_{0 \leq t < \infty} \sup_{|u| \leq A} \left| \sum_{2 \leq i \leq [ns]} (F(t \hat{\eta}_i(u)) - F(t \eta_i(u))) \right| = O_p(1).
\]
(6.64)

Let \( A > 0 \) and choose \( 0 < T_1 < T_2 < \infty \) such that
\[
\sup_{0 < t < 2T_1} tf(t) \leq A \quad \text{and} \quad \sup_{T_2/2 < t < \infty} tf(t) \leq \Delta .
\]
(6.65)

Using a two-term Taylor expansion we get that
\[
\sup_{|u| \leq A} \left| \eta_i(u) - 1 - n^{-1/2} u \left( \frac{w_i'(\theta)}{w_i(\theta)} \right)^T \right| \leq A^2 \frac{1}{n} \hat{Z}_{i,c},
\]
(6.66)
where
\[
\bar{Z}_{i,c} = \sup_{v \in U} \left| w_i''(v) \right| \sup_{|z-\theta| \leq c} \frac{w_i(z)}{w_i(\theta)}.
\]  
(6.67)

Assuming that \( c \) is small enough,
\[
E\bar{Z}_{i,c}^8 < \infty
\]  
(6.68)

(cf. Lemma 4.2). If follows from Lemma 4.2 that
\[
P \left\{ \max_{2 \leq i \leq n} \frac{|\eta_i(u) - 1|}{n^{-1/4}} > n^{-1/4} \right\} \leq C_{17}/n^2
\]
and thus (6.66) and (6.68) imply
\[
P \left\{ \max_{2 \leq i \leq n} \sup_{|u| \leq A} \left| \eta_i(u) - 1 \right| > n^{-1/4} \right\} \leq C_{18}/n^2
\]
and consequently by the Borel–Cantelli lemma
\[
\max_{2 \leq i \leq n} \sup_{|u| \leq A} \left| \eta_i(u) - 1 \right| = O(n^{-1/4}) \quad \text{a.s.}
\]  
(6.69)

If \( \max_{2 \leq i \leq n} \sup_{|u| \leq A} \left| \eta_i(u) - 1 \right| \leq n^{-1/8} \) holds, then by the mean value theorem we have with some \( \eta_i^* \)
\[
\sup_{0 \leq t \leq T_1} \sup_{|u| \leq A} \sum_{2 \leq i \leq n} |F(\eta_i(u)) - F(t)|
= \sup_{0 \leq t \leq T_1} \sup_{|u| \leq A} \sum_{2 \leq i \leq n} f(\eta_i^*) t |\eta_i(u) - 1|
\leq \sup_{0 \leq s \leq 2T_1} \sup_{0 \leq t \leq T} t/(t(1-n^{-1/8})) \sum_{2 \leq i \leq n} \left\{ n^{-1/2} \left| \frac{w_i'(\theta)}{w_i(\theta)} \right| + \frac{A^2}{2} \frac{1}{n} \bar{Z}_{i,c} \right\}
= \Delta n^{1/2} O(1) \quad \text{a.s.}
\]  
(6.70)

since by the ergodic theorem
\[
\frac{1}{n} \sum_{2 \leq i \leq n} \frac{w_i'(\theta)}{w_i(\theta)} = O(1) \quad \text{a.s.} \quad \text{and} \quad \frac{1}{n} \sum_{2 \leq i \leq n} \bar{Z}_{i,c} = O(1) \quad \text{a.s.}
\]  
(6.71)

The estimates in (6.69) and (6.70) imply
\[
\sup_{0 \leq t \leq T_1} \sup_{|u| \leq A} \sum_{2 \leq i \leq n} |F(\eta_i(u)) - F(t)| = \Delta n^{1/2} O(1) \quad \text{a.s.}
\]  
(6.72)
and similar arguments give
\[
\sup_{T_1 \leq t < \infty} \sup_{|u| \leq A} \sum_{2 \leq i \leq n} |F(t \eta_i(u)) - F(t)| = \Delta n^{1/2} O(1) \quad \text{a.s.} \tag{6.73}
\]

The Taylor formula yields
\[
F(t \eta_i(u)) - F(t) = tf(t)(\eta_i(u) - 1) + t(f(t \eta_i^*(u)) - f(t))(\eta_i(u) - 1) \tag{6.74}
\]
with some \( \eta_i^* = \eta_i^*(u) \) between \( \eta_i(u) \) and 1. By assumption (2.9) \( f \) is uniformly continuous on \([T_1/2, 2T_2]\) and therefore by (6.69)
\[
\sup_{T_1 \leq t \leq T_2} \sup_{|u| \leq A} \sum_{2 \leq i \leq n} |f(t \eta_i^*(u)) - f(t)| = \omega P(1). \tag{6.75}
\]

It follows from (6.66) and (6.71) and (6.75) that
\[
\sup_{|u| \leq A} \sup_{T_1 \leq t \leq T_2} \sum_{2 \leq i \leq n} t|f(t \eta_i^*(u)) - f(t)||\eta_i(u) - 1| = \omega P(n^{1/2}). \tag{6.76}
\]

Using again (6.66) and (6.71) we conclude
\[
\sup_{|u| \leq A} \sup_{T_1 \leq t \leq T_2} t f(t) \max_{2 \leq k \leq n} \sum_{2 \leq i \leq k} (\eta_i(u) - 1) - n^{-1/2} u \sum_{2 \leq i \leq k} (\frac{w_i'(\theta)}{w_i(\theta)})^T = O_P(1). \tag{6.77}
\]

By the ergodic theorem we have
\[
\frac{1}{n} \sum_{2 \leq i \leq n} \frac{w_i'(\theta)}{w_i(\theta)} \rightarrow d_0 \quad \text{a.s.}
\]
and therefore by (6.77)
\[
\max_{2 \leq k \leq n} \sup_{|u| \leq A} \sup_{T_1 \leq t \leq T_2} \left| t f(t) \left\{ \sum_{2 \leq i \leq k} (\eta_i(u) - 1) - kn^{-1/2} u d_0^T \right\} \right| = \omega P(n^{1/2}). \tag{6.78}
\]

The result in (6.63) follows from (6.64), (6.72), (6.73) and (6.78).

**Lemma 6.6.** If (2.1), (2.3), (2.4), (2.6)–(2.9) and (6.60) hold, then
\[
\sup_{0 \leq t < 1} \sup_{0 \leq s \leq t} \left| \tilde{z}_n(t,s) - (z_n(t,s) + st f(t)n^{1/2}(\tilde{\theta}_n - \theta)d_0^T) \right| = \omega P(1),
\]
where \( d_0 \) is defined in (2.10).

**Proof.** It follows from Lemmas 6.4 and 6.5.

**Lemma 6.7.** If (2.11)–(2.13) hold, then
\[
\{(z_n(t,s),n^{1/2}(\tilde{\theta}_n - \theta)d_0^T), \quad 0 \leq t < \infty, \quad 0 \leq s \leq 1\}
\]
\[
\rightarrow \{(K(F(t),s),\tau), \quad 0 \leq t < \infty, \quad 0 \leq s \leq 1\},
\]
where the convergence is in the Skorokhod space $D([0, \infty] \times [0, 1])$ and $(K(F(t), s), \tau)$, $0 \leq t < \infty$, $0 \leq s \leq 1$ is Gaussian with covariance structure given by (2.17)–(2.20).

**Proof.** By (2.11) it is enough to prove

$$\left\{ \left( \frac{2}{n}, \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq p+q+1} g_j(\epsilon_i) d_f(\epsilon_{i-1}, \epsilon_{i-2}, \ldots) d_j \right), \right.$$\[0 \leq t < \infty, \ 0 \leq s \leq 1 \}

$$\rightarrow \{(K(F(t), s), \tau), \ 0 \leq t < \infty, \ 0 \leq s \leq 1 \}. \quad (6.79)$$

We must show tightness and the convergence of the finite-dimensional distributions. The tightness follows from the fact that $\{x_n(t, s)\}$ converges weakly (cf. Csörgő and Révész, 1981).

The proof of the convergence of the finite dimensional distributions is based on the Cramér–Wold device (cf. Billingsley, 1968, p. 48). It is clear that $\{\sum_{1 \leq j \leq p+q+1} g_j(\epsilon_i) \ell_j(\epsilon_{i-1}, \epsilon_{i-2}, \ldots) d_j\}$ is stationary, ergodic and

$$E \left( \sum_{1 \leq j \leq p+q+1} g_j(\epsilon_i) \ell_j(\epsilon_{i-1}, \epsilon_{i-2}, \ldots) d_j | \mathcal{F}_j \right) = 0,$$

by (2.12), where $\mathcal{F}_j$ is the $\sigma$-algebra generated by $\{\epsilon_i, -\infty < i \leq j\}$. Now the convergence of the finite dimensional distributions is a consequence of Theorem 23.1 of Billingsley (1968, p. 206).

**Proof of Theorem 2.2.** It follows immediately from Lemmas 6.6 and 6.7.

**References**


Lumsdaine, R.L., 1996. Consistency and asymptotic normality of the quasi-maximum likelihood estimator in IGARCH(1,1) and covariance stationary GARCH(1,1) models. Econometrica 64, 575–596.


