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The General Common Nonnegative-Definite and Positive-Definite Solutions to the Matrix Equations $AXA^* = BB^*$ and $CXC^* = DD^*$

XIAN ZHANG*
Department of Mathematics, Heilongjiang University
Harbin, 150080, P.R. China
and
School of Mechanical and Manufacturing Engineering
The Queen's University of Belfast, Stranmillis Road
Belfast, BT9 5AH, United Kingdom
x.zhang@qub.ac.uk

MEI-YU CHENG
Department of Mathematics, Heilongjiang University
Harbin, 150080, P.R. China

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Abstract—We give necessary and sufficient conditions for the existence of a common nonnegative-definite (positive-definite) solution to the pair of matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$, and derive a representation of the general common nonnegative-definite (positive-definite) solution to these two equations when they have such common solutions. This paper can be viewed as a supplementary version of that derived by Young *et al.* [1] since Groß [2] has given a counterexample to point out a mistake in their basic Theorem 1. The presented example shows the advantage of the proposed approach. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Let $\mathbf{C}^{m\times n}$ be the set of all $m \times n$ complex matrices. We denote by $\mathbf{U}_n$ the subset of $\mathbf{C}^{n\times n}$ consisting of all unitary matrices. For $X \in \mathbf{C}^{m\times n}$, let X^*, $\mathcal{N}(X)$, and $\mathcal{R}(X)$ be the conjugate transpose, the nullspace, and the column space of X, respectively. We denote by I_n and O the $n \times n$ identity matrix and the zero matrix, respectively.

Some authors have established the problem for determining the general Hermitian nonnegative-definite solution to the matrix equation

$$AXA^* = BB^*, \tag{1}$$

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*Author to whom all correspondence should be addressed.

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with known matrices $A, B \in \mathbf{C}^{m\times n}$. For instance, Baksalary [3], Groß [2], Khatri and Mitra [4], and Zhang and Cheng [5] have derived the general Hermitian nonnegative-definite solution to the matrix equation (1), respectively. Moreover, Dai and Lancaster [6] have studied the similar problem and emphasized the importance of (1) within the real setting. More generally, Young *et al.* [1] have obtained a representation of the general common Hermitian nonnegative-definite solution to the pair of matrix equations (1) and

$$CXC^* = DD^*, \tag{2}$$

with known matrices $C, D \in \mathbf{C}^{p\times n}$. But Groß [2] has given a counterexample to point out that their representation is not true. However, up to now, nobody (we knew) has obtained a correct representation for the general common Hermitian nonnegative-definite solution to the pair of matrix equations (1) and (2).

The first aim of this paper is to determine the general common nonnegative-definite solution to the pair of matrix equations (1) and (2). To guarantee the consistencies of the matrix equations (1) and (2), in the following we assume

$$\mathcal{R}(B) \subseteq \mathcal{R}(A), \qquad \mathcal{R}(D) \subseteq \mathcal{R}(C). \tag{3}$$

The problem for determining the general common nonnegative-definite solution to the pair of matrix equations (1) and (2) can be precisely stated as follows.

PROBLEM 1. *Given matrices $A, B \in \mathbf{C}^{m\times n}$ and $C, D \in \mathbf{C}^{p\times n}$ satisfying (3). Determine necessary and sufficient conditions for the existence of a common Hermitian nonnegative-definite solution to the pair of matrix equations (1) and (2). Furthermore, give a representation of the general common Hermitian nonnegative-definite solution to these two equations when they have such common solutions.*

The second aim of this paper is to determine the general common positive-definite solution to the pair of matrix equations (1) and (2), i.e., specify the freedoms in the common Hermitian nonnegative-definite solutions to the pair of matrix equations (1) and (2) such that they are positive-definite. The problem can be precisely stated as follows.

PROBLEM 2. *Given matrices $A, B \in \mathbf{C}^{m\times n}$ and $C, D \in \mathbf{C}^{p\times n}$ satisfying (3). Determine necessary and sufficient conditions for the existence of a common Hermitian positive-definite solution to the pair of matrix equations (1) and (2). Furthermore, give a representation of the general common Hermitian positive-definite solution to these two equations when they have such common solutions.*

The solutions to Problems 1 and 2 are, respectively, established in Sections 2 and 3. In Section 4, an example is presented to illustrate the proposed solutions.

2. SOLUTION TO PROBLEM 1

This section considers solution of Problem 1 proposed in Section 1. We first introduce the following three lemmas.

LEMMA 1. *(See [1, Lemma 2; 7, p. 17].) Given matrices F, $G \in \mathbf{C}^{m\times n}$. Then $FF^* = GG^*$ if and only if $G = FT$ for some $T \in \mathbf{U}_n$.*

LEMMA 2. *(See [8, p. 270].) Given matrices $M \in \mathbf{C}^{m\times p}$ and $N \in \mathbf{C}^{m\times n}$. Let M^- be an arbitrary but fixed generalized inverse of M. Then the matrix equation $MX = N$ has a solution if and only if $MM^-N = N$. When this condition is met, the general solution to the equation is given by*

$$X = M^-N + \left(I - M^-M\right)Y,$$

where Y is free to vary over $\mathbf{C}^{p\times n}$.

LEMMA 3. *(See [2, p. 124; 3, Theorem 1].) Given matrices $A, B \in \mathbf{C}^{m\times n}$ satisfying $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Let A^- be an arbitrary but fixed generalized inverse of A. Then a representation of the general Hermitian nonnegative-definite solution to the matrix equation (1) is given by*

$$X = \left[A^-B + (I_n - A^-A)\,Y\right]\left[A^-B + (I_n - A^-A)\,Y\right]^*, \tag{4}$$

where Y is free to vary over $\mathbf{C}^{n\times n}$.

Based on Lemmas 1–3, the following lemma can be immediately derived.

LEMMA 4. *Suppose $A, B \in \mathbf{C}^{m\times n}$ and $C, D \in \mathbf{C}^{p\times n}$ satisfying (3). Let A^- and $[C(I_n - A^-A)]^-$ be arbitrary but fixed generalized inverses of A and $C(I_n - A^-A)$, respectively. Then the matrix X possessing the form (4) is a solution to the matrix equation (2) if and only if the matrix Y possesses the form*

$$Y = \left[C(I_n - A^-A)\right]^-(DT - CA^-B) + W - \left[C(I_n - A^-A)\right]^- C(I_n - A^-A)W, \tag{5}$$

where W is free to vary over $\mathbf{C}^{n\times n}$, and $T \in \mathbf{U}_n$ satisfies

$$\left[C(I_n - A^-A)\right]\left[C(I_n - A^-A)\right]^-(DT - CA^-B) = DT - CA^-B. \tag{6}$$

PROOF. *The "only if" part.* Suppose the matrix X possessing the form (4) is a solution to the matrix equation (2). Then

$$\left[CA^-B + C(I_n - A^-A)Y\right]\left[CA^-B + C(I_n - A^-A)Y\right]^* = DD^*.$$

Using Lemma 1 derives

$$CA^-B + C(I_n - A^-A)Y = DT$$

for some $T \in \mathbf{U}_n$. This, together with Lemma 2, implies relations (5) and (6).

The "if" part. Suppose the matrix Y possesses the form (5) with (6). It follows from (4) that

$$CXC^* = \left[CA^-B + C(I_n - A^-A)Y\right]\left[CA^-B + C(I_n - A^-A)Y\right]^*. \tag{7}$$

Substituting (5) into (7), we have

$$CXC^* = ZZ^* \tag{8}$$

with

$$Z = CA^-B + C(I_n - A^-A)\left[C(I_n - A^-A)\right]^-(DT - CA^-B). \tag{9}$$

Combining (9) and (6) yields $Z = DT$. This, together with (8), implies that the matrix X possessing the form (4) is a solution to the matrix equation (2).

The proof is completed. ∎

Combining Lemmas 3 and 4, the solution to Problem 1 can be stated as follows.

THEOREM 1. *Suppose the hypotheses of Lemma 4 are satisfied. Then the pair of matrix equations (1) and (2) has a common Hermitian nonnegative-definite solution if and only if there exists $T \in \mathbf{U}_n$ such that (6) holds. When the condition is met, the general common Hermitian nonnegative-definite solution to these two equations is given by (4) with (5), where W is free to vary over $\mathbf{C}^{n\times n}$, and $T \in \mathbf{U}_n$ is an arbitrary parameter matrix satisfying (6).*

3. SOLUTION TO PROBLEM 2

This section considers solution of Problem 2 proposed in Section 1. We first present the following two lemmas.

LEMMA 5. *Given matrices $M \in \mathbf{C}^{m\times n}$ and $N \in \mathbf{C}^{m\times p}$. Let M^- be an arbitrary but fixed generalized inverse of M. Further, assume*

$$MM^-N = N. \tag{10}$$

Then

$$\mathcal{R}\left(M^-NG\right) \cap \mathcal{R}\left((I_n - M^-M)H\right) = \{0\}, \tag{11}$$

for any $G \in \mathbf{C}^{p\times q}$ and $H \in \mathbf{C}^{m\times q}$.

PROOF. If $y \in \mathcal{R}(M^-NG) \cap \mathcal{R}((I_n - M^-M)H)$, then

$$y = M^-NGa \tag{12}$$

and

$$y = (I_n - M^-M)Hb \tag{13}$$

for some vectors a and b. It follows from (13) that $My = 0$. This, together with (12) and (10), gives $y = 0$. Therefore, (11) holds. ∎

LEMMA 6. *Suppose the hypotheses of Lemma 4 are satisfied, and the matrix X possesses the form (4) with (5) and (6). Then $\operatorname{rank} X = n$ if and only if*

$$\operatorname{rank} B = \operatorname{rank} A, \tag{14}$$

$$\operatorname{rank}\left[A^-A \quad Y\right] = n, \tag{15}$$

and

$$\mathcal{R}\left(\left(A^-B\right)^*\right) \cap \mathcal{R}\left(Y^*\left(I_n - A^-A\right)^*\right) = \{0\}. \tag{16}$$

PROOF. First, it follows from (4) that

$$\begin{aligned}\operatorname{rank} X &= \operatorname{rank}\left(A^-B + (I_n - A^-A)Y\right)\\ &\le \operatorname{rank}\left(A^-B\right) + \operatorname{rank}\left(\left(I_n - A^-A\right)Y\right)\\ &= \operatorname{rank} B + \operatorname{rank}\left(\left(I_n - A^-A\right)Y\right)\\ &\le \operatorname{rank} B + \operatorname{rank}\left(I_n - A^-A\right)\\ &= \operatorname{rank} B + n - \operatorname{rank} A.\end{aligned}$$

Therefore, $\operatorname{rank} X = n$ if and only if (14),

$$\operatorname{rank}\left(A^-B + \left(I_n - A^-A\right)Y\right) = \operatorname{rank}\left(A^-B\right) + \operatorname{rank}\left(\left(I_n - A^-A\right)Y\right) \tag{17}$$

and

$$\operatorname{rank}\left(\left(I_n - A^-A\right)Y\right) = \operatorname{rank}\left(I_n - A^-A\right) \tag{18}$$

are satisfied.

Second, applying Theorem 3.4.17 from [9], relation (18) is equivalent to

$$\mathcal{N}\left(Y^*\right) \cap \mathcal{R}\left(\left(I_n - A^-A\right)^*\right) = \{0\},$$

or equivalently,

$$\mathcal{R}(Y) \cap \mathcal{N}\left(I_n - A^-A\right) = \{0\}. \tag{19}$$

Since $\mathcal{N}(I_n - A^-A) = \mathcal{R}(A^-A)$, relation (19) is equivalent to (15).

Third, Lemma 5, [10], and (3), imply that (17) is equivalent to (16).

Combining the above three aspects completes the proof. ∎

Based on Lemmas 5 and 6, the solution to Problem 2 can be stated as follows.

THEOREM 2. *Suppose the hypotheses of Lemma 4 are satisfied, and the matrix X possesses the form (4) with (5) and (6). Then X is a common Hermitian positive-definite solution to the pair of matrix equations (1) and (2) if and only if relations (14)–(16) are satisfied.*

4. THE EXAMPLE

Consider the pair of matrix equations (1) and (2) with the parameter matrices (see [2])

$$A = [1 \quad 1], \qquad C = [0 \quad 1], \qquad B = D = [1 \quad 0].$$

Obviously, $m = p = 1$, $n = 2$, and (3) is satisfied. By choosing

$$A^- = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*,$$

we derive

$$C(I_2 - A^-A) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Again choosing

$$[C(I_2 - A^-A)]^- = [-1 \quad 1]^*,$$

we have that (6) holds for any $T \in \mathbf{U}_2$. We write

$$T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix},$$

where $t_i \in \mathbf{C}$, $i = 1, 2, 3, 4$, satisfy

$$|t_1|^2 + |t_2|^2 = 1, \qquad |t_3|^2 + |t_4|^2 = 1, \qquad t_1\bar{t}_3 + t_2\bar{t}_4 = 0.$$

Using Theorem 1, we derive that the general common nonnegative-definite solution to the pair of matrix equations is given by

$$X = \begin{bmatrix} (1-t_1)(1-\bar{t}_1) + |t_2|^2 & (1-t_1)\bar{t}_1 - |t_2|^2 \\ t_1(1-\bar{t}_1) - |t_2|^2 & 1 \end{bmatrix} = \begin{bmatrix} 2 - 2\,\mathrm{Re}(t_1) & \bar{t}_1 - 1 \\ t_1 - 1 & 1 \end{bmatrix}. \tag{20}$$

Furthermore, using Theorem 2, we derive that the general common positive-definite solution to the pair of matrix equations is given by

$$X = \begin{bmatrix} 2 - 2\,\mathrm{Re}(t_1) & \bar{t}_1 - 1 \\ t_1 - 1 & 1 \end{bmatrix}, \qquad |t_1| < 1.$$

REMARK 1. Let $t_1 = -z$ and $t_2 = u$ in (20). Then (20) turns into [2, equation (2.5)]. However, reference [2] did not give a representation of the general common nonnegative-definite solution to the pair of matrix equations (1) and (2) when the parameter matrices A, B, C, and D are arbitrarily chosen.

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