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Han's bijection via permutation codes

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ABSTRACT

We show that Han's bijection when restricted to permutations can be carried out in terms of the cyclic major code and the cyclic inversion code. In other words, it maps a permutation π with cyclic major code $(s_1, s_2, \ldots, s_n)$ to a permutation σ with cyclic inversion code $(s_1, s_2, \ldots, s_n)$. We also show that the fixed points of Han's map can be characterized by the strong fixed points of Foata's second fundamental transformation. The notion of strong fixed points is related to partial Foata maps introduced by Björner and Wachs.

1. Introduction

In his combinatorial proof of the fact that the Z-statistic introduced by Zeilberger and Bressoud [16] is Mahonian, Han [8] constructed a Foata-style bijection on words which maps the major index onto the Z-statistic. Since the Z-statistic and the inversion number coincide when restricted to permutations, Han's bijection maps the major index to the inversion number for permutations. Let H denote Han's bijection when restricted to permutations. Throughout this paper, by Han's bijection we always mean the map H. We shall show that the map H can be carried out by the cyclic major code and the cyclic inversion code.

The cyclic major code of a permutation can be described in terms of cyclic intervals, a notion also introduced by Han [9] in his study of the joint distribution of the excedance number and Denert's statistic. It should be noted that the cyclic inversion code in the context of this paper is the classical Lehmer code, but the cyclic major code is different from the well-studied major code as introduced by Rawlings [14]; see also [3,5,10,15].

Using the code representation, we show that the fixed points of Han's map can be characterized by the strong fixed points of Foata's second fundamental transformation. The notion of strong fixed points is related to partial Foata maps introduced by Björner and Wachs [1].

Let us give an overview of the background and definitions. Let $X = \{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}$ be a multiset with m_i i's and $m_1 + m_2 + \cdots + m_k = n$. The set of rearrangements of X is denoted by

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$R(X)$. When $m_1 = m_2 = \cdots = m_k = 1$, $R(X)$ reduces to the set S_n of permutations on $[n]$. For a word $w = w_1w_2\cdots w_n \in R(X)$, the descent set $\mathrm{Des}(w)$, the descent number $\mathrm{des}(w)$, the major index $\mathrm{maj}(w)$, the inversion number $\mathrm{inv}(w)$ and the Z-statistic $\mathrm{Z}(w)$ are defined by

$$\mathrm{Des}(w) = \{i|1 \le i \le n-1, w_i > w_{i+1}\},$$
$$\mathrm{des}(w) = \#\mathrm{Des}(w),$$
$$\mathrm{maj}(w) = \sum_{i\in \mathrm{Des}(w)} i,$$
$$\mathrm{inv}(w) = \#\{(i,j)|1 \le i < j \le n, w_i > w_j\},$$
$$\mathrm{Z}(w) = \sum_{i<j} \mathrm{maj}(w_{ij}),$$

where w_{ij} is a word obtained from w by deleting all elements except i and j and $\#S$ stands for the cardinality of a set S. For example, let $w = 211324314 \in R(1^3, 2^2, 3^2, 4^2)$. We have $\mathrm{Des}(w) = \{1, 4, 6, 7\}$, $\mathrm{des}(w) = 4$, $\mathrm{maj}(w) = 18$, $\mathrm{inv}(w) = 9$, and $\mathrm{Z}(w)$ can be computed as follows

$$\mathrm{maj}(21121) + \mathrm{maj}(11331) + \mathrm{maj}(11414) + \mathrm{maj}(2323) + \mathrm{maj}(2244) + \mathrm{maj}(3434) = 16.$$

A statistic is said to be Mahonian on $R(X)$ if it has the same distribution as the major index on $R(X)$. MacMahon [12,13] introduced the major index and proved that the major index is equidistributed with the inversion number for $R(X)$. Foata [4] found a combinatorial proof of this classical fact by constructing a bijection Φ, called the second fundamental transformation, which maps the major index to the inversion number, namely,

$$\mathrm{maj}(w) = \mathrm{inv}(\Phi(w)) \quad \text{for any } w \in R(X).$$

For completeness, we give a brief description of Foata's bijection [4]; see also [7,11]. Let $w = w_1w_2\cdots w_n$ be a word on a multiset X as defined above, and let x be an element in X. If $w_n \le x$, the x-factorization of w is defined as $w = v_1b_1\cdots v_pb_p$, where each b_i is less than or equal to x, and every element in v_i is greater than x. Note that v_i is allowed to be empty. Similarly, when $w_n > x$, the x-factorization of w is defined as $w = v_1b_1\cdots v_pb_p$, where each b_i is greater than x, and every element in v_i is less than or equal to x. In either case, set

$$\gamma_x(w) = b_1v_1\cdots b_pv_p, \qquad w' = w_1w_2\cdots w_{n-1}.$$

Then the second fundamental transformation Φ can be defined recursively by setting $\Phi(a) = a$ for each $a \in X$ and setting

$$\Phi(w) = \gamma_{w_n}(\Phi(w')) \cdot w_n$$

if w contains more than one element.

As an extension of the theorem of MacMohan, Björner and Wachs [1] considered the problem of finding subsets U of S_n for which the major index and inversion number are equidistributed. They introduced the kth partial Foata bijection $\phi_k: S_n \longrightarrow S_n$ for $1 \le k \le n$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. Define $\phi_1(\sigma) = \sigma$ and for $k > 1$ define

$$\phi_k(\sigma) = \gamma_{\sigma_k}(\sigma_1\sigma_2\cdots\sigma_{k-1}) \cdot \sigma_k\sigma_{k+1}\cdots\sigma_n.$$

It is easily seen that

$$\Phi = \phi_n \circ \phi_{n-1} \cdots \circ \phi_1.$$

A subset U of S_n is said to be a strong Foata class if

$$\phi_k(U) = U$$

for $1 \le k \le n$. A permutation σ is said to be a strong fixed point of Foata's map if

$$\phi_k(\sigma) = \sigma$$

for $1 \le k \le n$. As will be seen, the strong fixed points of Foata's map are closely related to the fixed points of Han's map.

The paper is organized as follows. In Section 2, we recall the construction of Han's map, and give a description of the cyclic major code and the cyclic inversion code. Then we give a reformulation of Han's map in terms of these two codes. In Section 3, we give a characterization of the fixed points of Han's map H. It turns out that a permutation is fixed by H if and only if it is a strong fixed points of Foata's map Φ.

2. Han's bijection via permutation codes

In this section, we are concerned with a reformulation of Han's bijection for permutations in terms of the cyclic major code and the cyclic inversion code. For completeness, let us give an overview of the map H.

Let $x \in [n]$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_{n-1}$ be a permutation on $\{1, 2, \ldots, x-1, x+1, \ldots, n\}$. Define $C^x(\sigma)$ as $\tau_1\tau_2\cdots\tau_{n-1}$, where $\tau_i = \sigma_i - x (\mathrm{mod}\ n)$, i.e.,

$$\tau_i = \begin{cases} \sigma_i - x + n, & \text{if } \sigma_i < x; \\ \sigma_i - x, & \text{if } \sigma_i > x, \end{cases}$$

and define $C_x(w)$ as the standardization of σ, i.e., $C_x(w) = \nu_1\nu_2\cdots\nu_{n-1} \in S_{n-1}$ with

$$\nu_i = \begin{cases} \sigma_i, & \text{if } \sigma_i < x; \\ \sigma_i - 1, & \text{if } \sigma_i > x. \end{cases}$$

Evidently, both C^x and C_x are bijections between permutations on $\{1, 2, \ldots, x-1, x+1, \ldots, n\}$ and S_{n-1}. So $(C^x)^{-1}$ and $(C_x)^{-1}$ are well defined. Han's bijection H can be defined by $H(1) = 1$ and

$$H(\sigma) = C_{\sigma_n}^{-1}(H(C^{\sigma_n}(\sigma'))) \cdot \sigma_n,$$

where $\sigma \in S_n$ with $n > 1$ and $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$.

We proceed to give the definition of cyclic intervals. Let $X = \{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}$ be a multiset. For $x, y \in X$, the cyclic interval $\rrbracket x, y\rrbracket$ is defined by Han [9] as

$$\rrbracket x, y\rrbracket = \begin{cases} \{z | z \in [k], x < z \le y\}, & \text{if } x \le y; \\ \{z | z \in [k], z > x \text{ or } z \le y\}, & \text{otherwise}. \end{cases}$$

Set $\rrbracket x, \infty\rrbracket = \{z | z \in [k], z > x\}$.

For any word $w = w_1w_2\cdots w_n$ on X and $1 \le i \le n$, define

$$t_i(w) = \#\{j | 1 \le j \le i-1, w_j \in \rrbracket w_i, \infty\rrbracket\},$$

and

$$s_i(w) = \#\{j | 1 \le j \le i-1, w_j \in \rrbracket w_i, w_{i+1}\rrbracket\},$$

where $w_{n+1} = \infty$.

For example, let $w = 312432143$. Then

$$(t_1(w), t_2(w), \ldots, t_9(w)) = (0, 1, 1, 0, 1, 3, 5, 0, 2),$$

and

$$(s_1(w), s_2(w), \ldots, s_9(w)) = (0, 0, 1, 3, 3, 4, 5, 6, 2).$$

The notion of cyclic intervals plays an important role in the proof of the fact that the bi-statistic (exc, Den) is equidistributed with (des, maj) on $R(X)$, where exc is the excedance number and Den is Denert's statistic; see [2,6,9].

We now give the definition of the cyclic major code also in terms of cyclic intervals. Meanwhile, the traditional inversion code can be described in this way. Let

$$E_n = \{(a_1, a_2, \ldots, a_n) \in Z^n | 0 \le a_i \le i-1, i = 1, 2, \ldots, n\}.$$

Keep in mind that the above definitions of $t_i(\sigma)$ and $s_i(\sigma)$ apply to permutations. It is well known that the map $I: S_n \longrightarrow E_n$ defined by

$$\sigma \longmapsto (t_1(\sigma), t_2(\sigma), \ldots, t_n(\sigma))$$

is a bijection known as the Lehmer code, which is often referred to as the inversion code. Note that

$$\sum_{i=1}^{n} t_i(\sigma) = \mathrm{inv}(\sigma).$$

On the other hand, it is easy to see that the map $M: S_n \longrightarrow E_n$ defined by

$$\sigma \longmapsto (s_1(\sigma), s_2(\sigma), \ldots, s_n(\sigma))$$

is also a bijection. We call $M(\sigma)$ the cyclic major code of σ. To recover σ from its cyclic major code $(s_1, s_2, \ldots, s_n)$, first let $\sigma_n = n - s_n$. Suppose that $\sigma_{k+1}, \ldots, \sigma_n$ have been determined by $s_{k+1}, \ldots, s_n$. Then delete the elements in the sequence

$$\sigma_{k+1}, \sigma_{k+1} - 1, \ldots, 1, n, (n-1), \ldots, (\sigma_{k+1}) + 1$$

that are equal to σ_j for some $j \geq k+1$ and set σ_k to be the $(s_k + 1)$th element in the resulting sequence. It has been shown by Han [9] that

$$\sum_{i=1}^{n} s_i(\sigma) = \mathrm{maj}(\sigma).$$

For example, $I(38516427) = (0, 0, 1, 3, 1, 3, 5, 1)$ and $M(38516427) = (0, 1, 1, 2, 3, 4, 4, 1)$.

The relation between these two codes is described below.

Proposition 2.1. *Let $\sigma \in S_n$. Suppose that $I(\sigma) = (t_1, t_2, \ldots, t_n)$ and $M(\sigma) = (s_1, s_2, \ldots, s_n)$. Then we have $s_n = t_n$, and for $1 \leq i < n$, $s_i = t_i - t_{i+1} (\mathrm{mod}\ i)$, that is,*

$$s_i = \begin{cases} t_i - t_{i+1}, & \text{if } t_i \geq t_{i+1}; \\ t_i - t_{i+1} + i, & \text{if } t_i < t_{i+1}. \end{cases}$$

Proof. It is clear that $s_n = t_n$. For $1 \leq i \leq n-1$, by the definition of $t_i(\sigma)$, we see that $t_i \geq t_{i+1}$ if and only if $\sigma_i < \sigma_{i+1}$. In this case,

$$\begin{aligned} s_i &= \#\{j \mid 1 \leq j \leq i-1, \sigma_i < \sigma_j < \sigma_{i+1}\} \\ &= \#\{j \mid 1 \leq j \leq i-1, \sigma_i < \sigma_j\} - \#\{j \mid 1 \leq j \leq i-1, \sigma_j > \sigma_{i+1}\} \\ &= t_i - t_{i+1}. \end{aligned}$$

If $t_i < t_{i+1}$, then $\sigma_i > \sigma_{i+1}$. Hence

$$\begin{aligned} s_i &= \#\{j \mid 1 \leq j \leq i-1, \sigma_i < \sigma_j \text{ or } \sigma_j < \sigma_{i+1}\} \\ &= \#\{j \mid 1 \leq j \leq i-1, \sigma_i < \sigma_j\} + \#\{j \mid 1 \leq j \leq i-1, \sigma_j < \sigma_{i+1}\} \\ &= t_i + \#\{j \mid 1 \leq j \leq i-1, \sigma_j < \sigma_{i+1}\} \\ &= t_i + i - \#\{j \mid 1 \leq j \leq i, \sigma_j > \sigma_{i+1}\} \\ &= t_i - t_{i+1} + i. \end{aligned}$$

This completes the proof. □

The following theorem states that Han's bijection H can be carried out in terms of the cyclic major code and the inversion code.

Theorem 2.2. *For each $n \geq 1$, we have*

$$H = I^{-1} \circ M.$$

In other words, H is a bijection on S_n with the property that

$$M(\sigma) = I(H(\sigma)).$$

Proof. We use induction on n. For $n = 1$, the theorem is obvious. Assume that $n > 1$. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, and let $M(\sigma) = (s_1(\sigma), s_2(\sigma), \ldots, s_n(\sigma))$. By definition, $s_n(\sigma) = \#\{\sigma_n + 1, \sigma_n + 2, \ldots, n\} = n - \sigma_n$. By the construction of H, we have

$$H(\sigma) = C_{\sigma_n}^{-1}(H(C^{\sigma_n}(\sigma'))) \cdot \sigma_n,$$

which implies $t_n(H(\sigma)) = n - \sigma_n$. Since the standardization of a permutation preserves the relative order, we find that

$$I(C_{\sigma_n}(\sigma_1\sigma_2\cdots\sigma_{n-1})) = (t_1(\sigma), t_2(\sigma), \ldots, t_{n-1}(\sigma)).$$

By induction, it suffices to show that

$$M(C^{\sigma_n}(\sigma_1\sigma_2\cdots\sigma_{n-1})) = (s_1(\sigma), s_2(\sigma), \ldots, s_{n-1}(\sigma)). \tag{2.1}$$

Suppose that $C^{\sigma_n}(\sigma_1\sigma_2\cdots\sigma_{n-1}) = \tau_1\tau_2\cdots\tau_{n-1}$. For the sake of presentation, let $\tau_n = \infty$. For $1 \le i \le n-1$ and $1 \le k \le i-1$, we claim that $\sigma_k \in]\!]\sigma_i, \sigma_{i+1}]\!]$ if and only if $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$. If it is true, then (2.1) follows immediately. This claim can be verified as follows.

(1) If $i \neq n-1$, there are two cases each of which has three subcases, namely,

(1a) $\sigma_n < \sigma_i < \sigma_{i+1}$;

(1b) $\sigma_i < \sigma_n < \sigma_{i+1}$;

(1c) $\sigma_i < \sigma_{i+1} < \sigma_n$;

and

(2a) $\sigma_n > \sigma_i > \sigma_{i+1}$;

(2b) $\sigma_i > \sigma_n > \sigma_{i+1}$;

(2c) $\sigma_i > \sigma_{i+1} > \sigma_n$.

We only give the proof of case (1b), the other cases can be justified by the same argument. Let us assume that $\sigma_i < \sigma_n < \sigma_{i+1}$. By definition, $\tau_i = n + \sigma_i - \sigma_n$, $\tau_{i+1} = \sigma_{i+1} - \sigma_n$, so we have $\tau_{i+1} < \tau_i$. Suppose that $\sigma_k \in]\!]\sigma_i, \sigma_{i+1}]\!]$. Then we deduce that $\sigma_i < \sigma_k < \sigma_{i+1}$ and

$$\tau_k = \begin{cases} \sigma_k - \sigma_n + n, & \text{if } \sigma_k < \sigma_n < \sigma_{i+1}; \\ \sigma_k - \sigma_n, & \text{if } \sigma_i < \sigma_n < \sigma_k. \end{cases}$$

If $\sigma_k < \sigma_n < \sigma_{i+1}$, then $\tau_k = \sigma_k - \sigma_n + n > \sigma_i - \sigma_n + n = \tau_i$, it follows that $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$; if $\sigma_i < \sigma_n < \sigma_k$, then $\tau_k = \sigma_k - \sigma_n < \sigma_{i+1} - \sigma_n = \tau_{i+1}$, which implies $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$. Conversely, if $\tau_k \in]\!]\tau_i, \tau_{i+1}]\!]$, then we deduce that $\tau_k > \tau_i$ or $\tau_k < \tau_{i+1}$. Assume that $\sigma_k \notin]\!]\sigma_i, \sigma_{i+1}]\!]$, then we have $\sigma_k < \sigma_i$ or $\sigma_k > \sigma_{i+1}$. Consequently,

$$\tau_k = \begin{cases} \sigma_k - \sigma_n + n, & \text{if } \sigma_k < \sigma_i < \sigma_n; \\ \sigma_k - \sigma_n, & \text{if } \sigma_k > \sigma_{i+1} > \sigma_n. \end{cases}$$

If $\sigma_k < \sigma_i$, then $\tau_k = \sigma_k - \sigma_n + n < \sigma_i - \sigma_n + n = \tau_i$. However, $\tau_k = \sigma_k + n - \sigma_n > \sigma_{i+1} - \sigma_n = \tau_{i+1}$, which is a contradiction. If $\sigma_k > \sigma_{i+1}$, then $\tau_k = \sigma_k - \sigma_n > \sigma_{i+1} - \sigma_n = \tau_{i+1}$, but now $\tau_k = \sigma_k - \sigma_n < \sigma_i - \sigma_n + n = \tau_i$, a contradiction too. So we reach the conclusion that $\sigma_k \in]\!]\sigma_i, \sigma_{i+1}]\!]$.

(2) If $i = n-1$, there are two cases, namely $\sigma_{n-1} > \sigma_n$ and $\sigma_{n-1} < \sigma_n$. For the first case, by definition we have $\tau_{n-1} = \sigma_{n-1} - \sigma_n$. It follows that

$$\begin{aligned} \sigma_k \in]\!]\sigma_{n-1}, \sigma_n]\!] &\Rightarrow \sigma_k > \sigma_{n-1} \quad \text{or} \quad \sigma_k < \sigma_n \\ &\Rightarrow \tau_k = \begin{cases} \sigma_k - \sigma_n, & \text{if } \sigma_k > \sigma_{n-1}; \\ \sigma_k - \sigma_n + n, & \text{if } \sigma_k < \sigma_n. \end{cases} \\ &\Rightarrow \tau_k > \tau_{n-1} \\ &\Rightarrow \tau_k \in]\!]\tau_{n-1}, \infty]\!]. \end{aligned}$$

Conversely, assume that $\tau_k \in]\!]\tau_{n-1}, \infty]\!]$, i.e., $\tau_k > \tau_{n-1} = \sigma_{n-1} - \sigma_n$. Suppose that $\sigma_k \notin]\!]\sigma_{n-1}, \sigma_n]\!]$, namely, $\sigma_n < \sigma_k < \sigma_{n-1}$. Then we have

$$\tau_k = \sigma_k - \sigma_n < \sigma_{n-1} - \sigma_n = \tau_{n-1},$$

a contradiction. This yields $\sigma_k \in]\!]\sigma_{n-1}, \sigma_n]\!]$. Similarly, one can verify the assertion for the case $\sigma_{n-1} < \sigma_n$. This completes the proof. $\square$

Table 2.1
The procedures to compute $H(\sigma)$ and $M(\sigma)$.

$\sigma = 392648517$	$I(H(\sigma)) = (0, 0, 1, 3, 1, 4, 3, 5, 2)$
$\downarrow$	$\Uparrow$
$\mathscr{C}^0(\sigma) = 39264851\mathbf{7}$	$C_7^{-1}(48617253) \cdot 7 = 496182537$
$\downarrow$	$\uparrow$
$\mathscr{C}^1(\sigma) = 5248617\mathbf{3}$	$C_3^{-1}(3751624) \cdot 3 = 48617253$
$\downarrow$	$\uparrow$
$\mathscr{C}^2(\sigma) = 271536\mathbf{4}$	$C_4^{-1}(364152) \cdot 4 = 3751624$
$\downarrow$	$\uparrow$
$\mathscr{C}^3(\sigma) = 53416\mathbf{2}$	$C_2^{-1}(25314) \cdot 2 = 364152$
$\downarrow$	$\uparrow$
$\mathscr{C}^4(\sigma) = 3125\mathbf{4}$	$C_4^{-1}(2431) \cdot 4 = 25314$
$\downarrow$	$\uparrow$
$\mathscr{C}^5(\sigma) = 423\mathbf{1}$	$C_1^{-1}(132) \cdot 1 = 2431$
$\downarrow$	$\uparrow$
$\mathscr{C}^6(\sigma) = 31\mathbf{2}$	$C_2^{-1}(12) \cdot 2 = 132$
$\downarrow$	$\uparrow$
$\mathscr{C}^7(\sigma) = 1\mathbf{2}$	$C_2^{-1}(1) \cdot 2 = 12$
$\downarrow$	$\uparrow$
$\mathscr{C}^8(\sigma) = \mathbf{1}$	$C_1^{-1}(\emptyset) \cdot 1 = 1$
$\downarrow$	$\uparrow$
$(1, 2, 2, 1, 4, 2, 4, 3, 7)$	$\emptyset$
$\Downarrow$	
$M(\sigma) =$ $(0, 0, 1, 3, 1, 4, 3, 5, 2)$	The construction of $H(\sigma)$

The following corollary provides an alternative way to compute the cyclic major code.

Corollary 2.3. *For any permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, define*

$$\mathscr{C}(\sigma) = C^{\sigma_n}(\sigma_1\sigma_2 \cdots \sigma_{n-1})$$

and define $L(\sigma) = \sigma_n$. Then we have

$$s_i(\sigma) = i - L(\mathscr{C}^{n-i}(\sigma)),$$

for $1 \le i \le n$, where $\mathscr{C}^0(\sigma) = \sigma$ and $\mathscr{C}^k(\sigma) = \mathscr{C}(\mathscr{C}^{k-1}(\sigma))$.

Proof. First we see that $\mathscr{C}^{n-i}(\sigma) \in S_i$. By the definition of $s_n(\sigma)$, we find

$$s_n(\sigma) = \#\{\sigma_n + 1, \ldots, n\} = n - \sigma_n = n - L(\sigma) = n - L(\mathscr{C}^0(\sigma)).$$

By the proof of Theorem 2.2, we deduce that

$$M(\mathscr{C}^{n-i}(\sigma)) = (s_1(\sigma), \ldots, s_i(\sigma)),$$

which implies that $s_i(\sigma) = i - L(\mathscr{C}^{n-i}(\sigma))$ for $i = 1, 2, \ldots, n$. $\square$

The sequence

$$L(\mathscr{C}^{n-1}(\sigma)), L(\mathscr{C}^{n-2}(\sigma)), \ldots, L(\mathscr{C}^0(\sigma))$$

gives an alternative way to compute the cyclic major code. It also facilitates the computation of $H(\sigma)$. For example, let $\sigma = 392648517$. We have

$$M(\sigma) = (0, 0, 1, 3, 1, 4, 3, 5, 2), \qquad H(\sigma) = 496182537,$$

(see Table 2.1).

The following corollary shows that Han's bijection H commutes with the complementation operator c, a property also satisfied by Foata's partial maps and thus by Foata's map Φ. For a permutation $\sigma \in S_n$, we define $c\sigma$ as $\tau_1\tau_2 \cdots \tau_n$, where $\tau_i = n+1-\sigma_i$. For a code $a = (a_1, a_2, \ldots, a_n) \in E_n$, we define $ca = (b_1, b_2, \ldots, b_n)$, where $b_i = i - 1 - a_i$.

Corollary 2.4. *For $\sigma \in S_n$, we have*

$$H(c\sigma) = cH(\sigma).$$

The above corollary can be easily verified by induction on n. It also follows from Theorem 2.2 and the relations

$$M(c\sigma) = c(M(\sigma)),$$
$$I(c\sigma) = c(I(\sigma)).$$

3. A characterization of fixed points

In this section, we give a characterization of the fixed points of Han's map H. As will be seen, the fixed points of Han's map are related to the strong fixed points of Foata's second fundamental transformation which are easier to characterize.

The notion of strong fixed points of Foata's map is related to the strong Foata classes introduced by Björner and Wachs [1]. A labeling w of a poset P is called recursive if every principal order ideal of P is labeled by a set of consecutive numbers. In particular, if P is a chain with n elements and $w : P \longrightarrow [n]$ is a labeling of P. Reading the labels from bottom to top, the labels form a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. It is easily seen that w is a recursive labeling of P if and only if for each $i \in [n]$, $\{\sigma_1, \sigma_2, \ldots, \sigma_i\}$ forms a set of consecutive numbers. By Theorem 4.2 in [1], we deduce that a permutation $\sigma \in S_n$ is a strong fixed point of Foata's map if and only if for each $i \in [n]$, $\{\sigma_1, \sigma_2, \ldots, \sigma_i\}$ forms a set of consecutive numbers. For example, $\sigma = 45367281 \in S_8$ is a strong fixed point of Foata's map, while $\pi = 34125678$ is not, since $\{\pi_1, \pi_2, \pi_3\} = \{1, 3, 4\}$ is not a set of consecutive numbers.

Theorem 3.1. *For each $\sigma \in S_n$, σ is a fixed point of H, i.e. $H(\sigma) = \sigma$, if and only if σ is a strong fixed point of Foata's map.*

Proof. Suppose that $H(\sigma) = \sigma$. By Theorem 2.2, we see that $I(\sigma) = M(\sigma)$. In particular, we have $s_{n-1}(\sigma) = t_{n-1}(\sigma)$. If $\sigma_{n-1} > \sigma_n$, by Corollary 2.3 we have

$$s_{n-1}(\sigma) = n - 1 - L(\mathscr{C}(\sigma)) = n - 1 - (\sigma_{n-1} - \sigma_n) = n - 1 + \sigma_n - \sigma_{n-1},$$

and by definition $t_{n-1}(\sigma) = n - \sigma_{n-1}$. It follows that $\sigma_n = 1$. If $\sigma_{n-1} < \sigma_n$, then $s_{n-1}(\sigma) = \sigma_n - \sigma_{n-1} - 1$ and $t_{n-1}(\sigma) = n - \sigma_{n-1} - 1$. Hence $\sigma_n = n$. Using relation (2.1), we get

$$M(\mathscr{C}(\sigma)) = (s_1(\sigma), s_2(\sigma), \ldots, s_{n-1}(\sigma)). \tag{3.2}$$

Moreover, when $\sigma_n = 1$ or $\sigma_n = n$, we have

$$\mathscr{C}(\sigma) = C_{\sigma_n}(\sigma_1 \cdots \sigma_{n-1}). \tag{3.3}$$

Combining (3.2), (3.3) and the fact that

$$I(C_{\sigma_n}(\sigma_1 \cdots \sigma_{n-1})) = (t_1(\sigma), t_2(\sigma), \ldots, t_{n-1}(\sigma)),$$

we obtain

$$M(\mathscr{C}(\sigma)) = I(\mathscr{C}(\sigma)).$$

By induction, we deduce that $\mathscr{C}(\sigma)$ is a strong fixed point of Foata's map. Consequently, by relation (3.3), we have

$$\{\sigma_1, \sigma_2, \ldots, \sigma_i\} = \begin{cases} \{(\mathscr{C}(\sigma))_1 + 1, (\mathscr{C}(\sigma))_2 + 1, \ldots, (\mathscr{C}(\sigma))_i + 1\}, & \text{if } \sigma_n = 1; \\ \{(\mathscr{C}(\sigma))_1, (\mathscr{C}(\sigma))_2, \ldots, (\mathscr{C}(\sigma))_i\}, & \text{if } \sigma_n = n, \end{cases}$$

which is a set of consecutive integers. Thus σ is a strong fixed point of Foata's map.

Conversely, suppose that $\sigma \in S_n$ is a strong fixed point of Foata's map. So $\{\sigma_1, \ldots, \sigma_{n-1}\}$ is a set of consecutive integers with $n - 1$ numbers in $[n]$. This implies that $\sigma_n = 1$ or $\sigma_n = n$. Hence

$$C^{\sigma_n}(\sigma') = C_{\sigma_n}(\sigma') = \begin{cases} (\sigma_1 - 1) \cdots (\sigma_{n-1} - 1), & \text{if } \sigma_n = 1; \\ \sigma_1 \cdots \sigma_{n-1}, & \text{if } \sigma_n = n, \end{cases}$$

where $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$. It follows that $C^{\sigma_n}(\sigma')$ is a strong fixed point of Foata's map. By induction, we deduce that

$$H(C^{\sigma_n}(\sigma')) = C^{\sigma_n}(\sigma').$$

Hence

$$\begin{aligned} H(\sigma) &= C_{\sigma_n}^{-1}(H(C^{\sigma_n}(\sigma'))) \cdot \sigma_n \\ &= C_{\sigma_n}^{-1}(C^{\sigma_n}(\sigma')) \cdot \sigma_n \\ &= C_{\sigma_n}^{-1}(C_{\sigma_n}(\sigma')) \cdot \sigma_n = \sigma' \cdot \sigma_n = \sigma, \end{aligned}$$

as desired. This completes the proof. $\square$

The following corollary gives another characterization of the fixed points of H in terms of the cyclic major code and the inversion code.

Corollary 3.2. *Let $\sigma \in S_n$. The following statements are equivalent:*

(1) *$M(\sigma) = I(\sigma)$, that is, σ is a fixed point of H.*
(2) *$I(\sigma) = (t_1(\sigma), t_2(\sigma), \ldots, t_n(\sigma))$ such that $t_i(\sigma) = 0$ or $i-1$ for each $i \in [n]$.*

Proof. It is easy to check that σ satisfies Condition (2) if σ is a strong fixed point of Foata's map. Conversely, suppose that $I(\sigma) = (t_1(\sigma), t_2(\sigma), \ldots, t_n(\sigma))$ with $t_i(\sigma) = 0$ or $i-1$ for each $i \in [n]$. We proceed by induction on n to show that σ is a strong fixed point of Foata's map. The statement is obvious for $n = 1$. Now we assume that the assertion holds for any permutation of length $n-1$ satisfying Condition (2). It is clear that

$$I(C_{\sigma_n}(\sigma')) = (t_1(\sigma), \ldots, t_{n-1}(\sigma)).$$

The inductive hypothesis implies that $C_{\sigma_n}(\sigma')$ is a strong fixed point of length $n-1$. Since $t_n = 0$ or $t_n = n-1$, we have $\sigma_n = 1$ or $\sigma_n = n$, and hence

$$C_{\sigma_n}(\sigma') = \begin{cases} (\sigma_1 - 1)\cdots(\sigma_{n-1} - 1), & \text{if } \sigma_n = 1; \\ \sigma_1 \cdots \sigma_{n-1}, & \text{if } \sigma_n = n. \end{cases}$$

It follows that σ is also a strong fixed point of Foata's map. Now the corollary is a consequence of Theorem 3.1. $\square$

Corollary 3.3. *For any $n \geq 1$, Han's map H has 2^{n-1} fixed points.*

By Theorem 3.1, we see that every fixed point of H is a fixed point of Φ, but the converse is not true. For example, let $\sigma = 14235 \in S_5$. Then σ is a fixed point of Φ, but it is not a fixed point of H.

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