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An Alternating Mesh Quality Metric Scheme for Efficient Mesh Quality Improvement☆

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Abstract

In the numerical solution of partial differential equations (PDEs), high-quality meshes are crucial for the stability, accuracy, and convergence of the associated PDE solver. Mesh quality improvement is often performed to improve the quality of meshes before use in numerical solution of the PDE. Mesh smoothing (performed via optimization) is one popular technique for improving the mesh quality; it does so by making adjustments to the vertex locations. When an inefficient mesh quality metric is used to design the optimization problem, and hence also to measure the mesh quality within the optimization procedure, convergence of the optimization method can be much slower than desired. However, for many applications, the choice of mesh quality metric and the optimization problem should be considered fixed. In this paper, we propose a simple mesh quality metric alternation scheme for use in the mesh optimization process. The idea is to alternate the use of the original inefficient mesh quality metric with a more efficient mesh quality metric on alternate iterations of the mesh optimization procedure in order to reduce the time to convergence, while solving the original mesh quality improvement problem. Typical results of using our application scheme to solve mesh quality improvement problems yield approximately 40-55% improvement in comparison to the original mesh optimization procedure. More frequent use of the efficient metric results in greater speed-ups.

Keywords: mesh quality improvement, mesh optimization, quality metric

1. Introduction

High-quality meshes are essential in the numerical solution of partial differential equations (PDEs) which arise in various science and engineering applications. In particular, the quality of the mesh has been shown to significantly impact the accuracy, stability, and convergence of the associated PDE solver. It is known that well-shaped elements are necessary for good numerical behavior of the PDE solver; even a few poorly-shaped elements can cause significant difficulty. Thus, mesh quality improvement is often applied before numerical solution of the PDE.

There are three main types of mesh quality improvement techniques: adaptivity [1, 2], smoothing [3, 4], and swapping [5, 6]. In mesh smoothing, the vertex coordinates are altered without changing the connectivity of the

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vertices. Optimization techniques (e.g., [7, 8, 9, 10, 11]) are typically used for the mesh smoothing process. When an optimization method is employed, an objective function which measures the overall mesh quality is designed. Hence, the choice of the mesh quality metric significantly affects the performance of the optimization method. In addition, the use of various mesh quality metrics and objective functions yields different final meshes.

There are numerous mesh quality metrics which have been proposed in the literature (e.g., [12, 13]). The use of several of the mesh quality metrics in an optimization procedure yield rather similar meshes, especially when meshes with isotropic elements are the goal. However, the amount of time required by the optimization solver depends significantly on the choice of metric. In particular, the use of an inefficient mesh quality metric can significantly increase the time to convergence of the optimization method when accurate mesh smoothing is desired.

One way to reduce the execution time of the optimization method would be to create a new optimization problem which employs a more efficient mesh quality metric. However, for many applications, the choice of mesh quality metric and the optimization problem should be considered fixed, as different element shapes are desirable for solving different PDEs.

Instead of changing the mesh optimization problem, we propose a simple mesh quality metric alternation scheme for use in the mesh optimization process. The idea is to alternate the use of the original, inefficient mesh quality metric with a more efficient mesh quality metric on alternate iterations of the mesh optimization procedure in order to reduce the time to convergence, while solving the original mesh quality improvement problem. Such a technique has the potential to make a significant impact on mesh optimization for problems in science and engineering where very large meshes and long optimization times are typical. For such applications, inaccurate mesh smoothing (corresponding to just a few iterations of smoothing) is currently used as an alternative to long optimization times, whereas additional mesh smoothing is desired.

2. Mesh Quality Improvement via Quality Metric Alternation

2.1. Quality Improvement Problem

The objective function used in this paper is

$$f(x) = \frac{1}{n} \sum_{1 \le i \le n} q_i^2, \tag{1}$$

where f is the overall mesh quality as measured by the average quality of the mesh elements, q_i is the quality of element i, and n is the number of mesh elements. The objective function is minimized, as lower values of q_i denote better quality elements for the metrics we employ.

When optimizing the mesh, the boundary vertices are held fixed, i.e., $x_{v_B} = x_{\overline{v_B}}$, where $x_{\overline{v_B}}$ are the boundary vertex coordinates. In addition, the initial meshes and subsequent meshes cannot contain any inverted elements. We employ a local implementation of the feasible Newton solver [8] in order to minimize (1).

2.2. Mesh Quality Metric Alternation

We propose a mesh quality metric alternation scheme for the solution of (1) whereby an inefficient metric (in the original optimization problem) is combined with a more efficient metric. The metrics are applied in an alternate fashion every other iteration. For example, alteration scheme A+B would use metric A on odd iterations and metric B on even iterations to perform the mesh optimization. However, metric A is always used to test for convergence of the method on each method, as the goal of the optimization is to minimize (1) as defined according to A. Hence, the goal of using scheme A+B is simply to improve the time to convergence (and not to change the optimization problem being solved). An important question considered in this paper is which metrics can be combined to create effective alternation schemes for mesh optimization? Clearly, the choice of metrics significantly impacts the time to convergence.

Table 1 shows five common mesh quality metrics used for mesh optimization: edge ratio [12], area [14], edge root mean square [12], inverse mean ratio [8], and aspect ratio [15]. Here L_{12}, L_{23}, and L_{13} are the lengths of the three edges in the triangle; L_{min} and L_{max} are the minimum and maximum edge lengths; S is the element's area; A is the Jacobian matrix for the physical triangle, and W is a Jacobian matrix for the reference triangle. The ranges of mesh quality metrics are: 1 to ∞ for EDGE, RMS, IMR, and AR, and 0 to ∞ for AREA. The ideal element of our mesh optimization problem is an equilateral triangle; hence lower values of the quality metrics yield meshes with better quality and it is appropriate to minimize (1).

Table 1: The mesh quality metric definitions

Quality Metric	Formula
edge ratio (EDGE)	$L_{\min}/L_{\max}$
area (AREA)	$S_{cur} - S_{avg}$
edge root mean square (RMS)	$\sqrt{\left(L_{12}^2 + L_{13}^2 + L_{23}^2\right)/3}$
inverse mean ratio (IMR)	$\|AW^{-1}\|_F^2 / \left(2\det(AW^{-1})\right)$
aspect ratio (AR)	$\sqrt{3}\left(L_{12}^2 + L_{13}^2 + L_{23}^2\right)/12S_i$

We propose six mesh quality metric alternations based upon these metrics: EDGE+IMR, AREA+IMR, RMS+IMR, EDGE+AR, AREA+AR, and RMS+AR, because the IMR and AR mesh quality metrics are faster than the EDGE, AREA, and RMS mesh quality metrics when used for mesh optimization.

There are many types of convergence criterion that can be applied for mesh optimization using the above mesh alternation schemes. For our purposes, convergence of the optimization scheme is said to occur when the optimization procedure has progressed at least 95% of the way to the desired level of mesh quality, i.e., when

$$f_{initial} - f_{current} > 0.95(f_{initial} - f_{desired}), \quad (2)$$

where $f_{initial}$ is the mesh quality of the initial input mesh, $f_{current}$ is the mesh quality of the current iteration, and $f_{desired}$ is the goal mesh quality. For our experiments, the desired quality, $f_{desired}$, is set to the quality of the fully converged local optimal solution.

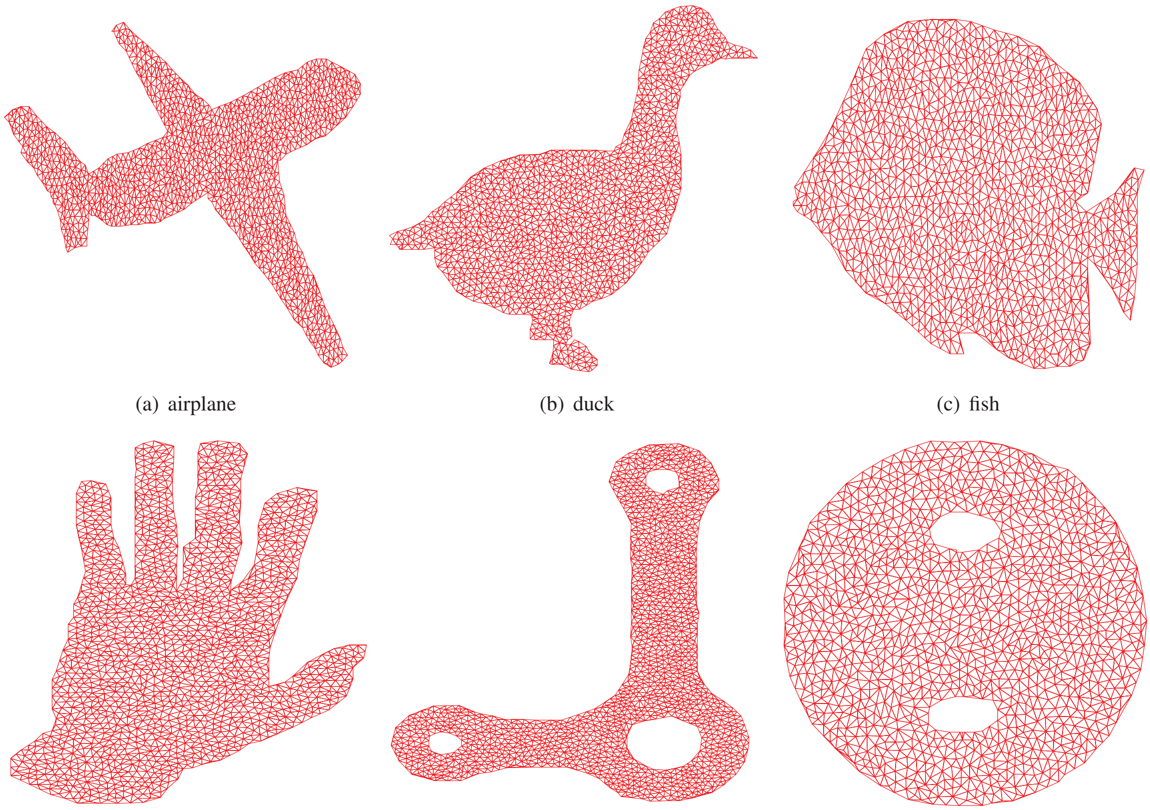
3. Numerical Experiments

To determine the impact of mesh quality alternation on the optimization process, we tested the six mesh quality metric alternations described above. Coarse approximations to the actual 2D meshes used in our experiments are shown in Figure 1. The meshes were generated by Triangle [16]. The interior mesh vertices were perturbed in order to design more challenging test cases. Table 2 gives their configurations. The feasible Newton method in the Mesh Quality Improvement Toolkit (Mesquite) Version 2.1.1 [17] was employed for our experiments. (Similar trends were observed when other solvers, e.g., quasi-Newton, were used to minimize (1).) The Mesquite code was modified to incorporate the alternation of the mesh quality metrics on every other iteration. The Solaris machine employed for the experiments was an UltraSPARC-III CPU with a 750MHz processor, 1GB SDRAM of memory, and 8MB L2 cache.

Table 2: The test mesh configurations

name	# of vertices	# of elements
airplane	72205	143205
duck	62372	123904
fish	57575	114546
hand	68807	136419
mechanic	69904	138610
twoholes	56088	111537

The goal of our experiment is to determine the impact that mesh quality metric alternation has on the time efficiency of the optimization procedure when used to minimize (1). The alternation schemes considered in this experiment are: EDGE+IMR, AREA+IMR, RMS+IMR, EDGE+AR, AREA+AR, and RMS+AR. These schemes are



(a) airplane (b) duck (c) fish

(d) hand (e) mechanic (f) twoholes

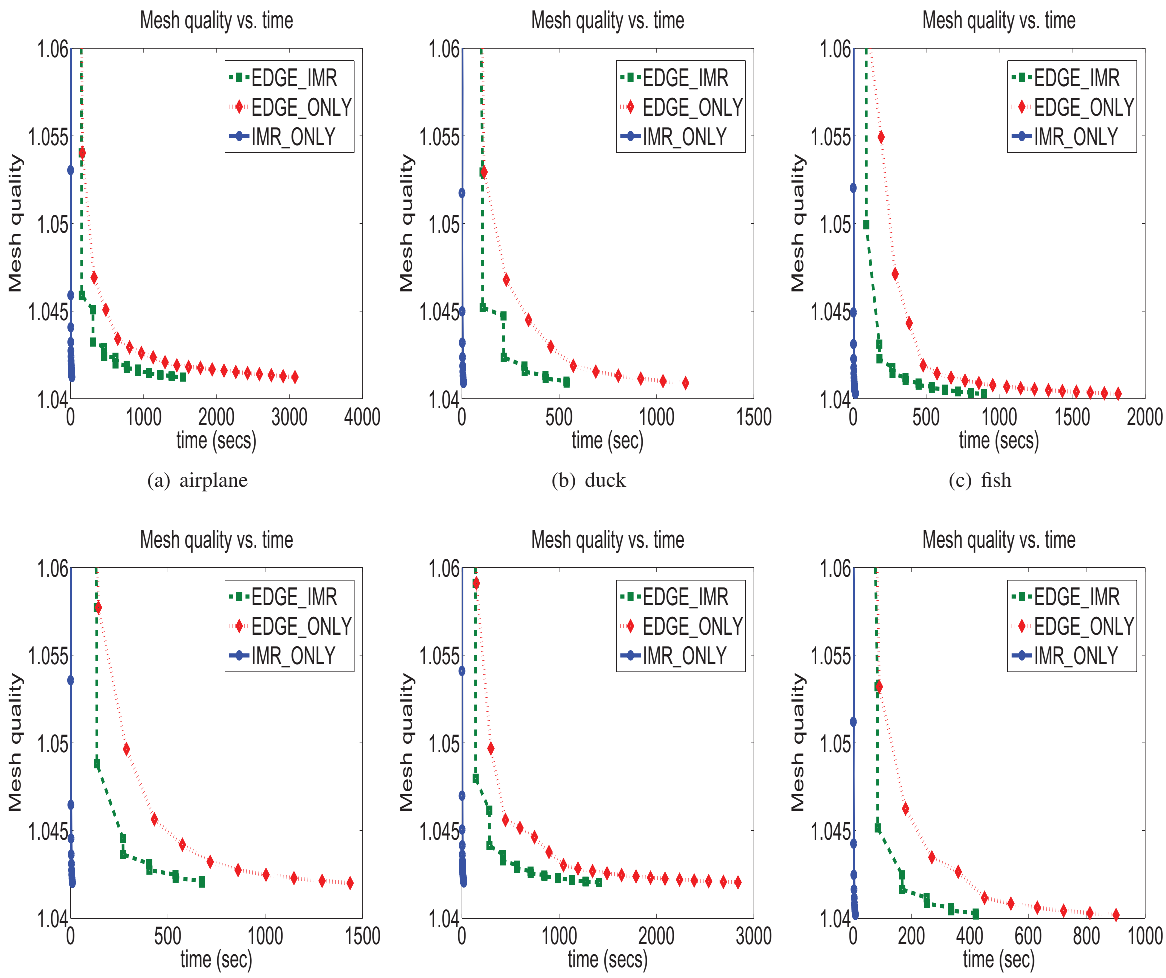
Figure 1: 2D meshes generated by Triangle [16]. These meshes are coarser, representative versions of the meshes used in the experiments.

designed to reduce the convergence time of the optimization solver when the EDGE, AREA, or RMS metrics are used to define (1).

Figure 2 shows the convergence history of the EDGE+IMR alternation scheme in comparison with the convergence histories of the EDGE and IMR schemes when used to optimize the quality of three test meshes. The three optimization schemes showed similar convergence trends on the various input meshes. The figure demonstrates that the EDGE+IMR metric was successful in reducing the convergence time over the corresponding EDGE solver by approximately 50% when (1) was minimized according to the EDGE metric. This reduction in time was possible through the alternation scheme since the IMR optimization solver demonstrated a much faster time to convergence did than the EDGE solver because it took less time per iteration.

Table 3 shows the total time required for mesh quality metric alternation with the EDGE and RMS metrics used as the base metrics. For the EDGE+IMR alternation scheme, the percentage reduction in the total time when compared with the original EDGE solver was: 54.7% for the airplane mesh, 52.9% for the duck mesh, 53% for the fish mesh, 52.9% for the hand mesh, 40% for the mechanic mesh, and 53.3% for the twoholes mesh. Here, the more efficient IMR metric, when used in combination with the less efficient EDGE metric, resulted in a faster solver than the EDGE solver. By combining two metrics, the convergence time can be reduced without altering the original optimization problem.

For the other combinations of quality metrics, alternation was also beneficial in reducing the time to convergence over the original optimization solver (based on the EDGE or RMS mesh quality metrics) in each case. At least 40% improvement was obtained via metric alternation. Such a reduction was possible because optimization with the IMR and AR mesh quality metrics was faster than optimization with either the EDGE and RMS mesh quality metrics. In



(a) airplane (b) duck (c) fish

(d) hand (e) mechanic (f) twoholes

Figure 2: Mesh quality vs. time plots for EDGE+IMR mesh quality metric alternation test. The plots are zoomed in and show only the convergence history at the beginning of the optimization. To reduce the convergence time of the EDGE optimization solver, alternation with the EDGE+IMR alternation scheme for optimization was applied. Approximately 50% improvement was observed in comparison with the EDGE optimization solver.

particular, mesh optimization with either the EDGE or RMS metrics took approximately 200 times as long to converge when compared to solvers based on IMR or AR. When the metrics were combined, the average time per iteration in the mesh quality metric alternation scheme was faster than that for the original scheme. Furthermore, the optimized meshes obtained via the alternation schemes achieved the desired levels of quality when measured according to the EDGE and RMS metrics.

The convergence behavior of the EDGE+AR, RMS+IMR, and RMS+AR alternation schemes was similar to that of the EDGE+IMR alternation scheme as was discussed above. However, two of the alternation schemes studied, i.e., AREA+IMR and AREA+AR, did not show much difference in the convergence time when compared with the other alternation schemes. Figure 3 and Table 4 show the convergence histories and timing results for the AREA+AR alternation scheme. (For the remainder of the paper, figures showing the results of optimizing the duck, hand, and twoholes meshes show similar convergence trends and are omitted due to space constraints.) The convergence times for the AREA+IMR and AREA+AR schemes were similar to those for the AREA solver. The timing results for the alternation schemes based on the AREA metric are shown in Table 4.

Table 3: Timing results for mesh optimization with EDGE and RMS mesh quality metrics and their mesh quality metric alternations: EDGE+IMR, EDGE+AR, RMS+IMR, and RMS+AR.

	Total time to convergence (secs)					
	EDGE	EDGE+IMR	EDGE+AR	RMS	RMS+IMR	RMS+AR
airplane	3399.36	1538.69	1625.29	3225.57	1529.34	1620.61
duck	2292.47	1077.72	1153.18	2830.56	1091.02	1147.5
fish	1911.24	897.54	964.74	1909.88	904.62	965.22
hand	2867.59	1350.37	1429.75	2875.84	1370.29	1451.6
mechanic	2986.64	1412.19	1512.19	3019.28	1424.02	1506.85
twoholes	1799.52	840.54	898.55	1800.32	831.96	916.27

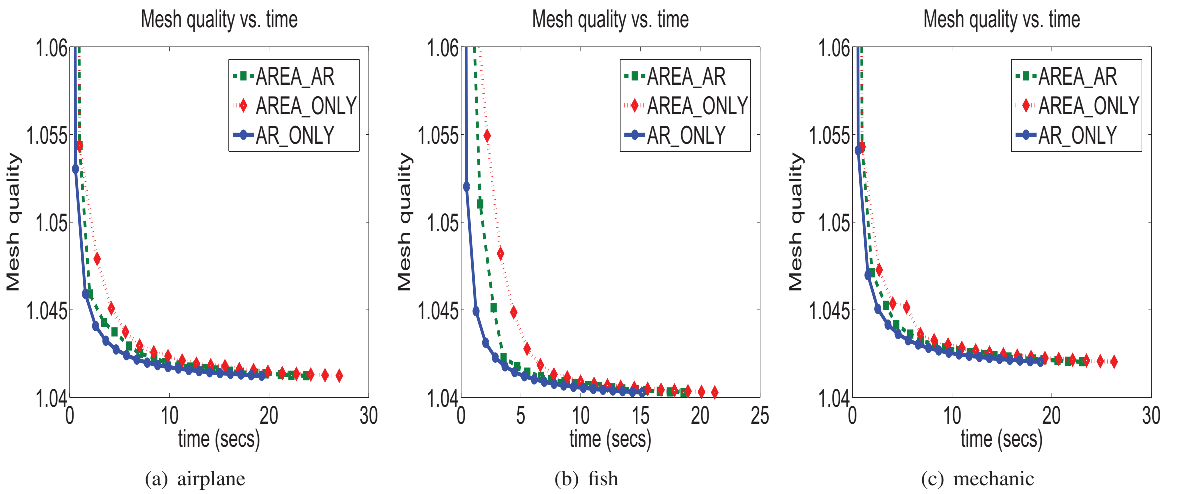


Figure 3: Mesh quality vs. time plots for AREA+AR mesh quality metric alternation test. The plots are zoomed in and show only the convergence history at the beginning of the optimization. To improve the time efficiency of mesh optimization process with AREA mesh quality metric, AR mesh quality metric was combined.

The convergence times of the AREA+AR alternation scheme were very similar to those of the AREA solver. In particular, speed-ups of only 12-13% were observed. The percentage of reduction in the convergence times for the individual meshes are as follows: 12.9% for the airplane mesh, 13.2% for the duck mesh, 13.1% for the fish mesh, 12.7% for the hand mesh, 12.4% for the mechanic mesh, and 13% for the twoholes mesh, respectively. The smaller reductions in the time to convergence were not unexpected, as the AREA and AR optimization solvers only had a 1% difference in their times to convergence. This experiment demonstrates that mesh quality metric alternation is more beneficial when two metrics are combined which correspond to optimization solvers with a large difference in their times to convergence. Thus, appropriate selection of mesh quality metrics for the alternation schemes are required.

Figure 4 shows the elemental mesh quality distributions for the mechanic mesh after mesh optimization was performed with the EDGE+IMR alternation scheme and the EDGE solver. (The distribution for the initial mesh is also shown for comparison purposes.) Clearly, the optimized meshes which result from the two schemes are somewhat different. In particular, the distribution for the EDGE+IMR mesh has more better quality elements despite the fact that the two meshes are of similar average quality. The difference in distributions is due to the two different paths that the optimization routines take to convergence. However, both optimization schemes yielded meshes that are of good quality and met the requirements for convergence.

The optimized meshes resulting from the other alternation schemes, such as EDGE+AR, RMS+IMR, and RMS+AR, have similar mesh quality distributions to the EDGE+IMR alternation scheme distribution. However, mesh optimization with alternation schemes based on the AREA metric showed different mesh quality distributions.

Table 4: Timing results for mesh optimization with the AREA mesh quality metric and its mesh quality metric alternations AREA+IMR and AREA+AR.

	Total time to convergence (secs)		
	AREA	AREA+IMR	AREA+AR
airplane	28.45	24.02	24.78
duck	24.41	20.52	21.2
fish	22.36	18.68	19.44
hand	26.98	22.8	23.55
mechanic	27.51	23.27	24.1
twoholes	21.77	18.22	18.94

Table 5: Total number of iterations and average time per iteration results for mesh quality metric alternations.

(a) EDGE mesh quality metric alternation

	# of iterations			average time per iteration (secs)		
	EDGE	EDGE+IMR	EDGE+AR	EDGE	EDGE+IMR	EDGE+AR
airplane	19	19	19	178.9135	80.98368	85.54158
duck	21	20	20	109.1652	53.886	57.659
fish	16	16	16	119.4525	56.09625	60.29625
hand	13	13	13	220.5838	103.8746	109.9808
mechanic	27	27	27	110.6163	52.30333	56.0071
twoholes	26	26	26	69.21231	32.32846	34.55962

(b) RMS mesh quality metric alternation

	# of iterations			average time per iteration (secs)		
	RMS	RMS+IMR	RMS+AR	RMS	RMS+IMR	RMS+AR
airplane	22	22	22	146.6168	69.51545	73.66409
duck	19	19	19	148.9768	57.42211	60.39632
fish	21	21	21	90.94667	43.07714	45.96286
hand	17	17	17	169.1671	80.60529	85.38824
mechanic	23	23	23	131.273	61.91391	65.51522
twoholes	19	19	19	94.75368	43.78737	48.22474

(c) AREA mesh quality metric alternation

	# of iterations			average time per iteration (secs)		
	AREA	AREA+IMR	AREA+AR	AREA	AREA+IMR	AREA+AR
airplane	20	20	20	1.4225	1.201	1.239
duck	21	21	21	1.162381	0.977143	1.009524
fish	19	19	19	1.176842	0.983158	1.023158
hand	16	16	16	1.68625	1.425	1.471875
mechanic	21	21	21	1.31	1.108095	1.147619
twoholes	20	20	20	1.0885	0.911	0.947

Figure 5 shows the elemental mesh quality distribution for the fish mesh after optimization was performed via the AREA+AR scheme. In this case, the meshes optimized by the AREA+AR alternation scheme and the AREA solver exhibited similar quality distributions.

As the results of the above experiments demonstrate, in order to yield a large reduction in the time to convergence, the mesh quality metric for the optimization solver should be combined with a quality metric for which the corresponding optimization solver is significantly faster to converge. A further reduction in the time to convergence

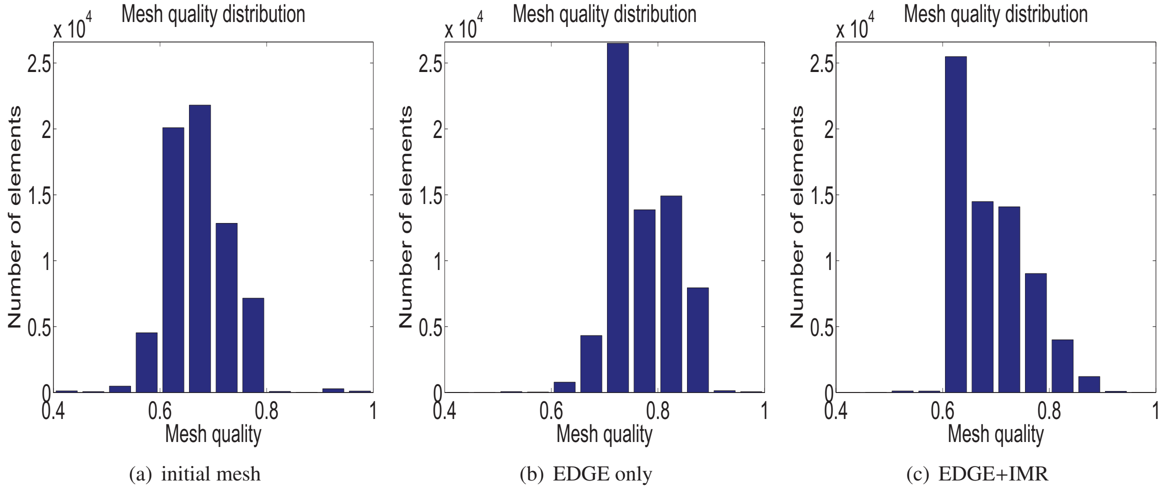


Figure 4: Mesh quality distribution (0.4 to 1) of the mechanic mesh (a) before and (b)-(c) after mesh optimization was performed via the (b) EDGE and (c) EDGE+IMR optimization solvers.

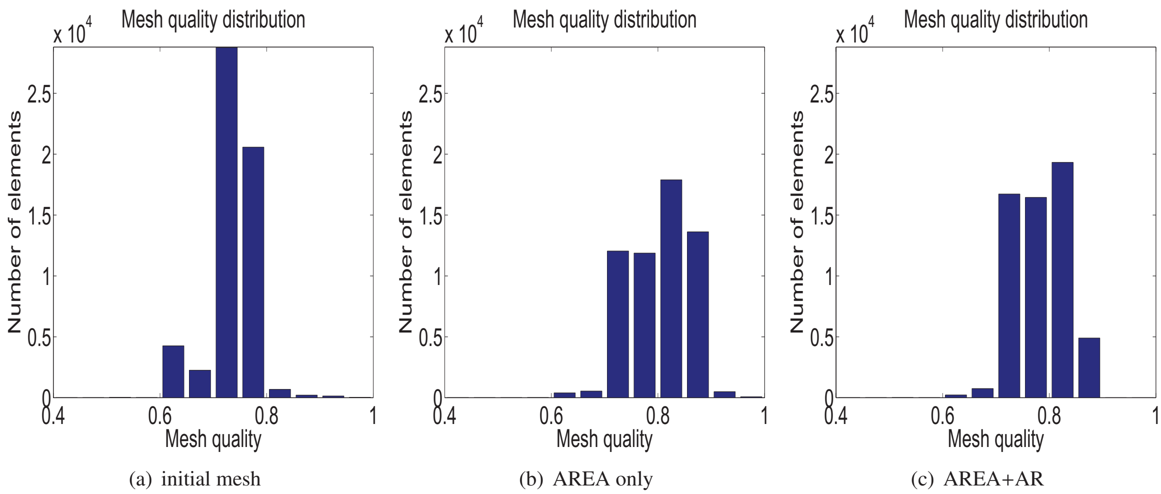
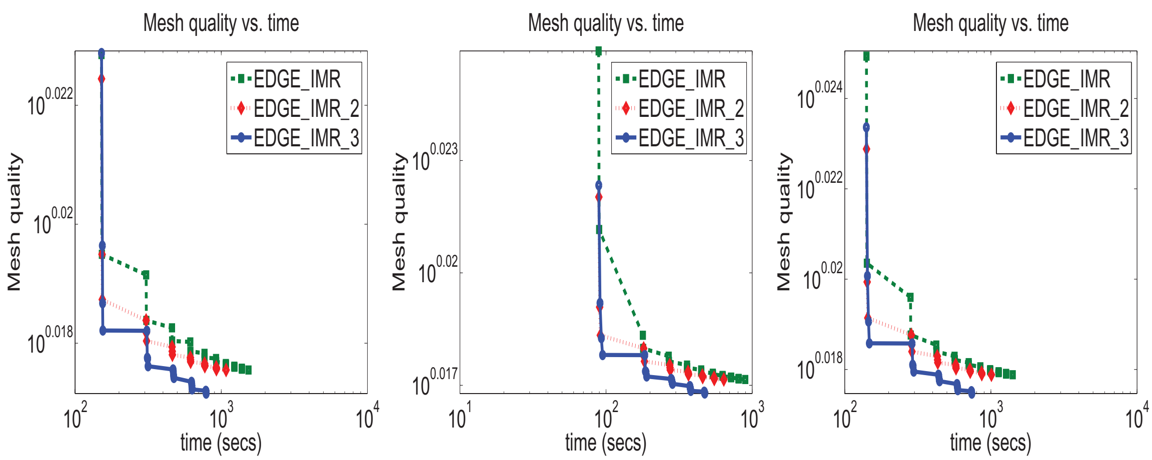


Figure 5: Mesh quality distribution (0.4 to 1) of the fish mesh (a) before and (b)-(c) after mesh optimization was performed via the (b) AREA and (c) AREA+AR optimization solvers.

may be possible through an increased frequency in the use of the fast metric in the optimization process. In order to determine whether or not this is true, an experiment with the alternation schemes EDGE+IMR+IMR and EDGE+IMR+IMR+IMR was performed. Figure 6 shows the convergence histories for these alternation schemes as applied to three test meshes.

As can be seen in Figure 6, the time to convergence decreased with an increase in frequency of application of the IMR metric in the alternation scheme. A significant reduction in the convergence time was observed when comparing the results of the EDGE+IMR and EDGE+IMR+IMR alternation schemes. The times to convergence for the EDGE+IMR+IMR alternation scheme were as follows: 1079.86 seconds for the airplane mesh, 760.63 seconds for the duck mesh, 640.01 seconds for the fish mesh, 964.69 seconds for the hand mesh, 1008.43 seconds for the mechanic mesh, and 597.87 seconds for the twoholes mesh. The corresponding percentage reductions in the optimization time are as follows: 29.8% for the airplane mesh, 29.3% for the duck mesh, 28.6% for the fish mesh, 28.5% for the hand mesh, 28.5% for the mechanic mesh, and 28.8% for the twoholes mesh, respectively. At least a 28% improvement



(a) airplane (b) fish (c) mechanic

Figure 6: Plots of mesh quality vs. time for the EDGE+IMR alternation scheme with varying frequency of use of the IMR metric. The plots are zoomed in and show only the convergence history at the beginning of the optimization. The performance of the EDGE+IMR, EDGE+IMR+IMR, and EDGE+IMR+IMR+IMR schemes was investigated in this experiment.

in time to convergence over the EDGE+IMR alternation scheme was obtained for the EDGE+IMR+IMR alternation scheme. Overall, the percentage decrease in the optimization time when using the EDGE+IMR+IMR alternation scheme instead of the EDGE solver for mesh optimization was approximately 67%.

Further improvement was observed when the EDGE+IMR+IMR+IMR alternation scheme was employed. However, the improvement over the EDGE+IMR+IMR alternation scheme was slightly smaller than the improvement that was observed between the EDGE+IMR+IMR and EDGE+IMR alternation schemes. The timing results for the EDGE+IMR+IMR+IMR mesh quality metric alternation were: 789.01 seconds for the airplane mesh, 561.38 seconds for the duck mesh, 473.49 seconds for the fish mesh, 708.55 seconds for the hand mesh, 740.53 seconds for the mechanic mesh, and 443.34 seconds for the twoholes mesh. These correspond to the following percentage reductions in the mesh optimization time: 27% for the airplane mesh, 26% for the duck mesh, 26% for the fish mesh, 27% for the hand mesh, 27% for the mechanic mesh, and 25% for the twoholes mesh, respectively. Thus, the EDGE+IMR+IMR+IMR alternation schemes exhibited a 77% reduction in the time to convergence when compared with the EDGE optimization solver in the minimization of (1). These results demonstrate that the efficient mesh quality metric can be applied for frequently in a mesh alternation scheme in order to reduce the time to convergence of the associated optimization solver.

4. Conclusions

The choice of mesh quality metric significantly impacts the time to convergence of the mesh optimization technique. In this paper, we proposed a mesh quality metric alternation scheme whereby the original (inefficient) mesh quality metric was alternated with a more efficient metric on alternate iterations of the optimization procedure. Its purpose is to reduce the execution time of the optimization solver, while solving the original mesh optimization problem.

Our experimental results show that the majority of mesh quality metric alternation schemes considered required less time than the original mesh optimization procedure. It was demonstrated that because the IMR and AMR metrics are more efficient than the EDGE and RMS metrics, the former metrics can be alternated with the latter metrics to reduce the time to convergence of the original optimization methods with the inefficient metrics. Mesh alternation schemes applying the EDGE+IMR, EDGE+AR, RMS+IMR, and RMS+AR combinations gave a 40-55% reduction in the optimization time. Increasing the frequency of the efficient metric further reduced the mesh optimization time.

For example, EDGE+IMR+IMR and EDGE+IMR+IMR+IMR converged faster than EDGE+IMR. The improvement rates from the original EDGE optimization were 67% and 77%, respectively, for the EDGE+IMR+IMR and EDGE+IMR+IMR+IMR alternation schemes. Alternation based on the AREA metric showed only 12% improvement because optimization involving AREA is not significantly slower than that of the more efficient metrics.

In conclusion, alternation of the inefficient metric from the original problem with a more efficient metric on alternate iterations can significantly reduce the time of the mesh optimization procedure. The greatest reductions occurred when two metrics corresponding to optimization solvers with very different convergence times were combined. Future research will focus on determining the optimal combination of quality metrics for a given mesh optimization problem and how changing the metric can cause a greater reduction in the mesh quality per step.

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