Computers and Mathematics with Applications 63 (2012) 1056–1065

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Advances in the Adomian decomposition method for solving two-point nonlinear boundary value problems with Neumann boundary conditions

Emad H. Aly [a,∗,1], Abdelhalim Ebaid [b], Randolph Rach [c]

[a] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Tabuk University, Tabuk, Saudi Arabia
[c] 316 S. Maple St., Hartford, MI 49057–1225, USA

ARTICLE INFO

Article history:
Received 26 July 2011
Received in revised form 2 December 2011
Accepted 5 December 2011

Keywords:
Adomian decomposition method
Two-point boundary value problems
Neumann boundary conditions

ABSTRACT

A new straightforward approach for solving ordinary and partial second-order boundary value problems with Neumann boundary conditions is introduced in this research. This approach depends mainly on the Adomian decomposition method with a new definition of the differential operator and its inverse, which has been modified for Neumann boundary conditions. The effectiveness of the proposed approach is verified by several linear and nonlinear examples.

1. Introduction

Over the past 25 years, the Adomian decomposition method (ADM) [1], which was first introduced by the American Physicist George Adomian, has been used to solve effectively and easily a large class of linear and nonlinear ordinary and partial differential equations. In his famous book, Adomian [1] showed the possibility of obtaining explicit solutions of wide varieties of physically significant problems. Moreover, in [2,3] Adomian indicated that no similarity reductions are used to solve Burger equation, where explicit solution was obtained using the t-partial solution. Adomian et al. [4] analyzed the mathematical models of the dynamic interaction of immune response with a population of bacteria, viruses, antigens, or tumor cells. The ADM was used also by Cherruault et al. [5], Kaya and El-Sayed [6], Biazar et al. [7], Hashim et al. [8], and Lesnic [9] to investigate analytically and numerically some other scientific models. Recently, Sweilam and Khader [10] applied the ADM to analyze the nonlinear vibrations of multiwalled carbon nanotubes.

In most cases, the ADM provides a rapidly convergent sequence of approximations, often requiring no more than just a few terms for high accuracy. In addition, the convergence of the ADM was discussed by Cherruault [11], Cherruault et al. [12], Cherruault and Adomian [13], and Cherruault et al. [14]. Moreover, many authors have found this method to be attractive for solving boundary value problems [15–33], because it can be used directly without restrictive assumptions, linearization or Green functions. For examples, Adomian and Rach [15] demonstrated how to solve nonlinear BVPs in several dimensions by the decomposition method. In their paper, they analyzed various ordinary and partial differential equations with Dirichlet and Neumann boundary conditions. In [16], Adomian solved the Thomas–Fermi equation subject to Dirichlet boundary conditions; however, his solution was depend upon evaluating the unknown constants of integration by applying the

∗ Corresponding author.
E-mail addresses: emad-aly@hotmail.com (E.H. Aly), halimgamil@yahoo.com (A. Ebaid).

1 Permanent address: Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11757, Cairo, Egypt.

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.12.010

boundary conditions to each evaluated approximate solution. Many other physical and engineering problems were solved by the ADM such as the nonlinear oscillator equation by Shawagfeh [17], the heat equation by Hadizadeh and Maleknejad [18], and Bratu-type equations by Wazwaz [20]. Benabidallah and Cherruault [21–23] used the ADM to solve classes of BVPs with Dirichlet boundary conditions and their analysis is discussed in the next section. Nonlinear boundary value problems of higher orders have been also investigated by Al-Hayani [26], Wazwaz [27,28], and Hashim [29]. Dehghan [30] applied the ADM to solve a two-dimensional parabolic equation subject to nonstandard boundary specifications.

In light of this introduction, it is observed that little attention was devoted for applying the ADM to boundary value problems with Neumann boundary conditions. Furthermore, the ADM was not applied in a direct manner to solve such kind of BVPs. So, the aim of this work is to introduce a direct approach for solving ordinary and partial second-order boundary value problems with Neumann boundary conditions.

2. ADM

Consider the two-point boundary value problem:

$$u''(x) + p(x)u'(x) + q(x)f\,(u(x)) = r(x), \quad x \in [a, b], \tag{1}$$

with Neumann boundary conditions

$$u'(a) = \alpha, \qquad u'(b) = \beta. \tag{2}$$

According to the standard ADM [1], the inverse operator $L_{xx}^{-1}[.] = \iint [.]dxdx$ is applied to Eq. (1) resulting in

$$u(x) = \phi_x + L_{xx}^{-1}\left(r(x) - p(x)u'(x) - q(x)f\,(u(x))\right), \tag{3}$$

where $\phi_x = c_1 + c_2x$, c_1 and c_2 are constants of integration. The method is then based on decomposing ϕ_x, the linear and nonlinear terms u and $f(u(x))$ as follows:

$$\begin{cases} \phi_x = \sum_{n=0}^{\infty} \phi_{x,n} = \sum_{n=0}^{\infty} (c_{1,n} + c_{2,n}x), \\ u(x) = \sum_{n=0}^{\infty} u_n(x), \\ f(u(x)) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \end{cases} \tag{4}$$

where A_n are the Adomian polynomials tailored to the specific nonlinearity and can be computed from the definitional formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{5}$$

The decomposition of the initial term is needed for nonlinear boundary value problems whether ordinary or partial differential equations are involved, however, it is unnecessary in linear ODEs; see [15] for details. Substituting (4) into (3), yields the following recursion scheme

$$\begin{cases} u_0(x) = \phi_{x,0} + L_{xx}^{-1} r(x), \\ u_{n+1}(x) = \phi_{x,n} - L_{xx}^{-1}\left(p(x)u_n'(x) + q(x)A_n\right), \quad n \geq 0. \end{cases} \tag{6}$$

The approximate solution of Eq. (1) is given by

$$\Phi_m(x) = \sum_{n=0}^{m-1} u_n(x) \quad \text{for } m > 0. \tag{7}$$

Thus $\Phi_1 = u_0$, $\Phi_2 = \Phi_1 + u_1$, $\Phi_3 = \Phi_2 + u_2$, etc., serve as approximate solutions of increasing accuracy as n increases and must, of course, satisfy the boundary conditions [15]. To evaluate values of the constants $c_{1,n}$ and $c_{2,n}$, we begin with $\Phi_1 = u_0 = c_{1,0} + c_{2,0}x + L_{xx}^{-1} r(x)$. Using the boundary conditions (2) results in

$$\begin{cases} c_{2,0} + \left[L_{xx}^{-1} r(x)\right]'_{x=a} = \alpha, \\ c_{2,0} + \left[L_{xx}^{-1} r(x)\right]'_{x=b} = \beta. \end{cases} \tag{8}$$

It should be noted that by solving these equations with respect to $c_{2,0}$, its value is obtained, while $c_{1,0}$ remains unknown and consequently u_0 is not completely determined; hence we cannot proceed with the standard ADM. Recently, an alternate procedure is proposed in [21,22] to obtain the numerical solutions for a class of boundary value problems. The canonical form obtained in these studies is based on integrating equation (1) first from x to b and then integrating the resulting equation again from a to x. This procedure permits only the use of the boundary condition $u'(b) = \beta$ and assumes that $u(a) = \sum_{k=0}^{\infty} c_k$, where the values of $c_0, c_1, c_2, \ldots$, etc. are evaluated by using the other boundary condition $u'(a) = \alpha$. This procedure is of course simpler than the standard ADM discussed above; however, it does not use all of the boundary conditions in a direct way. To the best of the authors' knowledge, such a direct way has not been previously employed to solve second-order BVPs with Neumann boundary conditions by the ADM. The aim of the present work is therefore to provide a direct approach by introducing a new definition of the inverse linear operator. The proposed procedure is tested by investigating a number of two-point boundary value problems with Neumann boundary conditions.

3. The inverse linear operator

Theorem 1. *If $u'(a) = \alpha$ and $u'(b) = \beta$ are Neumann boundary conditions of a second-order ordinary differential equation, then*

$$L_{xx}^{-1}u''(x) = u(x) - (x - \Omega)u'(a) - \frac{\Omega}{2}u'(b) - \frac{1}{\Omega}\int_0^{\Omega} u(x)dx, \quad a \le x \le b, \tag{9}$$

where $L_{xx}^{-1}[.]$ is defined by

$$L_{xx}^{-1}[.] = \int_{\Omega}^{x} dx' \int_a^{x'} [.]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx' \left(x' \int_b^{x'} [.]dx'' \right), \tag{10}$$

where Ω is an arbitrary finite constant.

Proof. Suppose that

$$L_{xx}^{-1}[.] = \int_{\Omega}^{x} dx' \int_a^{x'} [.]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx' \left(x' \int_b^{x'} [.]dx'' \right),$$

then

$$\begin{aligned} L_{xx}^{-1}[u''(x)] &= \int_{\Omega}^{x} dx' \int_a^{x'} [u''(x)]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx' \left(x' \int_b^{x'} [u''(x)]dx'' \right) \\ &= \int_{\Omega}^{x} [u'(x') - u'(a)]dx' + \frac{1}{\Omega}\int_0^{\Omega} x' \left[u'(x') - u'(b) \right] dx' \\ &= u(x) - u(\Omega) - (x - \Omega)u'(a) + \frac{1}{\Omega}\int_0^{\Omega} x'u'(x')dx' - \frac{1}{\Omega}\left[\frac{\Omega^2}{2}u'(b) \right] \\ &= u(x) - u(\Omega) - (x - \Omega)u'(a) + \frac{1}{\Omega}\int_0^{\Omega} x'u'(x')dx' - \frac{\Omega}{2}u'(b) \\ &= u(x) - u(\Omega) - (x - \Omega)u'(a) + \frac{1}{\Omega}\left[\Omega u(\Omega) - \int_0^{\Omega} u(x')dx' \right] - \frac{\Omega}{2}u'(b) \\ &= u(x) - u(\Omega) - (x - \Omega)u'(a) + \left[u(\Omega) - \frac{1}{\Omega}\int_0^{\Omega} u(x')dx' \right] - \frac{\Omega}{2}u'(b) \\ &= u(x) - (x - \Omega)u'(a) - \frac{\Omega}{2}u'(b) - \frac{1}{\Omega}\int_0^{\Omega} u(x)dx. \quad \square \end{aligned}$$

It should be noted that this theorem can also be readily extended to include second-order partial differential equations.

4. A new approach for solving second-order ODEs and PDEs

In this section, we establish two algorithms for solving linear and nonlinear second-order ordinary and partial differential equations with Neumann boundary conditions. First, we rewrite Eq. (1) in the form:

$$u''(x) = r(x) - p(x)u'(x) - q(x)f(u(x)). \tag{11}$$

Now, on applying the operator $L_{xx}^{-1}(.)$ given by Eq. (10) to the last equation, we obtain

$$u(x) = (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega}\int_0^{\Omega} u(x)dx + L_{xx}^{-1}\left(r(x) - p(x)u'(x) - q(x)f(u(x)) \right), \tag{12}$$

where u and $f(u(x))$ are decomposed by Eq. (4), the A_n are computed using formula (5). Substituting (4) into (12), and according to the ADM, the solution $u(x)$ can be directly computed in the nonlinear case by using the following modified recursion scheme

$$\begin{cases} u_0 = (x-\Omega)u'(a) + \dfrac{\Omega}{2}u'(b) + L_{xx}^{-1}\left[r(x)\right], \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^{\Omega} u_n(x)dx - L_{xx}^{-1}\left[p(x)u_n'(x) + q(x)A_n\right], \quad n \geq 0, \end{cases} \tag{13}$$

and in the linear case as follows

$$\begin{cases} u_0 = (x-\Omega)u'(a) + \dfrac{\Omega}{2}u'(b) + L_{xx}^{-1}\left[r(x)\right], \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^{\Omega} u_n(x)dx - L_{xx}^{-1}\left[p(x)u_n'(x) + q(x)u_n\right], \quad n \geq 0, \end{cases} \tag{14}$$

where algorithms (13) and (10), (14) and (10) comprise an advanced Adomian decomposition method (AADM) for nonlinear and linear cases, respectively. We conclude this section by noting that the approximate analytic solution is given by

$$\Phi_n(x; \Omega) = \sum_{i=0}^{n-1} u_i(x; \Omega). \tag{15}$$

The last equation modifies Eq. (7) by considering Φ_n to be a parametric function of Ω, which plays an important role in obtaining a class of solutions as shown in the next section.

5. Test examples

In this section, we show that the AADM is effective in studying the analytical and numerical solutions for second-order boundary value problems with appropriate Neumann boundary conditions. However, it should be noted that the AADM gives a unique solution, when $\Omega \to 0$, in the presence of the dependent variable, say u, explicitly of the ODE structure, as in Examples 1–4, and a class of solutions in the absence of it, as in Examples 5 and 6.

5.1. Example 1

Consider the following linear ordinary boundary value problem [34]

$$\begin{cases} u''(x) + u(x) + x = 0, \quad 0 \leq x \leq 1, \\ u'(0) = -1 + \csc(1), \qquad u'(1) = -1 + \cot(1). \end{cases} \tag{16}$$

On applying the AADM for this problem, we obtain

$$\begin{cases} u_0 = (x-\Omega)\left(-1+\csc(1)\right) + \dfrac{\Omega}{2}\left(-1+\cot(1)\right) + \dfrac{1}{24}\left(-4x^3 + 6\Omega + \Omega^3\right), \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^{\Omega} u_n(x)dx - \int_{\Omega}^{x} dx' \int_0^{x'} [u_n(x'')]dx'' - \dfrac{1}{\Omega}\int_0^{\Omega} dx' \left(x' \int_1^{x'} [u_n(x'')]dx''\right), \quad n \geq 0. \end{cases} \tag{17}$$

Using this algorithm, we find that the approximate solutions $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ as $\Omega \to 0$ are given by

$$\begin{cases} \Phi_2(x) = -x + \csc(1)\left(x - \dfrac{x^3}{3!}\right) + \dfrac{x^5}{5!}, \\ \Phi_3(x) = -x + \csc(1)\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}\right) - \dfrac{x^7}{7!}, \\ \Phi_4(x) = -x + \csc(1)\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}\right) + \dfrac{x^9}{9!}. \end{cases} \tag{18}$$

It should be noted that problem (16) has a unique solution, given by $u(x) = -x + \csc(1)\sin(x)$. We now have the following Maclaurin series of the exact solution

$$u(x) = -x + \csc(1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right). \tag{19}$$

From the approximate solutions given in (18) we note that $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ agree with its Maclaurin series (19) up to x^3, x^5 and x^7, respectively. Therefore, on evaluating more terms of the decomposition series, the Maclaurin series (19)

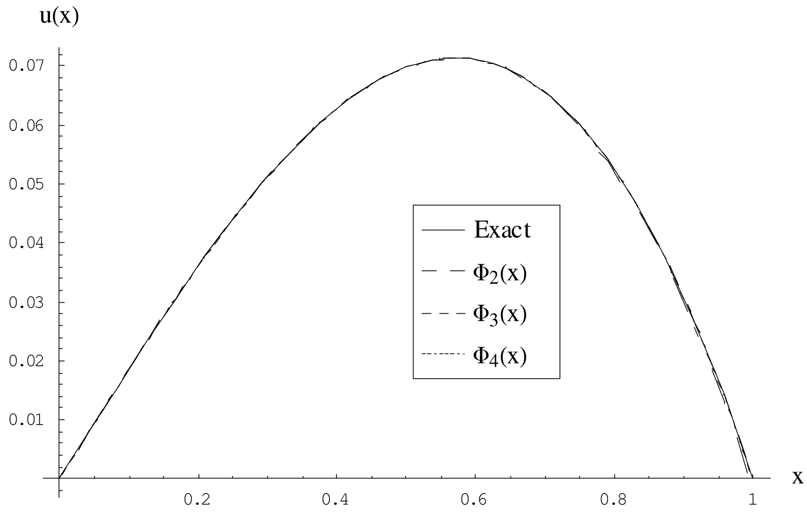

Fig. 1. Results of Example 1, comparison of the exact solution with Φ_1, Φ_2 and Φ_3.

is obtained in the limit. To numerically verify if the proposed approach leads to accurate solutions, we compare the approximate solutions $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ as $\Omega \to 0$ with the exact solution in Fig. 1. As mentioned in the beginning of this section, the numerical results show that a unique solution with good approximation is achieved using only a few terms of the decomposition series solution.

5.2. Example 2

Bratu model appears in a number of applications such as fuel ignition in the thermal combustion theory and also in the Chandrasekhar model of the expansion of the universe [20]. Therefore, on considering the following Bratu problem [20]

$$\begin{cases} u''(x) - 2\,\mathrm{e}^{u} = 0, & 0 \le x \le 1, \\ u'(0) = 0, & u'(1) = 2\tan(1). \end{cases} \tag{20}$$

and applying the AADM to it, we obtain the following recursion scheme

$$\begin{cases} u_0 = \Omega \tan(1), \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^{\Omega} u_n(x)dx + \int_{\Omega}^{x} dx' \int_0^{x'} \left[2A_n(x'')\right] dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx' \left(x' \int_1^{x'} \left[2A_n(x'')\right] dx'' \right), \end{cases} \tag{21}$$

where $n \ge 0$ and A_n are the Adomian polynomials of the nonlinear term $f(u) = \mathrm{e}^u$. On using the definitional formula (5), the first four terms of A_n are given by

$$\begin{cases} A_0 = \mathrm{e}^{u_0}, \\ A_1 = u_1 \mathrm{e}^{u_0}, \\ A_2 = \left(u_2 + \dfrac{u_1^2}{2} \right) \mathrm{e}^{u_0}, \\ A_3 = \left(u_3 + u_1 u_2 + \dfrac{u_1^3}{6} \right) \mathrm{e}^{u_0}. \end{cases} \tag{22}$$

Using algorithm (21), we find that the approximate solutions $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ as $\Omega \to 0$ are given by

$$\begin{cases} \Phi_2(x) = x^2, \\ \Phi_3(x) = x^2 + \dfrac{x^4}{6}, \\ \Phi_4(x) = x^2 + \dfrac{x^4}{6} + \dfrac{2x^6}{45}. \end{cases} \tag{23}$$

Problem (20) has the following exact solution $u(x) = -2\log[\cos(x)]$, where the Maclaurin series of this solution is given by

$$u(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \frac{62x^{10}}{14\,175} + \cdots. \tag{24}$$

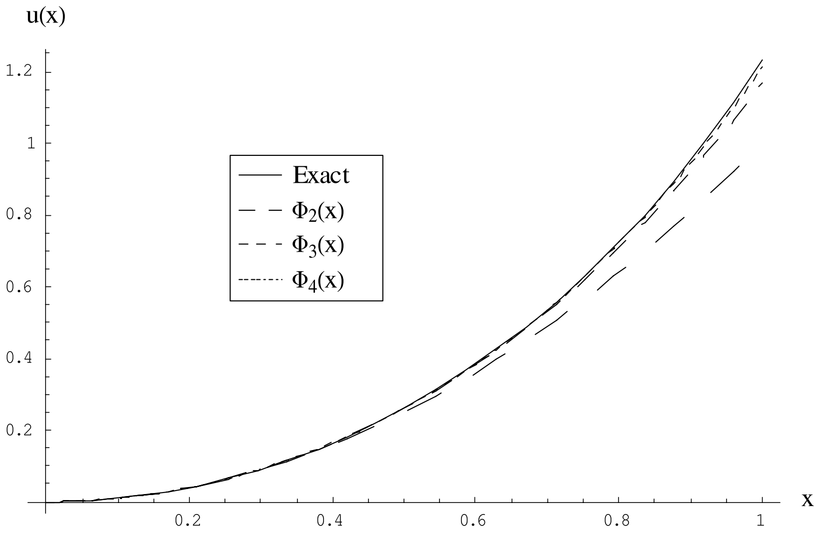

Fig. 2. Results of Example 2, comparison of the exact solution with Φ_2, Φ_3 and Φ_4.

The approximate solutions $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ given in (23) agree with the Maclaurin series expansion of the exact solution (24) up to x^2, x^4 and x^6, respectively. Therefore, computing more terms of the decomposition series solution, as $\Omega \to 0$, leads to obtaining the Maclaurin series of the exact solution. Moreover, Fig. 2 shows that a numerical solution with good accuracy is achieved using only a few terms of the decomposition series solution. Note especially that the curve of the approximate solution $\Phi_4(x)$ is nearly identical to the exact solution.

5.3. Example 3

Consider the nonlinear oscillator equation [17]

$$\begin{cases} u'' + \omega^2 u = \lambda u^m, \quad 0 \le x \le 1, \\ u'(0) = 1, \qquad u'(1) = cn\left(1|\dfrac{1}{4}\right) dn\left(1|\dfrac{1}{4}\right), \end{cases} \tag{25}$$

where m is a positive integer. This problem has the exact solution $u = sn\left(x|\frac{1}{4}\right)$ when $m = 3$ (Duffing oscillator), $\lambda = \frac{1}{2}$ and $\omega^2 = \frac{5}{4}$, where sn, cn, dn are Jacobi elliptic functions [17]. Using the AADM, the solution can be directly computed by using the following modified recursion scheme

$$\begin{cases} u_0 = x + \dfrac{\Omega}{2}\left[cn\left(1|\dfrac{1}{4}\right) dn\left(1|\dfrac{1}{4}\right) - 2\right], \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^{\Omega} u_n(x)dx + L_{xx}^{-1}\left[\lambda A_n - \omega^2 u_n\right], \quad n \ge 0. \end{cases} \tag{26}$$

The first few terms of Adomian polynomials for the nonlinear term u^m can be obtained from (5) as $A_0 = u_0^m$, $A_1 = m u_0^{m-1} u_1$, and $A_2 = m u_0^{m-1} u_2 + \frac{1}{2} m(m-1) u_0^{m-2} u_1^2$. The approximate solutions $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ as $\Omega \to 0$ are given by

$$\begin{cases} \Phi_2(x) = x - \dfrac{5x^3}{24} + \dfrac{x^5}{40}, \\ \Phi_3(x) = x - \dfrac{5x^3}{24} + \dfrac{73x^5}{1920} - \dfrac{11x^7}{1344} + \dfrac{x^9}{1920}, \\ \Phi_4(x) = x - \dfrac{5x^3}{24} + \dfrac{73x^5}{1920} - \dfrac{79x^7}{9216} + \dfrac{593x^9}{322\,560} - \dfrac{307x^{11}}{1\,182\,720} + \dfrac{11x^{13}}{998\,400}. \end{cases} \tag{27}$$

The Maclaurin series of the exact solution is given by

$$u(x) = x - \frac{5x^3}{24} + \frac{73x^5}{1920} - \frac{79x^7}{9216} + \frac{24\,487x^9}{13\,271\,040} - \cdots. \tag{28}$$

Again in this example, the approximate solutions $\Phi_2(x)$, $\Phi_3(x)$ and $\Phi_4(x)$ agree with the Maclaurin series expansion of the exact solution given in (28) up to x^3, x^5 and x^7, respectively. Furthermore, the numerical results presented in Fig. 3 show that a good approximation is achieved using only a few terms of the decomposition series solution.

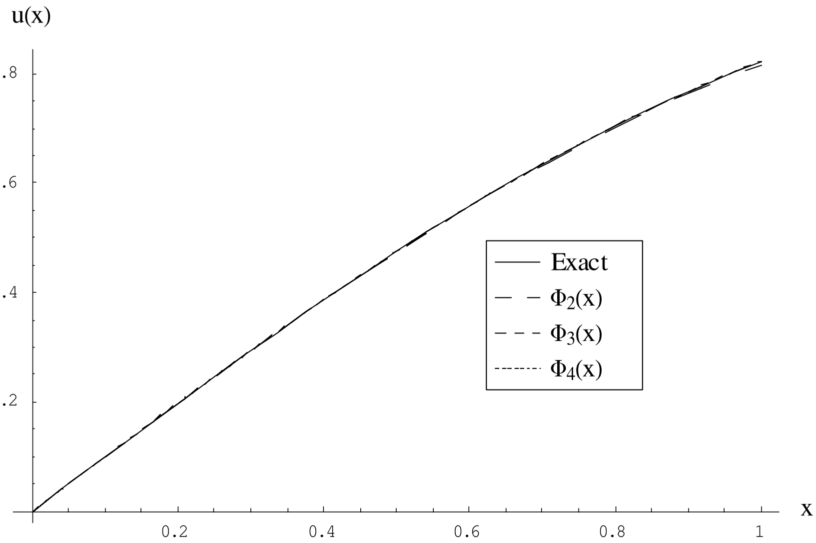

Fig. 3. Results of Example 3, comparison of the exact solution with Φ_2, Φ_3 and Φ_4.

5.4. Example 4

Consider the nonlinear Burger equation

$$\begin{cases} u'' + uu' + u = \frac{1}{2}\sin(2x), & 0 \le x \le \frac{\pi}{2}, \\ u'(0) = 1, \qquad u'\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{29}$$

Considering the AADM, the solution can be computed by using the following modified recursion scheme

$$\begin{cases} u_0 = x - \Omega - \frac{1}{16\Omega}\left[2\Omega\sin(2x) + \cos(2\Omega) - 4\Omega x + 6\Omega^2 - 1\right], \\ u_{n+1} = \frac{1}{\Omega}\int_0^\Omega u_n(x)dx - L_{xx}^{-1}\left[u_n + \sum_{k=0}^{n} u_k u'_{n-k}\right], \quad n \ge 0. \end{cases} \tag{30}$$

Using MATHEMATICA and algorithm (30), we obtain the approximate analytic solutions $\Phi_2(x)$, $\Phi_3(x)$, $\Phi_4(x)$ and $\Phi_5(x)$ when $\Omega \to 0$ in terms of the trigonometric functions sine and cosine. In addition, when using the Maclaurin expansions of the sine and cosine functions, these approximate solutions or Φ's can be written as infinite power series of x as follows:

$$\begin{cases} \Phi_2(x) = x - \frac{x^3}{3!} - \frac{3x^5}{40} + \cdots, \\ \Phi_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{109x^7}{5040} + \cdots, \\ \Phi_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{703x^9}{120\,960} + \cdots, \\ \Phi_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{8887x^{11}}{5\,702\,400} + \cdots. \end{cases} \tag{31}$$

It is clear from (31) that by evaluating more terms we can obtain closer and closer approximations to the Maclaurin expansions of the exact solution of problem (29), that is $u(x) = \sin(x)$.

5.5. Example 5

Consider the following linear ordinary boundary value problem

$$\begin{cases} u'' = e^x, \\ u'(0) = 1, \qquad u'(1) = e. \end{cases} \tag{32}$$

On applying the AADM for this problem, we obtain the following recursion scheme

$$\begin{cases} u_0 = e^x + \mu, \qquad \text{where } \mu = \dfrac{1 - e^{\Omega}}{\Omega}, \\ u_{n+1} = \dfrac{1}{\Omega} \displaystyle\int_0^{\Omega} u_n dx. \end{cases} \tag{33}$$

From this recurrence relation, we obtain $u_n = 0$, for all $n \geq 1$, consequently the exact solution is given by $u = u_0$. This example shows that the AADM can give a class of solutions.

5.6. Example 6

On considering the following nonlinear BVP

$$\begin{cases} u'' - (u')^2 = 0, \\ u'(0) = -1, \qquad u'(1) = -\dfrac{1}{2} \end{cases} \tag{34}$$

and applying the AADM, we obtain the following recursion scheme

$$\begin{cases} u_0 = -x + \mu_0; \qquad A_n = \displaystyle\sum_{i=1}^{n} u'_{n-i} u'_i \\ u_{n+1} = \displaystyle\int_{\Omega}^{x} \left(\int_0^x A_n dx \right) dx + \frac{1}{\Omega} \int_0^{\Omega} \left(x \int_1^x A_n dx \right) dx + \frac{1}{\Omega} \int_0^{\Omega} u_n dx. \end{cases} \tag{35}$$

From this algorithm, we can easily obtain the following approximants

$$\begin{cases} \Phi_2(x) = -x + \dfrac{x^2}{2} + \mu_1, \\ \Phi_3(x) = -x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \mu_2, \\ \Phi_4(x) = -x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{x^4}{4} + \mu_3, \\ \Phi_5(x) = -x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{x^4}{4} - \dfrac{x^5}{5} + \mu_4, \end{cases} \tag{36}$$

where the μ_i are constants in terms of Ω. As $n \to \infty$, then $u = -\log(x + 1) + \mu$, which is a class of exact solutions, where $\mu = \text{limit } \mu_i$ as $n \to \infty$.

Remark 1. Before applying the AADM technique to PDEs in the next three examples, it should be noted that Adomian and Rach [31] and Wazwaz [35] have proved that partial solutions are equal in the decomposition method for linear or nonlinear partial differential equations. Therefore, we shall consider the boundary conditions but not the initial conditions for studying the possibility of calculating a class of solutions using our new approach.

5.7. Example 7

Consider the following linear partial boundary value problem for the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \ t \geq 0, \\ u_x(0, t) = e^t, \qquad u_x(1, t) = e^t \cosh(1). \end{cases} \tag{37}$$

The heat problem (37) has a class of exact solutions given by $u(x, t) = e^t \sinh(x) + \mu_4$, where μ_4 is an arbitrary constant. Applying the AADM, we obtain the following recursion scheme

$$\begin{cases} u_0 = \left((x - \Omega) + \dfrac{\Omega}{2} \cosh(1) \right) e^t, \\ u_{n+1} = \dfrac{1}{\Omega} \displaystyle\int_0^{\Omega} u_n(x, t) dx + \int_{\Omega}^{x} dx' \int_0^{x'} \frac{\partial u_n(x'', t)}{\partial t} dx'' + \frac{1}{\Omega} \int_0^{\Omega} dx' \left(x' \int_1^{x'} \frac{\partial u_n(x'', t)}{\partial t} dx'' \right), \end{cases} \tag{38}$$

where $n \geq 0$. As in Example 6, the arbitrary constant μ_4 also generates a class of exact solutions. It is therefore sufficient to show that the exact solution $u(x,t)$ when $\mu_4 = 0$ matches the exact solution obtained through our algorithm (38) as $\Omega \to 0$. Using algorithm (38), the approximate analytic solution $\Phi_6(x,t)$ as $\Omega \to 0$ is given by

$$\Phi_6(x,t) = e^t \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} \right). \tag{39}$$

By evaluating more terms, we find that

$$u(x,t) = \lim_{n\to\infty} \Phi_n(x,t) = e^t \sinh(x), \tag{40}$$

which is the exact solution of the given linear heat equation with the specified Neumann boundary conditions when μ_4 vanishes.

5.8. Example 8

Consider the following inhomogeneous wave equation [36]

$$\begin{cases} u_{tt} = u_{xx} + 2\pi^2 e^{-\pi t} \sin(\pi x), \quad 0 \leq x \leq 1,\ t \geq 0, \\ u_x(0,t) = \pi e^{-\pi t}, \qquad u_x(1,t) = -\pi e^{-\pi t}. \end{cases} \tag{41}$$

Proceeding as above, we obtain the following recursion scheme

$$\begin{cases} u_0 = (x-\Omega)\pi e^{-\pi t} - \dfrac{\Omega}{2}\pi e^{-\pi t} + \dfrac{1}{\pi\Omega}\left[2\cos(\pi\Omega) + 2\pi\Omega\sin(\pi x) + \pi^2\Omega(3\Omega - 2x) - 2\right] e^{-\pi t}, \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^\Omega u_n(x,t)dx + \int_\Omega^x dx' \int_0^{x'} \frac{\partial^2 u_n(x'',t)}{\partial t^2} dx'' + \frac{1}{\Omega}\int_0^\Omega dx' \left(x' \int_1^{x'} \frac{\partial^2 u_n(x'',t)}{\partial t^2} dx'' \right), \end{cases} \tag{42}$$

where $n \geq 0$. Again, as in the last example, $\Phi_5(x,t)$ can be written as infinite power series of x as follows:

$$\Phi_5(x,t) = e^{-\pi t}\left[\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \frac{(\pi x)^9}{9!} - \frac{(\pi x)^{11}}{19\,958\,400} + \cdots \right]. \tag{43}$$

Evaluating more terms therefore results in

$$u(x,t) = \lim_{n\to\infty} \Phi_n(x,t) = e^{-\pi t}\sin(\pi x), \tag{44}$$

which matches the exact solution $u(x,t) = e^{-\pi t}\sin(\pi x) + \mu_5 t + \mu_6$ when μ_5 and μ_6 vanish, where μ_5 and μ_6 are arbitrary constants.

5.9. Example 9

Consider the following nonlinear Burger equation [37]

$$\begin{cases} u_t + uu_x - u_{xx} = 0, \quad 0 \leq x \leq 2,\ t \geq 0, \\ u_x(0,t) = \dfrac{1}{t} - \dfrac{\pi^2}{2t^2}, \qquad u_x(2,t) = \dfrac{1}{t} - \dfrac{\pi^2}{2t^2}\operatorname{sech}^2\left(\dfrac{\pi}{t}\right). \end{cases} \tag{45}$$

Proceeding as before, we obtain the following recursion scheme

$$\begin{cases} u_0 = (x-\Omega)\left(\dfrac{1}{t} - \dfrac{\pi^2}{2t^2}\right) + \dfrac{\Omega}{2}\left[\dfrac{1}{t} - \dfrac{\pi^2}{2t^2}\operatorname{sech}^2\left(\dfrac{\pi}{t}\right)\right], \\ u_{n+1} = \dfrac{1}{\Omega}\displaystyle\int_0^\Omega u_n(x,t)dx + L_{xx}^{-1}\left[\frac{\partial u_n}{\partial t} + \sum_{k=0}^n u_k u_{n-k}\right], \quad n \geq 0. \end{cases} \tag{46}$$

Using this algorithm, the approximate analytic solution $\Phi_5(x,t)$ as $\Omega \to 0$ is given by

$$\Phi_5(x,t) = \frac{x}{t} - \frac{\pi}{t}\left(\left(\frac{\pi x}{2t}\right) - \frac{1}{3}\left(\frac{\pi x}{2t}\right)^3 + \frac{2}{15}\left(\frac{\pi x}{2t}\right)^5 - \frac{17}{315}\left(\frac{\pi x}{2t}\right)^7 + \frac{62}{2835}\left(\frac{\pi x}{2t}\right)^9 \right), \tag{47}$$

which agrees up to x^9 with the Maclaurin series expansion of the exact solution $u = \frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2t}\right) + \mu_7$, when μ_7 goes to zero, where μ_7 is an arbitrary constant.

Remark 2. As shown from the results of Examples 1 through 9, and in many similar problems, three terms are usually sufficient to closely approximate the exact solution. However, four or more terms can be readily and easily computed as demonstrated in other examples. It should be noted that in most cases a few terms are adequate to achieve practical solutions for engineering design and systems analysis.

6. Conclusion

In this research, the advanced Adomian decomposition method (AADM) for solving two-point BVPs with Neumann boundary conditions has been introduced. This extension is based on a new definition of the inverse linear operator. The main advantage of this approach is the direct way of dealing with the Neumann boundary conditions. In addition, a unique solution is resulted or a class of approximate solutions are otherwise obtained. The AADM is validated by discussing several linear and nonlinear ordinary and partial two-point boundary value problems. It is shown that for a sufficiently small number of components, the approximate and exact solutions become nearly identical.

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