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Hyperelliptic Jacobians as billiard algebra of pencils of quadrics: Beyond Poncelet porisms

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Abstract

The thirty years old programme of Griffiths and Harris of understanding higher-dimensional analogues of Poncelet-type problems and synthetic approach to higher genera addition theorems has been settled and completed in this paper. Starting with the observation of the billiard nature of some classical constructions and configurations, we construct *the billiard algebra*, that is a group structure on the set T of lines simultaneously tangent to $d-1$ quadrics from a given confocal family in the d-dimensional Euclidean space. Using this tool, the related results of Reid, Donagi and Knörrer are further developed, realized and simplified. We derive a fundamental property of T: any two lines from this set can be obtained from each other by at most $d-1$ billiard reflections at some quadrics from the confocal family. We introduce two hierarchies of notions: s-*skew lines* in T and s-*weak Poncelet trajectories*, $s = -1, 0, \dots, d-2$. The interrelations between billiard dynamics, linear subspaces of intersections of quadrics and hyperelliptic Jacobians developed in this paper enabled us to obtain higher-dimensional and higher-genera generalizations of several classical genus 1 results: Cayley's theorem, Weyr's theorem, Griffiths–Harris theorem and Darboux theorem.

Keywords: Poncelet theorem; Pencils of quadrics; Billiard; Closed billiard trajectories; Cayley's theorem; Weyr's theorem; Griffiths–Harris theorem; Hyperelliptic curve; Hyperelliptic Jacobian

Contents

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1. Introduction

Having in mind the geometric interpretation of the group structure on a cubic (see Fig. 1), the question of finding an analogous construction of the group structure in higher genera arises. In this paper, we show that such a geometric construction exists in the case of hyperelliptic Jacobians. Our ideas are continuation of those of Reid, Donagi and Knörrer, see [11,21,31] and also [10,27]. Further development, realization, simplification and visualization of their constructions is obtained by using the ideas of billiard dynamics on pencils of quadrics developed in [15].

The key projective-geometric feature of that billiard dynamics is the Double Reflection Theorem, see Theorem 7 below. There are four lines belonging to a certain linear space and forming *the Double Reflection Configuration*: these four lines reflect to each other according to **the billiard law at some confocal quadrics**. This billiard configuration appears to be the exact genus two generalization of three points belonging to a line in the elliptic case.

In higher genera, we construct the corresponding more general billiard configuration, again by using Double Reflection Theorem. This configuration, which we call *s-brush*, is in one of the equivalent formulations, a certain billiard trajectory of length $s \leqslant g$ and the sum of s elements in the brush is, roughly speaking, the final segment of that billiard trajectory.

The milestones of this paper are [21] and [15] and the key observation giving a link between them is *that the correspondence* $g \mapsto g'$ *in Lemma* 4.1 *and Corollary* 4.2 *from* [21] *is* **the billiard map at the quadric $\mathcal{Q}_\lambda$**.

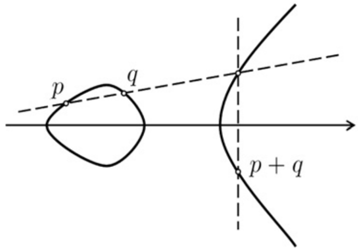

Fig. 1. The group law on the cubic curve.

Thus, after observing and understanding the billiard nature behind the constructions of [11,21,31], we become able to use the billiard tools to construct and study hyperelliptic Jacobians, and particularly their real part. Any real hyperelliptic Jacobian may be realized as a set T of lines simultaneously tangent to given $d-1$ quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_{d-1}$ of some confocal family in the d-dimensional Euclidean space. It is well known that such a set T is invariant under the billiard dynamics determined by quadrics from the confocal family. By using Double Reflection Theorem and some other billiard constructions we construct a group structure on T, *a billiard algebra*. The usage of billiard dynamics in algebro-geometric considerations appears to be, as usually in such a situation, of a two-way benefit. We derive a fundamental property of T: *any two lines in T can be obtained from each other by at most $d-1$ billiard reflections at some quadrics from the confocal family*. The last fact opens a possibility to introduce new hierarchies of notions: of *s-skew lines in T, $s=-1,0,\dots,d-2$* and of *s-weak Poncelet trajectories of length n*. The last are natural quasi-periodic generalizations of Poncelet polygons. By using billiard algebra, we obtain complete analytical description for them. These results are further generalizations of our recent description of Cayley's type of Poncelet polygons in arbitrary dimension, see [15]. Let us emphasize that the method used here, based on billiard algebra differs from the methods exposed in [15].

The interrelations between billiard dynamics, subspaces of intersections of quadrics and hyperelliptic Jacobians developed in this paper, enable us to obtain higher-dimensional generalizations of several classical results. To demonstrate the power of our method, we present here generalizations of Weyr's Poncelet theorem (see [35]) and also Griffiths–Harris Space Poncelet theorem (see [16]) in arbitrary dimension are derived and presented here. We also give an arbitrary-dimensional generalization of the Darboux theorem [9]. Let us mention that one of the plane versions of Darboux theorem has been recently rediscovered, with some improvements, in [32]. In that paper, the obtained configuration is called *Poncelet grid*, while we name it here *Poncelet–Darboux grid*, which is, by our opinion, more historically justified.

The paper is organized as follows. Section 2 consists of preliminaries. It starts with a review of confocal families of quadrics and their most important properties, in the d-dimensional Euclidean and projective space. The Poncelet theorem in the three-dimensional projective space over an arbitrary field is proved, giving accent on projective definitions and methods. We review the One Reflection Theorem and the Double Reflection Theorem (DRT). The latter appears to be one of the main projective tools used in the paper. The notions of virtual reflections and virtual reflection configuration are recalled and some of their basic properties from [15] are reviewed. We recall the definition of the generalized Cayley's curve from [15], and discuss its form in the real case.

Section 3 is the kernel of the paper. In Section 3.1, relying on the simple Lemma 2, we begin to build a bridge between Knörrer's constructions from [21] and the billiard notions from our previous article [15] and this paper. Then, we construct a birational morphism between the generalized Cayley's curve and the Reid–Donagi–Knörrer curve (see [11,21,31]). In Section 3.2, we construct a group structure on the set of lines in $\mathbf{E}^3$ that are simultaneously tangent to two given confocal quadrics. This group structure is naturally connected with the billiard law. Using its properties, some beautiful properties of confocal families are derived, e.g. Theorem 10. In Section 3.3, set T of lines simultaneously tangent to given $d-1$ quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_{d-1}$ of some confocal family in $\mathbf{E}^d$ is in several steps endowed with a group structure called *billiard algebra*. In this construction, we first define the addition operation on the set of all finite billiard trajectories with the same initial line. Then, observing equivalence relations α and β in the set of finite billiard trajectories, we prove that the addition operation is compatible with them, and that the operation on the equivalence classes has additional algebraic properties. At the end, we are ready to use these constructions in order to obtain the billiard algebra on the set of lines.

Section 4 is aimed to illustrate the billiard algebra introduced in Section 3. Section 4.1 starts with introduction of the notions of *s-skew lines* and *s-weak Poncelet trajectories*. We prove the Poncelet type theorem for such trajectories and deduce the corresponding analytic condition of Cayley's type. In Section 4.2, higher-dimensional generalizations of some classical results connected to Poncelet's porism are stated and proved. We introduce *generalized Weyr chains* and show their correspondence with Poncelet polygons (Theorem 15 and Proposition 10). This correspondence is then used to demonstrate the generalization to higher dimensions of the Griffiths–Harris space version of Poncelet theorem (Theorem 16). Section 4.3 starts with recalling and a few comments on Darboux's theorem on grids of conics associated to a Poncelet polygon on an arbitrary Liouville surface. Then, for the case when the surface is the Euclidean plane, we prove in Theorems 17 and 18 a generalization of the Darboux's theorem. The essential generalization to higher dimensions is demonstrated in Theorem 19.

Section 5 contains concluding remarks.

2. Preliminaries

2.1. *Confocal families of quadrics and billiards in Euclidean space*

In this section, we are going to define families of confocal quadrics in the d-dimensional Euclidean space $\mathbf{E}^d$ (see [1,20]) and summarize their basic properties connected with the billiard reflection law.

Definition 1. *A family of confocal quadrics* in the d-dimensional Euclidean space $\mathbf{E}^d$ is a family of the form:

$$\mathcal{Q}_\lambda: \quad \frac{x_1^2}{a_1-\lambda} + \cdots + \frac{x_d^2}{a_d-\lambda} = 1 \quad (\lambda \in \mathbf{R}), \tag{1}$$

where $a_1, \dots, a_d$ are real constants.

Let us notice that a family of confocal quadrics in the Euclidean space is determined by only one quadric.

From now on, we are going to consider the nondegenerate case when $a_1, \dots, a_d$ are all distinct.

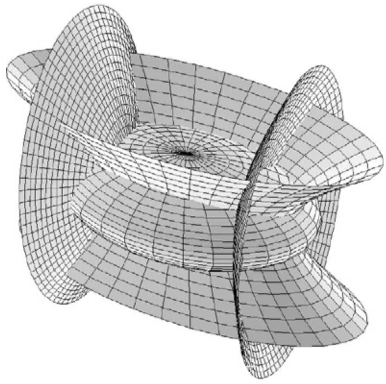

Fig. 2. Three confocal quadrics in $\mathbf{E}^3$.

Theorem 1 *(Jacobi). Any point of the d-dimensional Euclidean space is the intersection of exactly d quadrics of the confocal family* (1). *The quadrics are perpendicular to each other at the intersecting points. (See Fig.* 2*.)*

From here, it follows that to every point $x \in \mathbf{E}^d$ we may associate a d-tuple of distinct parameters $(\lambda_1, \dots, \lambda_d)$, such that x belongs to quadrics $\mathcal{Q}_{\lambda_1}, \dots, \mathcal{Q}_{\lambda_d}$. Additionally, if we order $a_1 < a_2 < \dots < a_d$ and $\lambda_1 < \lambda_2 \dots < \lambda_d$, then:

$$\lambda_1 \leqslant a_1 < \lambda_2 \leqslant a_2 < \dots < \lambda_d \leqslant a_d.$$

Definition 2. *Jacobi elliptic coordinates* of point $x = (x_1, \dots, x_d) \in \mathbf{E}^d$ are values $(\lambda_1, \dots, \lambda_d)$ that satisfy:

$$\frac{x_1^2}{a_1 - \lambda_i} + \dots + \frac{x_d^2}{a_d - \lambda_i} = 1, \quad 1 \leqslant i \leqslant d.$$

Theorem 2 *(Chasles). Any line in $\mathbf{E}^d$ is tangent to exactly $d - 1$ quadrics from a given confocal family. Tangent hyper-planes to these quadrics, constructed at the points of tangency with the line, are orthogonal to each other.*

Definition 3. We say that two lines ℓ_1, ℓ_2 intersecting in point $P \in \mathcal{Q}$ *satisfy the billiard reflection law* on quadric $\mathcal{Q}$ if they are coplanar with the line normal to $\mathcal{Q}$ at P, determine the congruent angles with this line.

A billiard trajectory within $\mathcal{Q}$ is a polygonal line with vertices on $\mathcal{Q}$, such that each pair of its consecutive segments satisfies the billiard reflection law on $\mathcal{Q}$ and lies on the same side of the tangent hyper-plane to $\mathcal{Q}$ at their common endpoint.

The billiard reflection on a quadric is tightly connected with the corresponding confocal family.

Theorem 3. *Two lines that satisfy the reflection law on a quadric $\mathcal{Q}$ in $\mathbf{E}^d$ are tangent to the same $d - 1$ quadrics confocal with $\mathcal{Q}$.*

From the previous theorem, it follows that all segment of a billiard trajectory within $\mathcal{Q}$ are tangent to the same $d-1$ quadrics confocal with $\mathcal{Q}$. We call these $d-1$ quadrics *caustics* of the given trajectory.

Theorem 4 *(Generalized Poncelet theorem). Consider a closed billiard trajectory within quadric $\mathcal{Q}$ in $\mathbf{E}^d$. Then all other billiard trajectories within $\mathcal{Q}$, that share the same $d-1$ caustics, are also closed. Moreover, all these closed trajectories have the same number of vertices.*

This theorem, concerning the plane case ($d=2$), is due to Jean-Victor Poncelet and dates from the beginning of the XIXth century [28]. Its generalization to the three-dimensional case is proved by Darboux [9] at the end of the same century. The generalization to an arbitrary dimension is obtained in [7] at the end of XXth century.

Having the generalized Poncelet theorem in mind, it is natural to ask about an explicit condition on a quadric and a set of confocal caustics, such that the corresponding billiard trajectories are closed. This condition, for $d=2$, was given by Cayley [6]. The condition for an arbitrary d is obtained by the authors of this paper in [12,13].

Theorem 5 *(Generalized Cayley condition). The condition on a billiard trajectory inside ellipsoid $\mathcal{Q}_0$ in $\mathbf{E}^d$, with nondegenerate caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from the family* (1)*, to be periodic with period $n \geqslant d$ is:*

$$\operatorname{rank}\begin{pmatrix} B_{n+1} & B_n & \dots & B_{d+1} \\ B_{n+2} & B_{n+1} & \dots & B_{d+2} \\ \dots & & & \\ \dots & & & \\ B_{2n-1} & B_{2n-2} & \dots & B_{n+d-1} \end{pmatrix} < n-d+1,$$

where $\sqrt{(x-a_1)\dots(x-a_d)(x-\alpha_1)(x-\alpha_{d-1})} = B_0 + B_1x + B_2x^2 + \cdots$ and all $a_1, \dots, a_d$ are distinct and positive.

2.2. Poncelet theorem in projective space over an arbitrary field

In this section, we are going to present the generalization of notions of confocal families of quadrics and related billiard trajectories to the projective space over an arbitrary field of characteristic not equal to 2. Also, we give a proof of the Full Poncelet theorem in the three-dimensional projective space, based on [4,7,8].

We begin with the definition of confocal family of quadrics in the projective space, as it was done in [8].

Definition 4. A family of quadrics in the projective space is *confocal* if for any hyper-plane the set of poles with respect to these quadrics is a line.

It can be shown that the dual to a confocal family is a pencil of quadrics. In contrast to the Euclidean case, notice that it follows from there that a confocal family in the projective space is determined by two quadrics.

On the other hand, if we introduce the metrics in a convenient manner in the real projective space $\mathbf{RP}^d$, the affine part of a given confocal family in $\mathbf{RP}^d$ will satisfy Definition 1. In this sense, Definitions 1 and 4 are equivalent.

As well as in the Euclidean case, any line in $\mathbf{P}^d$ is also tangent to $d-1$ quadrics from a given confocal family. We will refer this statement by *Chasles theorem*, similarly as in the Euclidean case, although the second part of Theorem 2 has no sense without metrics.

Now, we are able to define the billiard reflection in a pure projective manner, without metrics.

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadrics that meet transversely. Denote by u the tangent plane to $\mathcal{Q}_1$ at point x and by z the pole of u with respect to $\mathcal{Q}_2$. Suppose lines ℓ_1 and ℓ_2 intersect at x, and the plane containing these two lines meet u along ℓ.

Definition 5. If lines ℓ_1, ℓ_2, xz, ℓ are coplanar and harmonically conjugated, we say that rays ℓ_1 and ℓ_2 *obey the reflection law* at the point x of the quadric $\mathcal{Q}_1$ with respect to the confocal family which contains $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If we introduce a coordinate system in which quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are confocal in the Euclidean sense, reflection defined in this way is same as the standard one.

Theorem 6 *(One Reflection Theorem). Suppose rays ℓ_1 and ℓ_2 obey the reflection law at x of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let ℓ_1 intersects $\mathcal{Q}_2$ at y_1' and y_1, u is a tangent plane to $\mathcal{Q}_1$ at x, and z its pole with respect to $\mathcal{Q}_2$. Then lines $y_1'z$ and y_1z respectively contain intersecting points y_2' and y_2 of ray ℓ_2 with $\mathcal{Q}_2$. Converse is also true.*

Corollary 1. *Let rays ℓ_1 and ℓ_2 obey the reflection law of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then ℓ_1 is tangent to $\mathcal{Q}_2$ if and only if is tangent ℓ_2 to $\mathcal{Q}_2$; ℓ_1 intersects $\mathcal{Q}_2$ at two points if and only if ℓ_2 intersects $\mathcal{Q}_2$ at two points.*

Next assertion is crucial for proof of the Poncelet theorem.

Theorem 7 *(Double Reflection Theorem). Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ are given quadrics and ℓ_1 line intersecting $\mathcal{Q}_1$ at the point x_1 and $\mathcal{Q}_2$ at y_1. Let u_1, v_1 be tangent planes to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at points x_1, y_1 respectively, and z_1, w_1 their poles with respect to $\mathcal{Q}_2$ and $\mathcal{Q}_1$. Denote by x_2 second intersecting point of the line w_1x_1 with $\mathcal{Q}_1$, by y_2 intersection of y_1z_1 with $\mathcal{Q}_2$ and by ℓ_2, ℓ_1', ℓ_2' lines x_1y_2, y_1x_2, x_2y_2. Then pairs ℓ_1, ℓ_2; ℓ_1, ℓ_1'; ℓ_2, ℓ_2'; ℓ_1', ℓ_2' obey the reflection law at points x_1 (of $\mathcal{Q}_1$), y_1 (of $\mathcal{Q}_2$), y_2 (of $\mathcal{Q}_2$), x_2 (of $\mathcal{Q}_1$) respectively.*

Corollary 2. *If the line ℓ_1 is tangent to a quadric $\mathcal{Q}$ confocal with $\mathcal{Q}_1$ and $\mathcal{Q}_2$, then rays ℓ_2, ℓ_1', ℓ_2' also touch $\mathcal{Q}$.*

Using Double Reflection Theorem, we can prove the Full Poncelet theorem in the three-dimensional space. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_m, \mathcal{Q}$ be confocal quadrics and $\ell_1, \dots, \ell_m$ ($\ell_{m+1} = \ell_1$), lines such that pairs ℓ_i, ℓ_{i+1} obey the reflection law at point x_i of $\mathcal{Q}_i$. Let u_i be a plane tangent to $\mathcal{Q}_i$ at x_i, z_i its pole with respect to $\mathcal{Q}$ and $z_i \notin \mathcal{Q}$. If line ℓ_1 intersects $\mathcal{Q}$ at y_1, y_1', then, by One Reflection Theorem, the second intersection point y_2 of line y_1z_1 with quadrics $\mathcal{Q}$ belongs to ℓ_2. Similarly, having point y_i, we construct the point $y_{i+1} \in \ell_{i+1}$. It follows that $y_{m+1} = y_1$ or $y_{m+1} = y_1'$. Suppose $y_{m+1} = y_1$. It can be proved that, if, for a given polygon $x_1 \dots x_m$ and quadric $\mathcal{Q}$, $y_{m+1} = y_1$ holds, then $y_{m+1} = y_1$ for any surface $\mathcal{Q}$ from the confocal family.

Suppose ℓ_i' are rays ℓ_i reflected of $\mathcal{Q}$ at points y_i. By Double Reflection Theorem, ℓ_i' and ℓ_{i+1}' meet at point $x_i' \in \mathcal{Q}_i$ and obey the reflection law. In this way, we obtained a new polygon

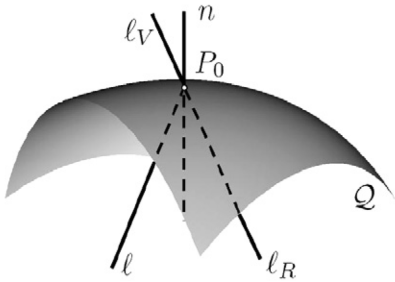

Fig. 3. Real and virtual reflection.

$x'_1 \dots x'_m$. If ℓ_1 touches quadrics $\mathcal{Q}'$, $\mathcal{Q}''$ confocal with $\{\mathcal{Q}_1, \mathcal{Q}\}$, then all sides of both polygons touch them. In this way, a two-parameter family of Poncelet polygons is obtained.

2.3. *Virtual billiard trajectories*

Apart from the real motion of the billiard particle in $\mathbf{E}^d$, it is of interest to consider *virtual reflections*. These reflections were discussed by Darboux in [9] (see Chapter XIV of Book IV in Volume 2). In this section, we review some recent results from [15].

Formally, in the Euclidean space, we can define the *virtual reflection* at the quadric $\mathcal{Q}$ as a map of a ray ℓ with the endpoint P_0 ($P_0 \in \mathcal{Q}$) to the ray complementary to the one obtained from ℓ by the real reflection from $\mathcal{Q}$ at the point P_0. In Fig. 3, ray ℓ_R is obtained from ℓ by the real reflection on the quadric surface $\mathcal{Q}$ at point P_0. Ray ℓ_V is obtained from ℓ by the virtual reflection. Line n is the normal to $\mathcal{Q}$ at P_0.

Let us remark that, in the case of real reflections, exactly one elliptic coordinate, the one corresponding to the quadric $\mathcal{Q}$, has a local extreme value at the point of reflection. On the other hand, on a virtual reflected ray, this coordinate is the only one not having a local extreme value at the point of reflection. In the 2-dimensional case, the virtual reflection can easily be described as the real reflection from the other confocal conic passing through the point P_0. In higher-dimensional cases, the virtual reflection can be regarded as the real reflection from the line normal to $\mathcal{Q}$ at P_0 (see Fig. 3).

The notions of real and virtual reflection cannot be straightforwardly extended to the projective space, since there we essentially use that the field of real numbers is naturally ordered. Nevertheless, it turns out that it is possible to define a certain configuration connected with real and virtual reflection, such that its properties remain in the projective case, too.

Let points $X_1, X_2; Y_1, Y_2$ belong to quadrics $\mathcal{Q}_1$, $\mathcal{Q}_2$ in $\mathbf{P}^d$.

Definition 6. We will say that the quadruple of points X_1, X_2, Y_1, Y_2 constitutes a *virtual reflection configuration (VRC)* if pairs of lines X_1Y_1, X_1Y_2; X_2Y_1, X_2Y_2; X_1Y_1, X_2Y_1; X_1Y_2, X_2Y_2 satisfy the reflection law at points X_1, X_2 of $\mathcal{Q}_1$ and Y_1, Y_2 of $\mathcal{Q}_2$ respectively, with respect to the confocal system determined by $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If, additionally, the tangent planes to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at $X_1, X_2; Y_1, Y_2$ belong to a pencil, we say that these points constitute a *double reflection configuration (DRC)* (see Fig. 4).

Now, the Darboux's statement can be generalized and proved as follows:

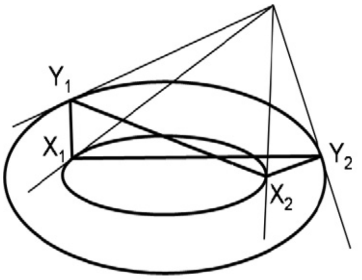

Fig. 4. Double Reflection Configuration.

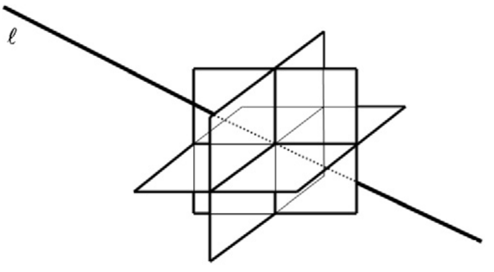

Fig. 5. Three points of the generalized Cayley curve in dimension 3.

Theorem 8. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be two quadrics in the projective space $\mathbf{P}^d$, X_1, X_2 points on $\mathcal{Q}_1$ and Y_1, Y_2 on $\mathcal{Q}_2$. If the tangent hyperplanes at these points to the quadrics belong to a pencil, then X_1, X_2, Y_1, Y_2 constitute a virtual reflection configuration.*

Furthermore, suppose that the projective space is defined over the field of reals. Introduce a coordinate system, such that $\mathcal{Q}_1$, $\mathcal{Q}_2$ become confocal ellipsoids in the Euclidean space. If $\mathcal{Q}_1$ is placed inside $\mathcal{Q}_2$, then the sides of the quadrilateral $X_1Y_1X_2Y_2$ obey the real reflection from $\mathcal{Q}_2$ and the virtual reflection from $\mathcal{Q}_2$.

We are going to conclude this section with the statement converse to the previous theorem.

Proposition 1. *Let pairs of points X_1, X_2 and Y_1, Y_2 belong to confocal ellipsoids $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in Euclidean space $\mathbf{E}^d$, and let α_1, α_2, β_1, β_2 be the corresponding tangent planes. If a quadruple X_1, X_2, Y_1, Y_2 is a virtual reflection configuration, then planes α_1, α_2, β_1, β_2 belong to a pencil.*

2.4. Generalized Cayley curve

Definition 7. Let ℓ be a line not contained in any quadric of the given confocal family in the projective space $\mathbf{P}^d$. The *generalized Cayley curve $\mathcal{C}_\ell$* is the variety of hyperplanes tangent to quadrics of the confocal family at the points of ℓ.

This curve is naturally embedded in the dual space $\mathbf{P}^{d*}$.

In Fig. 5, we see the planes which correspond to one point of the line ℓ in the 3-dimensional space.

Proposition 2. *The generalized Cayley curve is a hyperelliptic curve of genus $g = d - 1$, for $d \geqslant 3$. Its natural realization in $\mathbf{P}^{d*}$ is of degree $2d - 1$.*

The natural involution τ_ℓ on the generalized Cayley's curve $\mathcal{C}_\ell$ maps to each other the tangent planes at the points of intersection of ℓ with any quadric of the confocal family. It is easy to see that the fixed points of this involution are hyperplanes corresponding to the $d - 1$ quadrics that are touching ℓ and to $d + 1$ degenerate quadrics of the confocal family.

Now, let us make a few remarks on the real case.

Lemma 1. *Suppose a line ℓ is tangent to quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from the family* (1)*. Then Jacobian coordinates $(\lambda_1, \dots, \lambda_d)$ of any point on ℓ satisfy the inequalities $\mathcal{P}(\lambda_s) \geqslant 0$, $s = 1, \dots, d$, where*

$$\mathcal{P}(x) = (a_1 - x)\dots(a_d - x)(\alpha_1 - x)\dots(\alpha_{d-1} - x). \tag{2}$$

Proof. Let x be a point of ℓ, $(\lambda_1, \dots, \lambda_d)$ its Jacobian coordinates, and y a vector parallel to ℓ. The equation $Q_\lambda(x + ty) = 1$ is quadratic with respect to t. Its discriminant is:

$$\Phi_\lambda(x, y) = Q_\lambda(x, y)^2 - Q_\lambda(y)\bigl(Q_\lambda(x) - 1\bigr),$$

where

$$Q_\lambda(x, y) = \frac{x_1 y_1}{a_1 - \lambda} + \dots + \frac{x_d y_d}{a_d - \lambda}.$$

By [24],

$$\Phi_\lambda(x, y) = \frac{(\alpha_1 - \lambda)\dots(\alpha_{d-1} - \lambda)}{(a_1 - \lambda)\dots(a_d - \lambda)}.$$

For each of the coordinates $\lambda = \lambda_s$ $(1 \leqslant s \leqslant d)$, the quadratic equation has a solution $t = 0$; thus, the corresponding discriminants are nonnegative. This is obviously equivalent to $\mathcal{P}(\lambda_s) \geqslant 0$. $\square$

The equation of the generalized Cayley curve corresponding to a confocal family of the form (1) can be written as:

$$y^2 = \mathcal{P}(x).$$

It is important to note that the constants $\alpha_1, \dots, \alpha_{d-1}$, corresponding to the quadrics that are touching ℓ cannot take arbitrary values. More precisely, following [2,21], we can state:

Proposition 3. *There exists a line in $\mathbf{E}^d$ that is tangent to $d - 1$ distinct nondegenerate quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from the family* (1) *if and only if the set $\{a_1, \dots, a_d, \alpha_1, \dots, \alpha_{d-1}\}$ can be ordered as $b_1 < b_2 < \dots < b_{2d-1}$, such that $\alpha_j \in \{b_{2j-1}, b_{2j}\}$ $(1 \leqslant j \leqslant d - 1)$.*

It was observed in [15] that the generalized Cayley curve is isomorphic to the Veselov–Moser isospectral curve.

This curve is also naturally isomorphic to the curves studied by Knörrer, Donagi and Reid (RDK curves). We begin next section with the construction of this isomorphism, in order to establish the relationship between billiard law and algebraic structure on the Jacobian $\mathrm{Jac}(\mathcal{C}_\ell)$.

3. Billiard law and algebraic structure on the Abelian variety $\mathcal{A}_\ell$

The aim of this section is to construct in $\mathcal{A}_\ell$ an algebraic structure that is naturally connected with the billiard motion. In Section 3.1, we show that the generalized Cayley's curve is isomorphic to the RDK curve. Then, in order to get better understanding and intuition, we are going first, in Section 3.2 to describe in detail the billiard algebra, and prove some of its nice geometrical properties, for the case of dimension 3, i.e. when the corresponding curve is of genus 2. The general construction is given in Section 3.3.

3.1. Morphism between generalized Cayley's curve and RDK curve

Now we are going to establish the connection between generalized Cayley's curve defined above and the curves studied by Knörrer, Donagi, Reid and to trace out the relationship between billiard constructions and the algebraic structure of the corresponding Abelian varieties.

Suppose a line ℓ in $\mathbf{E}^d$ is tangent to quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from the confocal family (1). Denote by $\mathcal{A}_\ell$ the family of all lines which are tangent to the same $d-1$ quadrics. Note that according to the corollary of the One Reflection Theorem, the set $\mathcal{A}_\ell$ is invariant to the billiard reflection on any of the confocal quadrics.

We begin with the next simple observation.

Lemma 2. *Let the lines ℓ and ℓ' obey the reflection law at the point z of a quadric $\mathcal{Q}$ and suppose they are tangent to a confocal quadric $\mathcal{Q}_1$ at the points z_1 and z_2. Then the intersection of the tangent spaces $T_{z_1}\mathcal{Q}_1 \cap T_{z_2}\mathcal{Q}_1$ is contained in the tangent space $T_z\mathcal{Q}$.*

Proof. It follows from the One Reflection Theorem: since the poles z_1, z_2 and w of the planes $T_{z_1}\mathcal{Q}_1$, $T_{z_2}\mathcal{Q}_1$ and $T_z\mathcal{Q}$ with respect to the quadric $\mathcal{Q}_1$ are colinear, the planes belong to a pencil. $\square$

Following [21], together with $d-1$ affine confocal quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$, one can consider their projective closures $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$ and the intersection V of two quadrics in $\mathbf{P}^{2d-1}$:

$$x_1^2 + \dots + x_d^2 - y_1^2 - \dots - y_{d-1}^2 = 0, \tag{3}$$

$$a_1x_1^2 + \dots + a_dx_d^2 - \alpha_1 y_1^2 - \dots - \alpha_{d-1}y_{d-1}^2 = x_0^2. \tag{4}$$

Denote by $F = F(V)$ the set of all $(d-2)$-dimensional linear subspaces of V. For a given $L \in F$, denote by F_L the closure in F of the set $\{L' \in F \mid \dim L \cap L' = d-3\}$. It was shown in [31] that F_L is a nonsingular hyperelliptic curve of genus $d-1$. Note that for $d = 3$, i.e. when the curve $\mathcal{C}_\ell$ is of the genus 2, an isomorphism between $F(V)$ and the Jacobian of the hyperelliptic curve was established in [26].

The projection

$$\pi' : \mathbf{P}^{2d-1} \setminus \{(x, y) \mid x = 0\} \to \mathbf{P}^d, \quad \pi'(x, y) = x,$$

maps $L \in F(V)$ to a subspace $\pi'(L) \subset \mathbf{P}^d$ of the codimension 2. $\pi'(L)$ is tangent to the quadrics $\mathcal{Q}^{p*}_{\alpha_1}, \dots, \mathcal{Q}^{p*}_{\alpha_{d-1}}$ that are dual to $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$.

Thus, the space dual to $\pi'(L)$, denoted by $\pi^*(L)$, is a line tangent to the quadrics $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$.

We can reinterpret the generalized Cayley's curve $\mathcal{C}_\ell$, which is a family of tangent hyperplanes, as a set of lines from $\mathcal{A}_\ell$ which intersect ℓ. Namely, for almost every tangent hyperplane there is a unique line ℓ', obtained from ℓ by the billiard reflection. Having this identification in mind, it is easy to prove the following

Corollary 3. *There is a birational morphism between the generalized Cayley's curve $\mathcal{C}_\ell$ and Reid–Donagi–Knörrer's curve F_L, with $L = \pi^{*-1}(\ell)$, defined by*

$$j : \ell' \mapsto L', \quad L' = \pi^{*-1}(\ell'),$$

where ℓ' is a line obtained from ℓ by the billiard reflection on a confocal quadric.

Proof. It follows from the previous lemmata and Lemma 4.1 and Corollary 4.2 from [21]. □

Thus, Lemma 2 gives a link between the dynamics of ellipsoidal billiards and algebraic structure of certain Abelian varieties. This link provides a two way interaction: to apply algebraic methods in the study of the billiard motion, but also vice versa, to use billiard constructions in order to get more effective, more constructive and more observable understanding of the algebraic structure.

In the following section, we are going to use this link in constructing *the billiard algebra*, which is a group structure in $\mathcal{A}_\ell$.

3.2. Genus 2 case

Before we proceed in general case, we want to emphasize the billiard constructions involved in the first nontrivial case, of genus two.

3.2.1. Leading principle, definition and first properties of the operation

We formulate *the Leading Principle:*

The sum of the lines in any virtual reflection configuration is equal to zero if the four tangent planes at the points of reflection belong to a pencil.

Recall that, by Definition 6, such a configuration of four lines in a VRC, with tangent planes in a pencil, we are going to call *a Double Reflection Configuration (DRC).*

Neutral element

Let us fix a line $\mathcal{O} \in \mathcal{A}_\ell$.

Opposite element

First, we define $-\mathcal{O} := \mathcal{O}$.

For a given line $x \in \mathcal{C}_{\mathcal{O}}$, define

$$-x := \tau_{\mathcal{O}}(x),$$

where $\tau_{\mathcal{O}}$ is the hyperelliptic involution of the curve $\mathcal{C}_{\mathcal{O}}$.

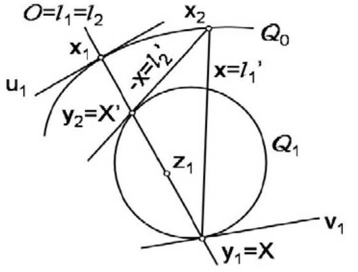

Fig. 6. Proposition 4.

Proposition 4. *For any $x \in \mathcal{C}_{\mathcal{O}}$, both lines x and $-x$ are obtained from $\mathcal{O}$ by the reflection on the same quadric $\mathcal{Q}_x$ from the confocal family.*

Moreover, let π be the unique plane orthogonal to $\mathcal{O}$ from the pencil determined by the tangent planes to $\mathcal{Q}_x$ at the intersection points with $\mathcal{O}$, and $\mathcal{Q}_{\mathcal{O}}$ the unique quadric from the confocal family such that π is tangent to it. Then the intersection point of π with $\mathcal{O}$ belongs to $\mathcal{Q}_{\mathcal{O}}$.

Proof. Follows from the degenerate case of Double Reflection Theorem applied on quadrics $\mathcal{Q}_x$ and $\mathcal{Q}_{\mathcal{O}}$ with $\mathcal{O} = l_1 = l_2$. See Fig. 6. □

Corollary 4. *Plane π and quadric $\mathcal{Q}_{\mathcal{O}}$ do not depend on x. Line $\mathcal{O}$ reflects to itself on $\mathcal{Q}_{\mathcal{O}}$.*

Suppose now that $x \in \mathcal{A}_\ell$, but x does not belong to $\mathcal{C}_{\mathcal{O}}$. Thus x does not intersect $\mathcal{O}$ and the two lines generate a linear projective space. This space intersects $\mathcal{A}_\ell$ along the divisor

$$\mathcal{O} + x + p + q.$$

(See, for example, [18,33].)

According to the Double Reflection Theorem, one can see that the lines $\mathcal{O}$, x, p, q form a double reflection configuration. We define $-x$ such that it forms a double reflection configuration with $\mathcal{O}$, $-p$, $-q$.

From the definition, it is immediately seen that $-(-x) = x$ for every $x \in \mathcal{A}_\ell$.

The following property is a consequence of Proposition 4 and the Double Reflection Theorem.

Proposition 5. *Lines x and $-x$ intersect each other and they satisfy the reflection law on $\mathcal{Q}_{\mathcal{O}}$.*

The following example is an illustration for the construction we have just made.

Example 1. Take the line $\mathcal{O}$ to be orthogonal to one of the coordinate hyperplanes. Then $\mathcal{Q}_{\mathcal{O}} = \pi$ coincide with this hyperplane and, additionally, among caustics $\mathcal{Q}_1, \dots, \mathcal{Q}_{d-1}$ there cannot be two quadrics of the same type.

According to [2], exactly the case of all quadrics of different type corresponds to the case where $\mathcal{A}_\ell$ consists of only one real connected component, isomorphic to the connected component of zero of $\mathrm{Jac}(\mathcal{C}_{\mathcal{O}})(\mathbf{R})$.

In this case, for any line $x \in \mathcal{A}_\ell$, the opposite element $-x$ may be defined as the line symmetric to x with respect to plane $\mathcal{Q}_{\mathcal{O}}$.

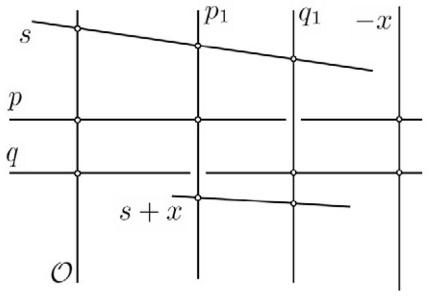

Fig. 7. Partial operation.

Addition

We are going to define operation

$$+ : \mathcal{A}_\ell \times \mathcal{A}_\ell \to \mathcal{A}_\ell.$$

Define $\mathcal{O} + x = x + \mathcal{O} = x$, for all $x \in \mathcal{A}_\ell$.

For $s_1, s_2 \in \mathcal{C}_\mathcal{O}$, define $s_1 + s_2$ as the line that forms a double reflection configuration with $-s_1$, $-s_2$, $\mathcal{O}$. Obviously, $s_1 + s_2 = s_2 + s_1$.

Notice that $-s_1, -s_2$ are unique lines from $\mathcal{A}_\ell$ that intersect both $s_1 + s_2$ and $\mathcal{O}$ (see [33]). Thus, we have:

Lemma 3. *Each line $x \in \mathcal{A}_\ell \setminus \mathcal{C}_\mathcal{O}$ can be in the unique way represented as the sum of two lines that intersect $\mathcal{O}$.*

Now, suppose $s \in \mathcal{C}_\mathcal{O}$, $x \in \mathcal{A}_\ell \setminus \mathcal{C}_\mathcal{O}$ (see Fig. 7). As already explained, take lines $p, q \in \mathcal{C}_\mathcal{O}$ such that $x = p + q$. Construct

$$p_1 = (-s) + (-p), \qquad q_1 = (-s) + (-q)$$

as above, since both pairs s, p and s, q intersect $\mathcal{O}$. Now both p_1 and q_1 intersect s. Thus, the three lines belong to a DRC with the fourth line z. We put by definition

$$s + x = x + s = z.$$

The following lemma gives a very important property of the operation.

Lemma 4. *Let s, x be lines in $\mathcal{C}_\mathcal{O}$, $\mathcal{A}_\ell$ respectively and $\mathcal{Q}_s$ the quadric from the confocal family such that s and $\mathcal{O}$ reflect to each other on it. Then the lines $s + x$ and $-x$ intersect each other, their intersection point belongs to quadric $\mathcal{Q}_s$, and these two lines satisfy the billiard reflection law on $\mathcal{Q}_s$.*

Proof. Follows from [11,21] and Lemma 2. □

Lemma 4 is going to play an important role in proving basic properties of the operation:

Lemma 5. *For all $p, q, s \in \mathcal{C}_\mathcal{O}$, the associative law holds: $p + (q + s) = (p + q) + s$.*

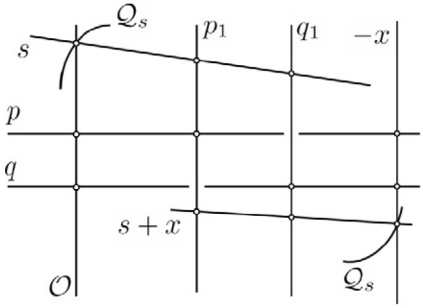

Fig. 8. Lemma 4.

Proof. Denoting, like in Fig. 8, by $-x$, p_1, q_1 lines forming DRCs with triplets $(\mathcal{O}, p, q)$, $(\mathcal{O}, p, s)$, $(\mathcal{O}, q, s)$, and applying Lemma 4, we see that $(p+q)+s$ is the unique line forming a DRC with triplets (p_1, q_1, s), $(p, p_1, -x)$, $(q, q_1, -x)$. The same holds for line $p+(q+s)$, thus the two lines coincide. □

Lemma 6. *Let $p, q, s \in \mathcal{C}_{\mathcal{O}}$. Then $-(p+q+s) = (-p) + (-q) + (-s)$.*

Proof. It is straightforward to prove that $p+q+s \in \mathcal{C}_{\mathcal{O}}$ if and only if two of the lines p, q, s are inverse to each other. In this case, the equality we need to prove immediately follows.

Thus, suppose $p+q+s$ does not intersect $\mathcal{O}$ and that a, b are lines forming a DRC with $p+q+s, \mathcal{O}$. By Lemma 4, we have that $p+q+s$ is the unique line intersecting $(-p)+(-q)$, $(-p)+(-s)$ and $(-q)+(-s)$. Reflecting p, q, s, a, b on $\mathcal{Q}_{\mathcal{O}}$ and applying the Double Reflection Theorem, we get that $-(p+q+s)$ intersects lines $p+q$, $p+s$, $q+s$. Thus, it is equal to $(-p)+(-q)+(-s)$, by Lemma 4. □

Suppose now two lines $x, y \in \mathcal{A}_\ell$ are given, none of which is intersecting $\mathcal{O}$. We define their sum as follows.

First, we represent x as a sum of two lines intersecting $\mathcal{O}$: $x = s_1 + s_2$, $s_1, s_2 \in \mathcal{C}_{\mathcal{O}}$. Then, we define:

$$x + y := s_1 + (s_2 + y).$$

We need to show that this definition is correct.

Lemma 7. *Let $s_1, s_2 \in \mathcal{C}_{\mathcal{O}}$, and $y \in \mathcal{A}_\ell$. Then*

$$s_1 + (s_2 + y) = s_2 + (s_1 + y).$$

Proof. If $y \in \mathcal{C}_{\mathcal{O}}$, it is enough to apply Lemma 5 and the commutativity property for the addition in $\mathcal{C}_{\mathcal{O}}$.

If $y \in \mathcal{A}_\ell \setminus \mathcal{C}_{\mathcal{O}}$, then the lines $-s_2 - y$ and y intersect with billiard reflection, as well as lines $-s_1 - y$ and y do. Thus, there is a unique, fourth line in the intersection of the space generated by $[-s_1 - y, y, -s_2 - y]$ and $\mathcal{A}_\ell$. At one hand, this line is equal to $s_2 - (-s_1 - y)$, at the other it is equal to $s_1 - (-s_2 - y)$. □

Now, from Lemmata 5–7, it follows that we have constructed on $\mathcal{A}_\ell$ a commutative group structure that is naturally connected with the billiard law.

3.2.2. Further properties of the operation

We started this section with the Leading Principle, and used it as a natural motivation for the construction of the algebra. Now, we are going to show that this is justified, i.e. that the statement formulated as the Leading Principle really holds in the group structure.

Theorem 9. *The sum of the lines in any double reflection configuration is equal to zero.*

Proof. Let a, b, c, d be the lines of a DRC. Take that pairs a, b and c, d satisfy the reflection law on $\mathcal{Q}_1$, and pairs b, c, d, a on $\mathcal{Q}_2$. Then, by Lemma 4, we have:

$$b = -a + s_1, \quad c = -b + s_2, \quad d = -c + \bar{s}_1, \quad a = -d + \bar{s}_2, \tag{5}$$

where s_i, $\bar{s}_i$ are obtained from $\mathcal{O}$ by the billiard reflection on $\mathcal{Q}_i$, $i = 1, 2$. Obviously, $\bar{s}_i \in \{s_i, -s_i\}$.

From (5), we get:

$$a + b + c + d = s_1 + \bar{s}_1 = s_2 + \bar{s}_2. \tag{6}$$

Thus, we only need to check the case when $\bar{s}_1 = s_1$ and $\bar{s}_2 = s_2$. Then we have: $s_2 = s_1 + s_1 + (-s_2)$ and, from the definition of the addition operation, it follows that lines $\mathcal{O}, s_1, -2s_1, s_2$ constitute a closed billiard trajectory with consecutive reflections on $\mathcal{Q}_1, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_2$. On the other hand, $(-s_1) + (-s_2)$ is the unique line in $\mathcal{A}_\ell$ that, besides $\mathcal{O}$, intersects both s_1 and s_2. Thus,

$$-2s_1 = (-s_1) + (-s_2) \quad \Rightarrow \quad s_1 = s_2 \quad \Rightarrow \quad \mathcal{Q}_1 = \mathcal{Q}_2.$$

This means that the double reflection configuration a, b, c, d is degenerated, i.e. two of the lines in the configuration coincide. Say $a = c$, then both b and d are obtained from a by the billiard reflection on $\mathcal{Q}_1$. So, $b = -a + s_1$, $d = -a - s_1$ and the theorem follows. □

Let us consider a billiard trajectory $\mathbf{t} = (\ell_0, \ell_1, \dots, \ell_n)$ with the initial line $\ell_0 = \mathcal{O}$ and the reflections on quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ from the confocal family. Applying Double Reflection Theorem to $\mathbf{t}$, we obtain different trajectories with sharing initial segment $\mathcal{O}$ and final segment ℓ_n. The reflections on each such a trajectory are also on the same set of quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_n$, but their order may be changed. Let us denote by $\ell_1^{(1)}, \dots, \ell_n^{(n)}$ all possible lines obtained after the first reflection on all these trajectories. Then, in our algebra, the final segment ℓ_n can be calculated from these ones:

Proposition 6. $\ell_n = (-1)^{n+1}(\ell_1^{(1)} + \dots + \ell_n^{(n)})$.

Proof. Follows from Theorem 9. □

The following interesting property also may be proved from Lemmata 5–7.

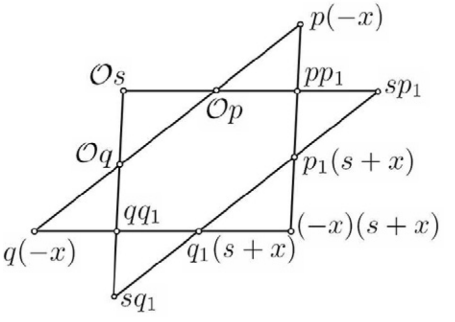

Fig. 9. The configuration of planes.

Proposition 7. *Let line $x \in \mathcal{A}_\ell$ be obtained from $\mathcal{O}$ by consecutive reflections on three quadrics $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$. Then the 6 lines obtained from $\mathcal{O}$ by reflections of the these quadrics may be divided into 2 groups such that:*

- *for each $i \in \{1,2,3\}$, the lines obtained from $\mathcal{O}$ by the reflections on $\mathcal{Q}_i$ are in different groups;*
- *all trajectories with three reflections on $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ starting with $\mathcal{O}$ and ending with x contain lines only from one of the groups.*

We will finish this subsection with a very beautiful and nontrivial theorem on confocal families of quadrics.

Theorem 10. *Let $\mathcal{F}$ be a family of confocal quadrics in $\mathbf{P}^3$. There exist configurations consisting of 12 planes in $\mathbf{P}^3$ with the following properties:*

- *The planes may be organized in 8 triplets, such that each plane in a triplet is tangent to a different quadric from $\mathcal{F}$ and the three touching points are collinear. Every plane in the configuration is a member of two triplets.*
- *The planes may be organized in 6 quadruplets, such that the planes in each quadruplet belong to a pencil and they are tangent to two different quadrics from $\mathcal{F}$. Every plane in the configuration is a member of two quadruplets.*

Moreover, such a configuration is determined by three planes tangent to three different quadrics from $\mathcal{F}$, with collinear touching points.

Proof. Denote by $\mathcal{O}$ the line containing the three touching points and p, q, s the lines obtained from $\mathcal{O}$ by the billiard reflection on the given quadrics from $\mathcal{F}$. We construct lines $p_1, q_1, -x, x+s$ as explained before Lemma 4 (see Fig. 8). The planes of the configuration are tangent to corresponding quadrics at points of the intersection of the lines.

The configuration of the planes in the dual space $\mathbf{P}^{3*}$ is shown in Fig. 9. Here, each plane is denoted by two lines that reflect to each other on. $\square$

It would be interesting to describe the variety of such configurations as a moduli-space.

3.3. General case

The genus two case we have just considered in detail gives us the necessary experience of using the billiard constructions to build a group structure. Moreover, it provides us with the case $n = 2$ in the general construction we are going to present now.

3.3.1. Billiard trajectories and effective divisors

Proposition 6 is going to serve as the motivation for defining the operation in the set $\mathcal{A}_\ell$ for higher-dimensional. Thus, let us now describe in detail the construction that preceded this proposition in Section 3.2.

Suppose that quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ from the confocal family in $\mathbf{E}^d$ are given. Let $\mathcal{O} = \ell_0$, $\ell_1, \dots, \ell_n$ be lines in $\mathcal{A}_\ell$ such that each pair of successive lines ℓ_i, ℓ_{i+1} satisfies the billiard reflection law at the quadric $\mathcal{Q}_{i+1}$ $(0 \leqslant i \leqslant n-1)$; thus the lines form billiard trajectory $\mathbf{t} = (\ell_0, \dots, \ell_n)$.

Let us, for $n \geqslant 1$, define lines $\ell_n^{(n)}, \ell_n^{(n-1)}, \dots, \ell_n^{(1)}$ by the procedure, as follows:

1. Set $\ell_n^{(n)} = \ell_n$.
2. $\ell_n^{(k)}$, for $n-1 \geqslant k \geqslant 1$ is the unique line that constitutes a DRC with ℓ_{k-1}, ℓ_k and $\ell_n^{(k+1)}$.

Let us note that, in this way, each line in the sequence

$$\ell_0,\ \ell_1,\ \dots,\ \ell_k,\ \ell_n^{(k+1)},\ \dots,\ \ell_n^{(n)}$$

is obtained from the previous one by the reflection from

$$\mathcal{Q}_1,\ \dots,\ \mathcal{Q}_k,\ \mathcal{Q}_n,\ \mathcal{Q}_{k+1},\ \dots,\ \mathcal{Q}_{n-1}$$

respectively.

Considering only the initial subsequences $\ell_0, \dots, \ell_k$ $(1 \leqslant k \leqslant n)$, we may in the same way define lines $\ell_k^{(1)}, \dots, \ell_k^{(k)}$. Notice that, for each k, $\ell_k^{(1)}$ intersects ℓ_0, and these two lines obey the reflection law on $\mathcal{Q}_k$.

Thus, we have constructed a mapping $\mathcal{D}$ from the set $\mathcal{TB}(\mathcal{O})$ of billiard trajectories with the fixed initial line $\ell_0 = \mathcal{O}$ to the ordered sets of lines from $\mathcal{A}_\ell$ which intersect $\mathcal{O}$:

$$\mathcal{D} : \mathbf{t} = (\ell_0, \dots, \ell_n) \mapsto \big(\ell_1^{(1)}, \dots, \ell_n^{(1)}\big),$$

where line $\ell_k^{(1)}$ intersects $\mathcal{O}$ according to the billiard law on the quadric $\mathcal{Q}_k$.

Doing the opposite procedure, we define morphism $\mathcal{B}$, the inverse of $\mathcal{D}$, which assigns to an n-tuple of lines intersecting $\mathcal{O}$ the unique billiard trajectory of length n with $\ell_0 = \mathcal{O}$ as the initial line:

$$\mathcal{B} : \big(\ell_1^{(1)}, \dots, \ell_n^{(1)}\big) \ \mapsto\ (\ell_0, \dots, \ell_n).$$

Hence, the mapping $\mathcal{B}$ gives *the billiard representation* of an ordered set of lines intersecting $\mathcal{O}$.

In order to consider just divisors of the curve $\mathcal{C}_\mathcal{O}$, instead of ordered n-tuples of lines, we need to introduce the following relation α between billiard trajectories: we say that two billiard trajectories are *α-equivalent* if one can be obtained from the other by a finite set of *double reflection moves*. The double reflection move transforms a trajectory $p_1 p_2 \dots p_{k-1} p_k p_{k+1} \dots p_n$ into trajectory $p_1 p_2 \dots p_{k-1} p'_k p_{k+1} \dots p_n$ if the lines p_{k-1}, p_k, p'_k, p_{k+1} form a double reflection configuration.

It follows directly from the Double Reflection Theorem that two trajectories $\mathbf{t}_1, \mathbf{t}_2$ with the same initial segment $\mathcal{O} = \ell_0$ are α-equivalent if and only the n-tuples $\mathcal{D}(\mathbf{t}_1)$ and $\mathcal{D}(\mathbf{t}_2)$ may be obtained from each other by a permutation.

Thus, the mapping $\mathcal{D}$ may be considered as a mapping from

$$\widehat{\mathcal{TB}}(\mathcal{O}) = \mathcal{TB}(\mathcal{O})/\alpha$$

to the set of positive divisors on $\mathcal{C}_{\mathcal{O}}$ and it represents *the divisor representation* of billiard trajectories.

The following lemma, which follows from [11], is a sort of Riemann–Roch theorem in this approach.

Lemma 8. *A minimal billiard trajectory of length s from x to y is unique, up the relation α. If there are two non α-equivalent trajectories of the same length $k > s$ from x to y, then there is a trajectory from x to y of length $k-2$.*

Now, we are able to introduce an operation, *summation*, in the set $\widehat{\mathcal{TB}}(\mathcal{O})$. Given two billiard trajectories $\mathbf{t}_1, \mathbf{t}_2 \in \widehat{\mathcal{TB}}(\mathcal{O})$, we define their sum by the equation:

$$\mathbf{t}_1 \oplus \mathbf{t}_2 := \mathcal{B}\big(\mathcal{D}(\mathbf{t}_1) + \mathcal{D}(\mathbf{t}_2)\big).$$

This operation is associative and commutative, according to the Double Reflection Theorem.

Theorem 11. *The set $\widehat{\mathcal{TB}}(\mathcal{O}) = \mathcal{TB}(\mathcal{O})/\alpha$ with the summation operation is a commutative semigroup. This semigroup is isomorphic to the semigroup of all effective divisors on the curve $\mathcal{C}_{\mathcal{O}}$ that do not contain the points corresponding to the caustics.*

Note that in $\widehat{\mathcal{TB}}(\mathcal{O})$, the trajectory consisting only of the single line $\mathcal{O}$ is a neutral element.

Define now the following equivalence relation in the set of all finite billiard trajectories.

Definition 8. Two billiard trajectories are *β-equivalent* if they have common initial and final segments, and they are of the same length.

The set of all classes of β-equivalent billiard trajectories of length n, with the fixed initial segment $\mathcal{O}$, we denote by $\widetilde{\mathcal{TB}}(\mathcal{O})(n)$, and

$$\widetilde{\mathcal{TB}}(\mathcal{O}) = \bigcup_n \widetilde{\mathcal{TB}}(\mathcal{O})(n).$$

Proposition 8. *The relation β is compatible with the addition of billiard trajectories.*

To prove this, we will need the following important lemma:

Lemma 9. *Let $\mathbf{t} = (\ell_1, \ell_2, \dots, \ell_{2k}, \ell_{2k+1} = \ell_1)$ be a closed billiard trajectory, and $p_1 \in \mathcal{A}_\ell$ a line that intersects ℓ_1. Construct iteratively lines $p_2, \dots, p_{2k+1}$ such that quadruples $p_i, \ell_i, p_{i+1}, \ell_{i+1}$ $(1 \leqslant i \leqslant 2k)$ form double reflection configurations. Then $p_{2k+1} = p_1$.*

Proof. We are going to proceed with the induction. For $k = 2$, the lines $\ell_1, \ell_2, \ell_3, \ell_4$ form a double reflection configuration, and the statement follows by the Double Reflection Theorem.

Now, suppose $k > 2$. $(\ell_1, \dots, \ell_k)$ and $(\ell_1 = \ell_{2k+1}, \ell_{2k}, \dots, \ell_k)$ are two billiard trajectories of the same length from ℓ_1 to ℓ_k. If they are α-equivalent, then the claim follows by the Double Reflection Theorem. If they are not α-equivalent, then, by Lemma 8, there is a billiard trajectory $\ell'_1 = \ell_1, \ell'_2, \dots, \ell'_{k-2} = \ell_k$. The statement now follows from the inductive hypothesis applied to trajectories

$$\left(\ell_1, \ell_2, \dots, \ell_k = \ell'_{k-2}, \ell'_{k-3}, \dots, \ell'_1 = \ell_1\right)$$

and

$$\left(\ell'_1, \ell'_2, \dots, \ell'_{k-2} = \ell_k, \ell_{k+1}, \dots, \ell_{2k+1} = \ell_1\right). \qquad \square$$

Proof of Proposition 8. We need to prove the following:

$$\mathbf{t}_1 \sim_\beta \mathbf{t}'_1,\ \mathbf{t}_2 \sim_\beta \mathbf{t}'_2 \quad \Rightarrow \quad \mathbf{t}_1 + \mathbf{t}_2 \sim_\beta \mathbf{t}'_1 + \mathbf{t}'_2.$$

Clearly, it is enough to prove this relation for the case when $\mathbf{t}_2 = \mathbf{t}'_2$ and the length of $\mathbf{t}_2$ is equal to 2. Suppose that $\mathbf{t}_2 = (\mathcal{O}, p)$, where p is obtained from $\mathcal{O}$ by the reflection on quadric $\mathcal{Q}_p$. Then trajectories $\mathbf{t}_1 + \mathbf{t}_2$, $\mathbf{t}_1 + \mathbf{t}'_2$ are obtained by adding one segment to $\mathbf{t}_1$, $\mathbf{t}'_1$ respectively. These segments satisfy the reflection law on $\mathcal{Q}_p$ with the final segments of $\mathbf{t}_1$, $\mathbf{t}'_1$. Since $\mathbf{t}_1 \sim_\beta \mathbf{t}'_1$, their final segments coincide and the statement follows from Lemma 9 applied to the trajectory

$$\left(\ell_1, \ell_2, \dots, \ell_n = \ell'_n, \ell'_{n-1}, \ell'_{n-2}, \dots, \ell'_1 = \ell_1\right),$$

where $\mathbf{t}_1 = (\ell_1, \dots, \ell_n)$, $\mathbf{t}_2 = (\ell'_1, \dots, \ell'_n)$. $\square$

3.3.2. The group structure in $\mathcal{A}_\ell$

We wish to use the constructed algebra on the set of billiard trajectories, in order to obtain an algebraic structure on $\mathcal{A}_\ell$, such that the operation is naturally connected with the billiard reflection law.

From [33] we get

Theorem 12. *For any two given lines x and y from $\mathcal{A}_\ell$, there is a system of at most $d-1$ quadrics from the confocal family, such that the line y is obtained from x by consecutive reflections on these quadrics.*

The divisor representation of the corresponding billiard trajectory of length $s \leqslant d-1$ will be called *the s-brush of y related to x*.

Now, we can define a group structure in $\mathcal{A}_\ell$ associated with a fixed line in this set.

Neutral element

Let us fix a line $\mathcal{O} \in \mathcal{A}_\ell$.

Inverse element

Let x be an arbitrary line in $\mathcal{A}_\ell$, and $\mathcal{D}(x)$ the divisor representation of the minimal billiard trajectory connecting $\mathcal{O}$ with x. Define $-x$ as the final segment of the billiard trajectory $\mathcal{B}(\tau\mathcal{D}(x))$, where τ is the hyperelliptic involution of $\mathcal{C}_{\mathcal{O}}$.

Addition

For two lines x and y from $\mathcal{A}_\ell$, denote their brushes related to $\mathcal{O}$ as S_1 and S_2. Define

$$x + y := (-1)^{|S_1|+|S_2|+1}\mathcal{E}\mathcal{B}\bigl(\tau^{|S_1|+1}(S_1), \tau^{|S_2|+1}(S_2)\bigr),$$

where $\mathcal{E}\mathcal{B}(\tau^{|S_1|+1}(S_1), \tau^{|S_2|+1}(S_2))$ is the final segment of the billiard trajectory $\mathcal{B}(\tau^{|S_1|+1}(S_1), \tau^{|S_2|+1}(S_2))$.

From all above, similarly as in the genus 2 case, we get

Theorem 13. *The set $\mathcal{A}_\ell$ with the operation defined above is an Abelian group.*

Let us observe that another kind of divisor representation of billiard trajectories may be introduced, that associates a divisor of degree 0 to a given trajectory. Such a representation was used in [9] and explicitly described in [14,15]. The positive part of this representation coincide with the divisor obtained by $\mathcal{B}$, and the negative part is invariant for the hyperelliptic involution. Moreover, as it follows from [9], two trajectories with the initial segment $\mathcal{O}$ will have the same final segments if and only if their representations are equivalent divisors. Thus, it follows that the group structure on $\mathcal{A}_\ell$ is isomorphic to the quotient of the Jacobian of the curve $\mathcal{C}_{\mathcal{O}}$ by a finite subgroup which is generated by the points corresponding to the caustic quadrics.

4. Billiard algebra, theorems of Poncelet type and their generalizations

4.1. Billiard algebra, weak Poncelet trajectories and theorems of Poncelet–Cayley's type

Now, as a consequence of Theorem 12 we are able to introduce the following hierarchies of notions.

Definition 9. For two given lines x and y from $\mathcal{A}_\ell$ we say that they are s-*skew* if s is the smallest number such that there exist a system of $s + 1 \leqslant d - 1$ quadrics $\mathcal{Q}_k$, $k = 1, \dots, s + 1$ from the confocal family, such that the line y is obtained from x by consecutive reflections on $\mathcal{Q}_k$. If the lines x and y intersect, they are 0-*skew*. They are (-1)-*skew* if they coincide.

Definition 10. Suppose that a system S of n quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ from the confocal family is given. For a system of lines $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_n$ in $\mathcal{A}_\ell$ such that each pair of successive lines $\mathcal{O}_i$, $\mathcal{O}_{i+1}$ satisfies the billiard reflection law at $\mathcal{Q}_{i+1}$ $(0 \leqslant i \leqslant n - 1)$, we say that it forms an *s-weak Poncelet trajectory of length n associated to the system S* if the lines $\mathcal{O}_0$ and $\mathcal{O}_n$ are s-skew.

For s-weak Poncelet trajectories we will also sometimes say $(d - s - 2)$-*resonant billiard trajectories*. Periodic trajectories or generalized classical Poncelet polygons are (-1)-weak Poncelet trajectories or, in other words, $(d - 1)$-resonant billiard trajectories, and they are described analytically in [14,15].

Our next goal is to get complete analytical description of s-weak Poncelet trajectories of length r, generalizing in such a way the results from [15]. Here, we are going to use fully the tools and the power of billiard algebra.

To fix the idea, let us consider first the system S consisting on r equal quadrics $\mathcal{Q}_1 = \cdots = \mathcal{Q}_r$ from the confocal family. Suppose a system of lines $\mathcal{O}_0$, $\mathcal{O}_1, \dots, \mathcal{O}_r$ in $\mathcal{A}_\ell$ forms an s-weak Poncelet trajectory of length r associated to the system S. Then

$$\mathcal{O}_r = r\mathcal{O}_1^{(1)},$$

with some line $\mathcal{O}_1^{(1)}$ which intersects $\mathcal{O}_0$. Again, from the condition that $\mathcal{O}_r$ and $\mathcal{O}_0$ are s-skew we get

$$\mathcal{O}_r = \mathcal{O}_1^{'(1)} + \cdots + \mathcal{O}_{s+1}^{'(1)}$$

with some lines $P_i = \mathcal{O}_i^{'(1)}$ which intersect $\mathcal{O}_0$. From the last two equations we come to the conclusion

Proposition 9. *The existence of an s-weak Poncelet trajectory of length r is equivalent to existence of a meromorphic function f on the hyperelliptic curve $\mathcal{C}_\ell$ such that f has a zero of order r at $P = \mathcal{O}_1^{(1)}$, a unique pole at "infinity" E, and the order of the pole is equal to $n = r + s + 1$.*

Now, we are going to derive explicit analytical condition of Cayley's type. As in [15] we consider the space $\mathcal{L}(nE)$ of all meromorphic functions on $\mathcal{C}_\ell$ with the unique pole at the infinity point E of the order not exceeding n. Let $(f_1, \dots, f_k)$ be one of the bases of this space, $k = \dim \mathcal{L}(nE)$. Consider the vectors

$$v_1, \dots, v_r \in \mathbf{C}^k,$$

where $v_i^j = f_j^{(i-1)}(P)$ and vectors

$$u_1, \dots, u_{s+1} \in \mathbf{C}^k,$$

with $u_i^j = f_j(P_i)$. From the condition (see [15])

$$\operatorname{rank}[v_1, \dots, v_r, u_1, \dots, u_{s+1}] < n - g + 1$$

we get the condition

$$\operatorname{rank}[v_1, \dots, v_r] < r + s - g + 2 = r + s - d + 3.$$

Now, we can rewrite it in the form usual for the Cayley's type conditions.

Theorem 14. *The existence of an s-weak Poncelet trajectory of length r is equivalent to:*

$$\operatorname{rank}\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{m+1} \\ B_{d+2} & B_{d+3} & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{d+m-s-2} & B_{d+m-s-1} & \dots & B_{r-1} \end{pmatrix} < m - d + 1,$$

when $r+s+1=2m$, and

$$\operatorname{rank}\begin{pmatrix} B_d & B_{d+1} & \dots & B_{m+1} \\ B_{d+1} & B_{d+2} & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{d+m-s-2} & B_{d+m-s-1} & \dots & B_{r-1} \end{pmatrix} < m-d+2,$$

when $r+s+1=2m+1$.

With $B_0, B_1, B_2, \dots,$ we denoted the coefficients in the Taylor expansion of function $y=\sqrt{\mathcal{P}(x)}$ in a neighborhood of P, where $y^2=\mathcal{P}(x)$ is the equation of the generalized Cayley curve, with the polynomial $\mathcal{P}(x)$ given by (2).

Proof. Denote by $\mathcal{L}((r+s+1)E)$ the linear space of all meromorphic functions on $\mathcal{C}_\ell$, having a unique pole at the infinity point E, with the order not exceeding $r+s+1$. By the Riemann–Roch theorem:

(i) $\dim\mathcal{L}((r+s+1)E)=[\frac{r+s+1}{2}]+1$ if $r+s+1\leqslant 2d-1$,
(ii) $\dim\mathcal{L}((r+s+1)E)=r+s-g+1$ if $r+s+1$ is even and greater than $2d-2$,
(iii) $\dim\mathcal{L}((r+s+1)E)=r+s-g+2$ if $r+s+1$ is even and greater than $2d-2$.

We may choose the following bases in each of the three cases:

(i) $1, x, \dots, x^m$, where $m=[\frac{r+s+1}{2}]\leqslant d$;
(ii) $1, x, \dots, x^m, y, xy, \dots, x^{m-d}$, for $r+s+1=2m\geqslant 2d-2$;
(iii) $1, x, \dots, x^m, y, xy, \dots, x^{m-d+1}$, for $r+s+1=2m+1\geqslant 2d-2$.

Now, the statement follows from the considerations preceding the theorem. $\square$

Example 2. For $s=-1$, the inequalities in Theorem 14 become the conditions for periodic billiard trajectories (see [14,15]).

Example 3. The condition for existence of a $(d-3)$-weak Poncelet trajectory of length r is equivalent to:

$$\det\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{m+1} \\ B_{d+2} & B_{d+3} & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{m+1} & B_{m+2} & \dots & B_{r-1} \end{pmatrix}=0, \quad r+d-2=2m;$$

$$\det\begin{pmatrix} B_d & B_{d+1} & \dots & B_{m+1} \\ B_{d+1} & B_{d+2} & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{m+1} & B_{m+2} & \dots & B_{r-1} \end{pmatrix}=0, \quad r+d-2=2m+1.$$

Example 4. If $r+s<2d-1$, then, as it follows from the proof of Theorem 14, an s-weak Poncelet trajectory of length r may exist only if the hyper-elliptic curve $\mathcal{C}_\ell$ is singular.

4.2. Remark on generalized Weyr's theorem and Griffiths–Harris space Poncelet theorem in higher dimensions

In this section we obtain higher-dimensional generalizations of the results of [16,19,35]. Nice exposition of those classical results one can find in [3]. The dual version of [16] is given in [7].

Each quadric $\mathcal{Q}$ in $\mathbf{P}^{2d-1}$ contains at most two unirational families of $(d-1)$-dimensional linear subspaces. Such unirational families are usually called *rulings of the quadric*.

Theorem 15. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be two general quadrics in $\mathbf{P}^{2d-1}$ with the smooth intersection V and $\mathcal{R}_1$, $\mathcal{R}_2$ their rulings. If there exists a closed chain*

$$L_1, L_2, \dots, L_{2n}, L_{2n+1} = L_1$$

of distinct $(d-1)$-dimensional linear subspaces, such that $L_{2i-1} \in \mathcal{R}_1$, $L_{2i} \in \mathcal{R}_2$ $(1 \leqslant i \leqslant n)$ and $L_j \cap L_{j+1} \in F(V)$ $(1 \leqslant j \leqslant 2n)$, then there are such closed chains of subspaces of length $2n$ through any point of $F(V)$.

Proof. Each of the unirational families $\mathcal{R}_i$ determines an involution τ_i on Abelian variety $F(V)$. Such an involution interchanges two $(d-2)$-intersections of an element of $\mathcal{R}_i$ with V. Denote by $\mathrm{tr} : F(V) \to F(V)$ their composition and by $L := L_{2n} \cap L_1 \in F(V)$. Since tr is a translation on $F(V)$ satisfying $\mathrm{tr}^n(L) = L$ we see that tr is of order n and the theorem follows. $\square$

Definition 11. We will call the chains considered in Theorem 15 *generalized Weyr's chains*.

The theorem can be adjusted for nonsmooth intersections, but we are not going into details. Instead, we consider the case of two quadrics (3) and (4) as in Section 3.1. By using the projection π^* we get:

Proposition 10. *A generalized Weyr chain of length $2n$ projects into a Poncelet polygon of length $2n$ circumscribing the quadrics $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$ and alternately inscribed into two fixed confocal quadrics (projections of $\mathcal{Q}_1, \mathcal{Q}_2$). Conversely, any such a Poncelet polygon of the length $2n$ circumscribing the quadrics $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$ and alternately inscribed into two fixed confocal quadrics can be lifted to a generalized Weyr chain of length $2n$.*

Proof. It follows from Lemma 4.1 and Corollary 4.2 of [21] and Lemma 2. $\square$

Thus, we obtained in a correspondence between generalized Weyr chains and Poncelet polygons subscribed in $d-1$ given quadrics and alternately inscribed in two quadrics from some confocal family. Such Poncelet polygons have been completely analytically described, among others, in [15] (see Example 4 from there).

Let us note that correspondence between classical Weyr's theorem in $\mathbf{P}^3$ and classical Poncelet theorem about two conics in a plane was observed by Hurwitz in [19]. Nevertheless the projection we used here is not a straightforward generalization of the one used by Hurwitz.

Polygonal lines, circumscribed about one conic and alternately inscribed in two conics, appear as an example in [34]. Our result, applied to the lowest dimension, gives the condition for the closeness of such a polygonal line, see Corollary 1 from [15].

Let us also mention that if rulings $\mathcal{R}_1$ and $\mathcal{R}_2$ are connected by a generalized Weyr's chain of length $2n$, then the same is true for $\mathcal{R}_2$, $\mathcal{R}_1$ and also for the pair $\mathcal{R}'_1$, $\mathcal{R}'_2$ of the complementary rulings of $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Now we are able to present a new higher-dimensional generalization of the Griffiths–Harris Space Poncelet theorem from [16].

Theorem 16. *Let $\mathcal{Q}_1^*$ and $\mathcal{Q}_2^*$ be the duals of two general quadrics in $\mathbf{P}^{2d-1}$ with the smooth intersection V. Denote by $\mathcal{R}_i$, $\mathcal{R}'_i$ pairs of unirational families of $(d-1)$-dimensional subspaces of $\mathcal{Q}_i^*$. Suppose there are generalized Weyr's chains between $\mathcal{R}_1$ and $\mathcal{R}_2$ and between $\mathcal{R}_1$ and $\mathcal{R}'_2$. Then there is a finite polyhedron inscribed and subscribed in both quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. There are infinitely many such polyhedra.*

The polyhedra from the previous theorem can be described in more details as arrangements of d-dimensional "faces" which are bitangents of the quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Their intersections are $(d-1)$-dimensional "edges" which alternately belong to the rulings of $\mathcal{Q}_1^*$ and $\mathcal{Q}_2^*$. The intersections of "edges" are $(d-2)$-dimensional "vertices" which belong to $F(V)$.

The proof of the last theorem is based on the following lemma from [11].

Lemma 10. *Let $\mathcal{Q} \subset \mathbf{P}^{2d-1}$ be a quadric of the rank not less than $2d-1$ and $x \subset \mathcal{Q}$ a linear subspace of the dimension $d-2$. If $\mathcal{Q}$ is singular, suppose additionally that x does not contain the vertex. Then, for each ruling $\mathcal{R}$ of $\mathcal{Q}$, there is a unique $(d-1)$-dimensional linear subspace $s = s(\mathcal{R}, x)$ such that $s \in \mathcal{R}$ and $x \subset s$.*

4.3. Poncelet–Darboux grid and higher-dimensional generalizations

4.3.1. Historical Remark: on Darboux's Heritage

In [9] (Volume 3, Book VI, Chapter I), Darboux proved that Liouville surfaces are exactly these having an orthogonal system of curves that can be regarded in two or, equivalently, infinitely many different ways, as geodesic conics. These coordinate curves are analogues of systems of confocal conics in the Euclidean plane. Knowing this, essential properties of conics may be generalized to all Liouville surfaces (see [9] and also [15] for a review and clarifications). Here is the citation of one of these properties:

> *"Considerons un polygone variable dont tous les côtés sont des lignes géodésiques tangentes a une même courbe coordonnée. Le dernier sommet de ce polygone décrira aussi une des courbes coordonnées et il en sera de même de points d'intersection de deux côtés quelconques de ce polygone."*

This statement is not only the generalization of Poncelet theorem to Liouville surfaces. Additionally, for a Poncelet polygon circumscribed about a fixed coordinate curve, with each vertex moving along one of the coordinate curves, Darboux stated here that the intersection point of two arbitrary sides of the polygon is also going to describe a coordinate curve.

Let us note that a weaker version of this claim, although with some improvements, has been recently rediscovered in [32], and a more elementary proof of the result from [32] has been published in [23]. The main theorem of [32] corresponds only to a fixed Poncelet polygon associated to a pair of ellipses in the Euclidean plane. In a sense, this gives a new argument in favor of our

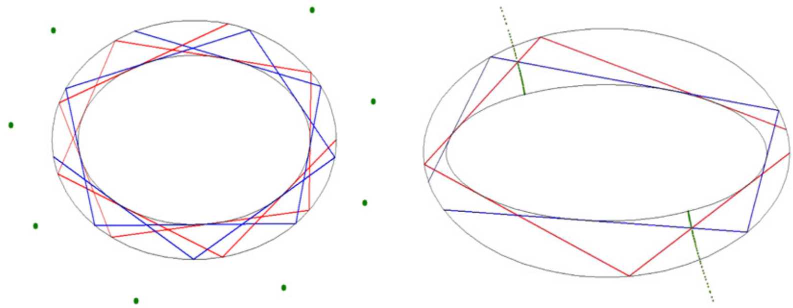

Fig. 10. Billiard trajectories with an ellipse as a caustic and the intersection points of the corresponding segments.

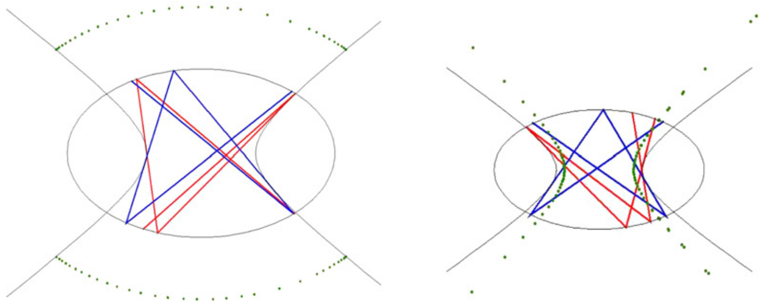

Fig. 11. Billiard trajectories with a hyperbola as a caustic and the intersection points of the corresponding segments.

observation in [14] and [15], that the work of Darboux connected with billiards and Poncelet theorem seems to be unknown to nowadays mathematicians.

This section is devoted to a multi-dimensional generalization of the Darboux theorem, related to billiard trajectories within an ellipsoid in the d-dimensional Euclidean space.

4.3.2. Poncelet–Darboux grid in Euclidean plane

Before starting with higher-dimensional generalizations, let us give some improvements together with a simpler proof of the statement corresponding to the plane case.

Theorem 17. *Let $\mathcal{E}$ be an ellipse in $\mathbf{E}^2$ and $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories $\mathcal{E}$, sharing the same caustic. Then all the points $a_m \cap b_m$ $(m \in \mathbf{Z})$ belong to one conic $\mathcal{K}$, confocal with $\mathcal{E}$.*

Moreover, under the additional assumption that the caustic is an ellipse, we have: if both trajectories are winding in the same direction about the caustic, then $\mathcal{K}$ is also an ellipse; if the trajectories are winding in opposite directions, then $\mathcal{K}$ is a hyperbola. (See Fig. 10.*)*

For a hyperbola as a caustic, it holds: if segments a_m, b_m intersect the long axis of $\mathcal{E}$ in the same direction, then $\mathcal{K}$ is a hyperbola, otherwise it is an ellipse. (See Fig. 11.*)*

Proof. The statement follows by application of the Double Reflection Theorem. Namely, the lines a_0 and b_0 intersect at a point that belongs to one ellipse and one hyperbola from the confocal family. They satisfy the reflection law on exactly one of these two curves, depending on the orientation of the billiard motion along the lines. Now, by the Double Reflection Theorem, a_1

and b_1 satisfy the reflection law on the same conic. The same is true, for any pair a_m, b_m, by the induction.

For the second part of the theorem, it is sufficient to observe that the winding direction about an ellipse is changed by the reflections on the hyperbolae, and preserved by the reflections on the ellipses from the confocal family. If an oriented line is placed between the foci, then the direction it is intersecting the axis containing the foci is changed by the reflections of the ellipses and preserved by the reflections on the hyperbolae. □

Proposition 11. *Let $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories within the ellipse $\mathcal{E}$, sharing the same caustic. If the caustic is an ellipse and the trajectories are winding in the opposite directions about it, then all the points $a_m \cap b_m$ $(m \in \mathbf{Z})$ are placed on two centrally symmetric half-branches of a hyperbola confocal with $\mathcal{E}$.*

Proof. Denote by $\mathcal{E}_c$ the ellipse which is the common caustic of the given billiard trajectories. It is possible to introduce a metric μ on $\mathcal{E}_c$, such that $\mu(AB) = \mu(CD)$ if and only if the tangent lines at A, B and at C, D intersect on the same ellipse of the confocal family. In particular, this means that the points where two consecutive billiard segments touch the caustic $\mathcal{E}'$ are always at the fixed distance from each other.

Take that a_0 and b_0 intersect on the upper left half-branch of the hyperbola $\mathcal{K}$ (see Fig. 10). Then it may be proved from the Double Reflection Theorem, that each touching points of these two segments with caustic $\mathcal{E}'$ are at equal μ-distances from the upper left intersection point of $\mathcal{K}$ and $\mathcal{E}_c$. Since μ is symmetric with respect to the coordinate center, they are also equally distanced from the lower right intersection point, but not from the other two intersection points. The same holds for any pair a_m, b_m. Thus, their intersections lies on two centrally symmetric half-branches of the hyperbola. □

Now, the generalized claim about Poncelet–Darboux grids is immediately following from Theorem 17.

Theorem 18. *Let $(\ell_m)_{m\in\mathbf{Z}}$ be the sequence of segments of a billiard trajectory within the ellipse $\mathcal{E}$. Then each of the sets $\mathrm{P}_k = \bigcup_{i-j=k} \ell_i \cap \ell_j$, $\mathrm{Q}_k = \bigcup_{i+j=k} \ell_i \cap \ell_j$ $(k \in \mathbf{Z})$, belongs to a single conic confocal to $\mathcal{E}$.*

If the caustic of the trajectory (ℓ_m) is an ellipse, then the sets P_k are placed on ellipses and Q_k on hyperbolae. If the caustic is a hyperbola, then the sets P_k, Q_k are placed on ellipses for k even and on hyperbolae for k odd.

Proof. To show the statement for P_k, take $a_m = \ell_m$, $b_m = \ell_{m+k}$ and apply Theorem 17. For Q_k, take $a_m = \ell_m$, $b_m = \ell_{k-m}$. □

Remark 1. Notice that Theorem 18 is more general then the one given in [23,32], since we do not suppose that the billiard trajectory is closed. Also, the statement can be formulated for an arbitrary conic, not only for an ellipse.

The main statement proved in [32] and [23] is a special case consequence of Theorem 18:

Corollary 5. *(See [23,32].) Let (ℓ_m) be a closed billiard trajectory within an ellipse, with the elliptical caustic. Each set P_k lie on an ellipse confocal to $\mathcal{E}$, and Q_k on a confocal hyperbola. (See Fig. 10.)*

Let us show one interesting property of Poncelet–Darboux grids.

Proposition 12. *Let (ℓ_m) be a billiard trajectory within ellipse $\mathcal{E}$, with the elliptical caustic $\mathcal{E}_c$. Then the ellipse containing the set P_k depends only on k, $\mathcal{E}$ and $\mathcal{E}_c$. In other words, this ellipse will remain the same for any choice of billiard trajectory within $\mathcal{E}$ with the caustic $\mathcal{E}_c$.*

Proof. This claim may be proved by the use of the Double Reflection Theorem, similarly as in Theorem 17. Nevertheless, we are going to show it in another way, that gives a possibility of explicit calculation of the ellipse containing P_k.

To the billiard within $\mathcal{E}$ with the fixed caustic $\mathcal{E}_c$, we may associate an elliptic curve (see [17,22,24,25]). Each point of the curve corresponds to a conic from the confocal family. On the other hand, the billiard motion may be viewed as the linear motion on the Jacobian of the curve (i.e. on the curve itself, since this is the elliptic case), with translation jumps corresponding to reflections. More precisely, the translation is exactly by this value on the elliptic curve, which is associated to the ellipse that a segment is reflected on.

Thus, if the point M on the elliptic curve is associated to the ellipse $\mathcal{E}$, then the set P_k is placed on the ellipse corresponding to kM. □

4.3.3. Grids in arbitrary dimension

Although Theorem 17 gives certain progress in understanding of Poncelet–Darboux grids, the essential breakthrough in this matter represents, in our opinion, the study of the higher-dimensional situation. This analysis is based on introduction of higher-dimensional analogues of grids and our notion of s-skew lines.

Theorem 19. *Let $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories within the ellipsoid $\mathcal{E}$ in $\mathbf{E}^d$, sharing the same $d-1$ caustics. Suppose the pair (a_0, b_0) is s-skew, and that by the sequence of reflections on quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$ the minimal billiard trajectory connecting a_0 to b_0 is realized.*

Then, each pair (a_m, b_m) is s-skew, and the minimal billiard trajectory connecting these two lines is determined by the sequence of reflections on the same quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$.

Proof. This may be proved by the use of the Double Reflection Theorem, similarly as in Theorem 17. □

This theorem also can be stated for an arbitrary quadric, not only for ellipsoids.

Proposition 13. *Let $\mathcal{E}$ be an ellipsoid in $\mathbf{E}^d$, $\mathcal{Q}_1, \dots, \mathcal{Q}_{d-1}$ quadrics confocal to $\mathcal{E}$, and k an integer.*

Suppose that there exists a trajectory (ℓ_m) of the billiard within $\mathcal{E}$, having the caustics $\mathcal{Q}_1, \dots, \mathcal{Q}_{d-1}$, such that the pair (ℓ_0, ℓ_k) is s-skew and that the minimal billiard trajectory connecting ℓ_0 to ℓ_k is realized by the sequence of reflections on quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$ confocal to $\mathcal{E}$.

Then, for any billiard trajectory $(\hat{\ell}_m)$ within $\mathcal{E}$ with the caustics $\mathcal{Q}_1, \dots, \mathcal{Q}_{d-1}$, the pairs of lines $(\hat{\ell}_m, \hat{\ell}_{m+k})$ are s-skew and the minimal billiard trajectory between $\hat{\ell}_m$ and $\hat{\ell}_{m+k}$ is realized by the sequence of reflections on $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$.

5. Conclusion

"One of the most important and also most beautiful theorems in classical geometry is that of Poncelet (...) His proof was synthetic and somewhat elaborate in what was to become the predominant style in projective geometry of last century. Slightly thereafter, Jacobi gave another argument based on the additional theorem for elliptic functions. In fact, as will be seen below, the Poncelet theorem and additional theorem are essentially equivalent, so that at least in principle Poncelet gave a synthetic derivation of the group law on an elliptic curve. Because of the appeal of the Poncelet theorem it seems reasonable to look for higher-dimensional analogues... Although this has not yet turned out to be the case in the Poncelet-type problems..."

These introductory words from [16] written by Griffiths and Harris exactly 30 years ago, could serve as a motto for the present paper, announcing the programme realized here (see [29,30,34] and also [5]).

Usually, the Poncelet theorem is associated with the points of finite order on an elliptic curve. However, our analysis from Section 3 shows that a Poncelet polygon should be understood as a pair of equivalent divisors and through a billiard analogue of the Abel theorem it can further be associated to a meromorphic function of a hyperelliptic curve. On the other hand, the billiard analogue of the Riemann–Roch theorem gives uniqueness of the billiard trajectory with minimal number of lines connecting two given lines in $\mathcal{A}_\ell$. Thus, the Poncelet story in higher dimensions is at least as rich as a story of meromorphic functions on hyperelliptic curves.

Such a deep relationship between hyperelliptic Jacobians and pencils of quadrics through integrable billiard systems opens a new view to the well known, but still amazing role of elliptic coordinates in the theory of integrable systems and in the separation of variables in Hamilton–Jacobi method. The question of synthetic approach to the addition law on nonhyperelliptic Jacobians remains open and could lead to new methods of separation of variables.

Table of notations

- $\mathcal{A}_\ell$ the family of all lines in $\mathbf{E}^d$ which are tangent to given $d-1$ confocal quadrics (see Section 3.1),
- $\mathcal{C}_\ell, \mathcal{C}_\mathcal{O}$ the generalized Cayley curve (see Definition 7 in Section 2.4),
- $\mathbf{E}^d$ the d-dimensional Euclidean space,
- $F(V)$ the set of all $(d-2)$-dimensional linear subspaces of the intersection V of two quadrics in $\mathbf{P}^{2d-1}$ (see Lemma 2 in Section 3.1),
- $\mathbf{P}^d$ the d-dimensional projective space,
- $\mathbf{P}^{d*}$ the dual d-dimensional projective space,
- $\mathcal{Q}$ a quadric,
- $\mathcal{Q}^*$ the quadric dual to $\mathcal{Q}$,
- $\mathcal{Q}_\lambda$ a quadric from the family (1) of confocal quadrics in $\mathbf{E}^d$ (see Definition 1 from Section 2.1).

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References

[1] V. Arnol'd, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1978.
[2] M. Audin, Courbes algébriques et systémes intégrables : géodesiques des quariques, Expo. Math. 12 (1994) 193–226.
[3] W. Barth, Th. Bauer, Poncelet theorems, Expo. Math. 14 (1996) 125–144.
[4] M. Berger, Geometry, Springer-Verlag, Berlin, 1987.
[5] V.M. Buchhstaber, V.Z. Enolskii, D.V. Leykin, Hyperelliptic Kleinian functions and applications, in: I.M. Krichever, S.P. Novikov (Eds.), Rev. Math. Math. Phys., vol. 10, Part 2, Gordon and Breach, 1997, pp. 1–125.
[6] A. Cayley, Developments on the porism of the in-and-circumscribed polygon, Philosophical Magazine 7 (1854) 339–345.
[7] S.-J. Chang, B. Crespi, K.-J. Shi, Elliptical billiard systems and the full Poncelet's theorem in n dimensions, J. Math. Phys. 34 (6) (1993) 2242–2256.
[8] S.-J. Chang, K.J. Shi, Billiard systems on quadric surfaces and the Poncelet theorem, J. Math. Phys. 30 (4) (1989) 798–804.
[9] G. Darboux, Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitesimal, vols. 2 and 3, in : Gauthier-Villars, Paris, 1887, 1889.
[10] U. Desale, S. Ramanan, Classification of vector bundles of rank 2 over hyperelliptic curves, Invent. Math. 38 (1977) 161–185.
[11] R. Donagi, Group law on the intersection of two quadrics, Ann. Sc. Norm. Super Pisa (1980) 217–239.
[12] V. Dragović, M. Radnović, Conditions of Cayley's type for ellipsoidal billiard, J. Math. Phys. 39 (1) (1998) 355–362.
[13] V. Dragović, M. Radnović, On periodical trajectories of the billiard systems within an ellipsoid in R^d and generalized Cayley's condition, J. Math. Phys. 39 (11) (1998) 5866–5869.
[14] V. Dragović, M. Radnović, Cayley-type conditions for billiards within k quadrics in $\mathbf{R}^d$, J. Phys. A 37 (2004) 1269–1276;
V. Dragović, M. Radnović, Corrigendum: Cayley-type conditions for billiards within k quadrics in $\mathbf{R}^d$, J. Phys. A 38 (2005) 7927, math-ph/0503053.
[15] V. Dragović, M. Radnović, Geometry of integrable billiards and pencils of quadrics, J. Math. Pures Appl. 85 (2006) 758–790, math-ph/0512049.
[16] P. Griffiths, J. Harris, A Poncelet theorem in space, Comment. Math. Helv. 52 (2) (1977) 145–160.
[17] P. Griffiths, J. Harris, On Cayley's explicit solution to Poncelet's porism, Enseign. Math. 24 (1–2) (1978) 31–40.
[18] P. Griffiths, J. Harris, Principles of Algebraic Geometry, 1978.
[19] A. Hurwitz, Über unendlich-vieldeutige geometrische Aufgaben, insbesondere aber die Schliesungsprobleme, Math. Ann. 15 (1879) 8–15.
[20] C. Jacobi, Vorlesungen über Dynamic, Gesammelte Werke, Supplementband, Berlin, 1884.
[21] H. Knörrer, Geodesics on the ellipsoid, Invent. Math. 59 (1980) 119–143.
[22] H. Lebesgue, Les coniques, Gauthier–Villars, Paris, 1942.
[23] M. Levi, S. Tabachnikov, The Poncelet grid and the billiard in an ellipse, Amer. Math. Monthly 114 (10) (2007) 895–908.
[24] J. Moser, Geometry of quadrics and spectral theory, in: The Chern Symposium, Springer-Verlag, New York/Berlin, 1980, pp. 147–188.
[25] J. Moser, A.P. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomials, Comm. Math. Phys. 139 (2) (1991) 217–243.
[26] M. Narasimhan, Ramanan, Moduli of vector bundles on compact Riemann surfaces, Ann. of Math. (2) 89 (1969) 14–51.

[27] P.E. Newstead, Stable bundles of rank 2 and odd degree over a curve of genus 2, Topology 7 (1968) 205–215.
[28] J.V. Poncelet, Traité des propriétés projectives des figures, Mett, Paris, 1822.
[29] E. Previato, The Poncelet's theorem and generalizations, Proc. Amer. Math. Soc. 127 (1999) 2547–2556.
[30] E. Previato, Some integrable billiards, in: S. Abenda, G. Gaeta, S. Walcher (Eds.), SPT2002: Symmetry and Perturbation Theory, World Scientific, Singapore, 2002, pp. 181–195.
[31] M. Reid, Cambridge thesis, June 1972, unpublished.
[32] R. Schwartz, The Poncelet grid, Adv. Geom. 7 (2) (2007) 157–175.
[33] A.N. Tyurin, On intersection of quadrics, Russian Math. Surveys 30 (1975) 51–105.
[34] A.P. Veselov, Integrable discrete-time systems and difference operators, Funct. Anal. Appl. 22 (1988) 83–93.
[35] E. Weyr, Über einige Statze von Steiner und ihren Zusammenhang mit der zwei und zweigliedrigen Verwandtschaft der Grundebilde ersten Grades, Crelles J. Reine Angew. Math. 71 (1870) 18–28.