

Curves of genus 3 over small finite fields

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Communicated by Prof. J.H. van Lint at the meeting of April 28, 2003

1. INTRODUCTION

The maximal number of rational points that a (smooth, geometrically irreducible) curve of genus g over a finite field \mathbb{F}_q of cardinality q can have, is denoted by $N_q(g)$. The interest in this number, particularly for fixed q as a function of g , arose primarily during the last two decades from applications to error correcting codes [Lach], [T-G-Z], [vL-vdG]. A lot of results on $N_q(g)$ for fixed q and ‘ g large’ are discussed in the second part of J-P. Serre’s 1985 Harvard lectures [Se3].

The first part of these Harvard lectures reverses the roles of q and g : fix the genus g and study the resulting function $N_q(g)$ of q . In terms of coding theory, this implies that one does not consider asymptotic results (in terms of increasing lengths of the codes), but rather one puts a constraint on the ‘complexity’ of the curves used for constructing codes.

It is a classical result that $N_q(0) = q + 1$: any curve of genus 0 containing a rational point, is isomorphic to the projective line. The determination of $N_q(1)$ is the work of Max Deuring, published in 1941 (see [De] and also [Wa]). Next, $N_q(2)$ was computed by J-P. Serre (1983) [Se1], [Se2], [Se3]; see also [Sh].

For $g \geq 3$, no general formula for $N_q(g)$ seems to be known. The tables [G-V] describe what is known about $N_q(g)$ for $g \leq 50$ and $q \in \{2, 3, 4, 8, 9, 16, 27, 32, 64, 81, 128\}$. J-P. Serre in [Se2, § 4] and [Se3, p. 64-65] presents the values of $N_q(3)$ for all $q \leq 25$. This table is slightly extended in [A-T2, p. 164].

The goal of the present note is twofold. On the one hand, we prove a ‘guess’ expressed by J-P. Serre on page 66 of [Se3]:

Proposition 1.1. *If C/\mathbb{F}_q is a curve of genus 3 with the property $\#C(\mathbb{F}_q) > 2q + 6$, then $q \in \{8, 9\}$ and C is isomorphic over \mathbb{F}_q to one of the following two curves.*

(1) ($q = 8$). *The plane curve over \mathbb{F}_8 given by $x^4 + y^4 + z^4 + x^2y^2 + y^2z^2 + x^2z^2 + x^2yz + xy^2z + xyz^2 = 0$, which has exactly 24 rational points over \mathbb{F}_8 ;*

(2) ($q = 9$). *The quartic Fermat curve over \mathbb{F}_9 given by $x^4 + y^4 + z^4 = 0$, which has exactly 28 rational points over \mathbb{F}_9 .*

The proof uses work of K.-O. Stöhr and J.F. Voloch [S-V], which allows one to translate the assumption into a property of the endomorphism ring of the jacobian of the curve. Also, a result of A. Hefez and J.F. Voloch [H-V] classifying plane quartics with the property that every tangent line is in fact a flex line, is used. A large part of the proof can be traced back to [Se3].

Secondly, some results on determining $N_q(3)$ for relatively small q are discussed. For this, various known upper bounds are recalled, and some explicit constructions of genus 3 curves for which it is easy to determine the number of rational points are presented. Using this, the following extension to Serre’s table was found.

Proposition 1.2. *The maximum number $N_q(3)$ of rational points on a curve of genus 3 over a finite field with q elements is, for $q < 100$, given in the following table.*

q	2	3	4	5	7	8	9
$N_q(3)$	7	10	14	16	20	24	28
q	11	13	16	17	19	23	25
$N_q(3)$	28	32	38	40	44	48	56
q	27	29	31	32	37	41	43
$N_q(3)$	56	60	62	64	72	78	80
q	47	49	53	59	61	64	67
$N_q(3)$	87	92	96	102	107	113	116
q	71	73	79	81	83	89	97
$N_q(3)$	120	122	131	136	136	144	155

Proposition 1.1 is proven in Section 3 below, and Proposition 1.2 in Sections 2 and 4.

This note originated in three lectures on the results of [A-T2] which I gave in the Mathematics Departments of Utrecht, Eindhoven and Nijmegen between October 2002 and January 2003. I thank the colloquium organizers Gunther Cornelissen and Johan van de Leur in Utrecht, the organizer Arjeh Cohen of the EIDMA seminar *Combinatorial Theory*, and Jozef Steenbrink in Nijmegen for offering me the opportunity to speak about these topics.

2. UPPER BOUNDS

Let q be the cardinality of some finite field \mathbb{F}_q . Suppose C/\mathbb{F}_q is a (smooth, complete, absolutely irreducible) curve of genus 3. We write N for the cardinality of the set $C(\mathbb{F}_q)$ of \mathbb{F}_q -rational points on C . Finally, we put $m := [2\sqrt{q}]$, the largest integer which is $\leq 2\sqrt{q}$. The following bounds for the number N are known.

Proposition 2.1. *With notations as above, one has*

- (a) $N \leq q^2 + q + 1$;
- (b) $N \leq 2q + 6$ except for $q = 8$ and for $q = 9$;
- (c) $N \leq q + 1 + 3m$;
- (d) $N \leq q + 3m - 1$ if $q = a^2 + 1$ for some integer a ;
- (e) $N \leq q + 3m - 1$ if $q = a^2 + 2$ for some integer $a \geq 2$;
- (f) $N \leq q + 3m - 2$ if $q = a^2 + a + 1$ for some integer a ;
- (g) $N \leq q + 3m - 2$ if $q = a^2 + a + 3$ for some integer $a \geq 3$.

The first assertion here follows from the well known fact that there are only two possibilities for a curve of genus 3. It can be hyperelliptic, which means that its canonical morphism to \mathbb{P}^2 has degree 2 and has as image a rational curve. This implies that $N \leq 2q + 2$, which is less than $q^2 + q + 1$ since $q \geq 2$. If the curve is not hyperelliptic, then the canonical morphism defines an isomorphism to a plane quartic curve. In this case of course N is at most the cardinality of $\mathbb{P}^2(\mathbb{F}_q)$, which is $q^2 + q + 1$.

Example 2.2. The equation $y^2z^2 + yz^3 + xy^3 + x^2y^2 + x^3z + xz^3 = 0$ defines a smooth quartic over \mathbb{F}_2 containing all 7 points of $\mathbb{P}^2(\mathbb{F}_2)$. Hence $N_2(3) = 7$, as is asserted in various tables such as [Se1], [Se3, p. 64-65], [vL-vdG, p. 71], [G-V]. Note that in [Se3, p. 64-65], Serre remarks that “*this list is not entirely guaranteed ...*”. Indeed, the curve $x^4 + y^4 + z^4 + x^2y^2 + y^2z^2 + x^2z^2 + x^2yz + xy^2z = 0$ over \mathbb{F}_2 given there, contains 1 rather than 7 rational points over \mathbb{F}_2 . An explicit smooth quartic over \mathbb{F}_2 containing 7 rational points is given by Serre in [Se3, part II p. 41] and [Se1, p. 401].

Item (b) in Proposition 2.1 follows from a beautiful geometric argument presented in [S-V] by K.-O. Stöhr and J.F. Voloch; see also [Se3, p. 64-66]. We recall it here, since it is relevant for the proof of Proposition 1.1 given in Section 3 below. The assertion is true for hyperelliptic curves. Hence we can and will assume that C is a plane quartic given by $F = 0$. Writing F_x , et cetera, for the partial derivative $\frac{\partial F}{\partial x}$, the line T_p in \mathbb{P}^2 tangent to C in a point $p \in C$ is given by $xF_x(p) + yF_y(p) + zF_z(p) = 0$. Write $\pi : C \rightarrow C$ for the Frobenius morphism over \mathbb{F}_q , which raises all coordinates to the q th power. Obviously, one has for a point $p \in C$ that

$$p \in C(\mathbb{F}_q) \Leftrightarrow \pi(p) = p$$

and

$$\pi(p) = p \Rightarrow \pi(p) \in T_p.$$

The right hand side of the latter implication can be written as

$$x(p)^q F_x(p) + y(p)^q F_y(p) + z(p)^q F_z(p) = 0$$

where $x(p), y(p), z(p)$ are the projective coordinates of the point p . Hence if we denote by C_q the plane curve of degree $q+3$ given by the equation $x^q F_x + y^q F_y + z^q F_z = 0$, then it follows that

$$C(\mathbb{F}_q) \subseteq C \cap C_q.$$

Now two possibilities arise.

If $C \subset C_q$, then C/\mathbb{F}_q is called *Frobenius non-classical*. It means that every $p \in C$ has the property that $\pi(p) \in T_p$. Proposition 1.1 asserts in particular, that this happens only for $q = 8$ and for $q = 9$, which we have excluded here.

If $C \not\subset C_q$, then the number of intersection points of C and C_q , counted with multiplicities, equals $4 \cdot (q+3)$ by Bézout's theorem [HAG, Ch. I, Cor. 7.8]. Note that at every point $p \in C(\mathbb{F}_q)$, the line T_p intersects both C and C_q at p with multiplicity ≥ 2 . Hence these points also have intersection multiplicity ≥ 2 in $C \cap C_q$. It follows that $2N \leq 4 \cdot (q+3)$, which is what we wanted to prove.

Item (c) in Proposition 2.1 is J-P. Serre's refinement [Se1] of the classical Hasse-Weil bound.

Items (d), (e), (f) and (g) can be found in Lauter's paper [Lau-Se, Thm. 1 and 2]. Basically, the proof extends an idea exposed in [Se3, p. 13-15], which Serre attributes to A. Beauville.

3. FROBENIUS NON-CLASSICAL QUARTICS

In this section we present a proof of Proposition 1.1. So we suppose C/\mathbb{F}_q is a curve of genus 3 satisfying $\#C(\mathbb{F}_q) > 2q+6$. Then in particular C is not hyperelliptic. Hence we may assume that C is a plane quartic curve. The Stöhr-Voloch result as given in the previous section shows that C is Frobenius non-classical. The tangent line T_x at a general point $x \in C$ then intersects C in x (with multiplicity ≥ 2), also in $\pi(x) \neq x$, and hence in a unique fourth point $\phi(x)$. The rational map ϕ extends to a morphism $\phi : C \rightarrow C$. It is non-constant because a smooth curve of positive genus cannot have the property that all its tangent lines meet in a single point (see [HAG, Ch. IV, Thm. 3.9]).

If x, y are two points on C with tangent lines T_x, T_y given by $\ell = 0$ and $m = 0$ respectively, then $f := \ell/m$ defines a function on C with divisor $\text{div}(f) = 2(x-y) + \pi(x-y) + \phi(x-y)$. In particular, this implies that in the endomorphism ring of the Jacobian $J(C)$ of C , the relation $2 + \pi + \phi = 0$ holds. We will distinguish two cases.

(1) If $\phi : C \rightarrow C$ is separable, then the Hurwitz formula implies that ϕ is an isomorphism. As a consequence, the eigenvalues of ϕ acting on a Tate module of $J(C)$ are roots of unity. Let ζ be such an eigenvalue. The equality $2 + \pi + \phi = 0$ shows that $-2 - \zeta$ is an eigenvalue of Frobenius, hence $(-2 - \zeta)(-2 - \zeta^{-1}) = q$. It follows that $q = |2 + \zeta|^2 \leq 9$. Moreover, since $\zeta +$

$\zeta^{-1} = (q - 5)/2$ is integral, it follows that q is odd. So $q \in \{3, 5, 7, 9\}$. If $q = 3$, then ζ would be a primitive third root of unity ω and the eigenvalues of π would be $-2 - \omega$ and $-2 - \omega^{-1}$, each with multiplicity 3. Hence the trace of Frobenius would be -9 which implies $C(\mathbb{F}_3) = 13$. However, it is well known that over \mathbb{F}_3 , a genus 3 curve cannot have more than 10 rational points ([Se2, § 4], [G-V] and also Proposition 2.1(f) above). If $q = 5$ or $q = 7$, then every eigenvalue of Frobenius is $-2 - \zeta$ for a primitive fourth respectively sixth root of unity ζ . This implies that the only eigenvalue of $\phi^{(q-1)/2}$ is -1 . Hence also $\phi^{(q-1)/2}$ acting on the space of regular 1-forms on C has as only eigenvalue -1 . Therefore C is hyperelliptic, contrary to our assumption. The remaining case is that $q = 9$ and $\zeta = 1$. It means that ϕ is the identity map, so every line tangent to C is in fact a flex line. As is shown in the paper by Hefez and Voloch [H-V], this implies in our situation that C is isomorphic to the Fermat curve given by $x^4 + y^4 + z^4 = 0$. We also have that every eigenvalue of Frobenius equals -3 , hence $C(\mathbb{F}_9) = 9 + 1 + 6 \cdot 3 = 28$.

(2) The remaining case is that $\phi : C \rightarrow C$ is inseparable. Then both π and ϕ are inseparable and hence so is $2 = -\pi - \phi$. This implies that q is even. Write π_2 for the map that raises coordinates to their second power. Since ϕ is inseparable, one can write $\phi = \psi\pi_2^n$ for some $n > 0$ and ψ separable. Put $\pi = \pi_2^m$, then with $k := \min(m, n)$ one finds $2 + (\pi_2^{(m-k)} + \psi\pi_2^{(n-k)})\pi_2^k = 0$. Comparing degrees (of isogenies between Jacobians) it follows that $2^{3k} \leq 2^6$, so $k = 1$ or $k = 2$. Therefore, considering a pair λ, μ of eigenvalues of π and ϕ , with $2 + \lambda + \mu = 0$ and $|\lambda|^2 = 2^k \leq |\mu|^2$, one finds, using that $|\mu|^2 = (2 + \lambda)(2 + \bar{\lambda})$ is a power of 2, that $q \leq 16$. On the other hand, a result of [H-V] says that any Frobenius non-classical curve of degree d over \mathbb{F}_q contains exactly $d(q - d + 2)$ rational points. In our case, this equals $4q - 8$, which is $> 2q + 6$ only when $q > 7$. On the other hand, this number cannot exceed the Hasse-Weil bound $q + 1 + 6\sqrt{q}$. Hence we must have $q \leq 9$.

It follows that $q = 8$. In this case, the possibilities for the pairs λ, μ show that either the eigenvalues of Frobenius are $-2 \pm 2i$ (each with multiplicity 3, which would yield the number of points $8 + 1 + 12 = 21 < 2q + 6$, or the eigenvalues $\lambda = (-5 \pm \sqrt{-7})/2$ (each with multiplicity 3), which yield as the number of points $8 + 1 + 15 = 24 = 4q - 8$. Such a λ corresponds to $\mu = (1 \mp \sqrt{-7})/2$. Note that $\mu^3 = \lambda$. Hence on an abelian 3-fold A over \mathbb{F}_2 where the Frobenius π_2 satisfies $\pi_2^2 - \pi_2 + 2 = 0$, we will have $\pi_2^3 + \pi + 2 = 0$. Over \mathbb{F}_8 this A will be isogenous to our $J(C)$. Note that using [Lau-Se, Appendix] and [Hof, § 6], there exists a unique Jacobian of a curve of genus 3 over \mathbb{F}_8 resp. over \mathbb{F}_2 , with a multiplicity 3 eigenvalue of Frobenius $\pm\lambda$ (resp. $\pm\mu$). Moreover, the curve over the larger field is obtained by extending scalars from the one over \mathbb{F}_2 . It follows that C can be defined over \mathbb{F}_2 , and in fact is given by a plane quartic curve over \mathbb{F}_2 with the property that for every point x on it, $x, \pi_2(x)$ and $\pi_2^3(x)$ are collinear. In terms of the coordinates (ξ, η, μ) of such a point, it means that the three vectors (ξ, η, μ) , (ξ^2, η^2, μ^2) and (ξ^8, η^8, μ^8) are linearly dependent. Thus, (ξ, η, μ) satisfies the degree 11 homogeneous polynomial equation

$$\det \begin{pmatrix} \xi & \xi^2 & \xi^8 \\ \eta & \eta^2 & \eta^8 \\ \mu & \mu^2 & \mu^8 \end{pmatrix} = 0.$$

Clearly, the left hand side here is divisible by all seven linear forms in ξ, η, μ over \mathbb{F}_2 . So the curve we look for will correspond to the remaining factor of degree $11 - 7 = 4$, which is

$$\xi^4 + \eta^4 + \mu^4 + \xi^2\eta^2 + \eta^2\mu^2 + \xi^2\mu^2 + \xi^2\eta\mu + \xi\eta^2\mu + \xi\eta\mu^2 = 0.$$

This indeed defines a smooth quartic, which by construction is Frobenius non-classical over \mathbb{F}_8 .

This finishes the proof of Proposition 1.1.

4. CONSTRUCTIONS OF CURVES WITH MANY POINTS

In this section we provide the information needed to verify Proposition 1.2. First of all, in most cases Proposition 2.1 shows that the values presented in our table are upper bounds for $N_q(3)$. To be precise, we mention in the following table which of the items (a)–(f) in Proposition 2.1 works for a given q .

q :	2	3	4	5	7	8	9
item(s) :	(a, d)	(f)	(b)	(b, d)	(b, f)	(c)	(c)
q :	11	13	16	17	19	23	25
item(s) :	(b, e)	(b, f)	(b)	(b, d)	(b, c)	(g)	(b, c)
q :	27	29	31	32	37	41	43
item(s) :	(e)	(c)	(f)	–	(d)	(c)	(f)
q :	47	49	53	59	61	64	67
item(s) :	(c)	(c)	(c)	(g)	(c)	(c)	(c)
q :	71	73	79	81	83	89	97
item(s) :	(c)	(f)	(c)	(c)	(e)	(c)	(c)

For $q = 32$, Proposition 2.1 only gives the bound $N \leq 66 = q + 1 + 3m$. However, the maximum in this case is 64, as is asserted in the tables [G-V]. A curve over \mathbb{F}_{32} containing 64 rational points is given in the thesis [Sem]. To show that this is indeed the maximum, one may use the following fact due to J-P. Serre.

Lemma 4.1. (Serre) *Let $q > 1$ be a power of a prime number and $m := \lfloor 2\sqrt{q} \rfloor$. If $4q - m^2 \leq 11$ then $N_q(3) \leq q + 3m - 1$.*

Proof. The fact that under the given conditions no genus 3 curve with exactly $q + 1 + 3m$ points exists, follows from [Lau-Se, App. 7.1]. The fact that no curve of genus 3 with exactly $q + 3m$ points exists, is proven in [Se2]. This implies the lemma. \square

What remains, is to show that for each $q < 100$ the given upper bound is sharp, i.e., there exists a genus 3 curve with that number of rational points. For each $q \leq 25$, this was done by Serre [Se3, p. 64-65] (note Example 2.2 given above,

which seems to be the only ‘misprint’ in Serre’s table). We will now discuss the cases $25 < q < 100$.

In the case $q = 27$, a genus 3 curve with exactly 56 points was constructed by Van der Geer and Van der Vlugt [G-V].

For $q \in \{29, 49, 53, 67, 71, 89\}$, a curve with $q + 1 + 3m$ points exists. In fact, such a curve is found in the family

$$C_\lambda : x^4 + y^4 + z^4 = (\lambda + 1)(x^2y^2 + y^2z^2 + x^2z^2)$$

as follows from the table given in [A-T2, § 4.2]. Specifically, one finds $\#C_2(\mathbb{F}_{29}) = 60$ and $\#C_{-1}(\mathbb{F}_{49}) = 92$ and $\#C_2(\mathbb{F}_{53}) = 96$ and $\#C_{30}(\mathbb{F}_{67}) = 116$ and $\#C_{37}(\mathbb{F}_{71}) = 120$ and $\#C_{13}(\mathbb{F}_{89}) = 144$. The same family C_λ yields an optimal curve for $q = 43$, with $q + 1 + 3(m - 1) = 80$ rational points, namely C_{10} .

The prime numbers $q \in \{31, 61, 73, 79, 97\}$ can be treated using the family

$$D_{a,b} : x^3z + y^3z + x^2y^2 + axyz^2 + bz^4 = 0$$

of curves with a noncyclic automorphism group of order 6. In this family one finds, for instance, $\#D_{4,2}(\mathbb{F}_{31}) = 62$ and $\#D_{29,34}(\mathbb{F}_{61}) = 107$ and $\#D_{2,48}(\mathbb{F}_{73}) = 122$ and $\#D_{11,8}(\mathbb{F}_{79}) = 131$ and $\#D_{56,79}(\mathbb{F}_{97}) = 155$.

The case $q = 32$ we already discussed.

For $q = 37$ and $q = 83$ one may proceed as follows. Consider the elliptic curves

$$E_\lambda : y^2 = x(x - 1)(x - \lambda)$$

and

$$E'_{a,b} : y^2 = x(x^2 + ax + b).$$

Denote by T resp. T' the point $(0, 0)$ on E_λ and $E'_{a,b}$, respectively. Let $P = (1, 0) \in E_\lambda$ and let $Q \neq T'$ be one of the points with y -coordinate 0 on $E'_{a,b}$. The triples $(0, T', T')$ and $(T, T', 0)$ and (P, Q, Q) generate a (rational) subgroup H of order 8 in $E_\lambda \times E'_{a,b} \times E'_{a,b}$. The quotient $(E_\lambda \times E'_{a,b} \times E'_{a,b})/H$ is an abelian variety. From [H-L-P, Prop. 15] it follows that this is in fact the jacobian of a genus 3 curve C over \mathbb{F}_q , provided that $\lambda(\lambda - 1)(a^2\lambda - 4b)$ is a nonzero square in \mathbb{F}_q . Under this condition, one has

$$\#C(\mathbb{F}_q) = \#E_\lambda(\mathbb{F}_q) + 2\#E'_{a,b}(\mathbb{F}_q) - 2q - 2.$$

In fact, the curve C here can be given by the equation

$$(\lambda - 1)(\lambda x^4 + by^4 + bz^4 + \lambda ax^2y^2 + \lambda ax^2z^2) = (\lambda a^2 - 2b(\lambda + 1))y^2z^2.$$

The choice $\lambda = 7$, $a = 0$ and $b = 2$ turns out to give a curve C over \mathbb{F}_{37} with exactly 72 rational points. Similarly, $\lambda = 5$ and $(a, b) = (4, 2)$ yields a genus 3 curve over \mathbb{F}_{83} containing 136 rational points.

For $q = 41$, consider the family

$$X_{a,b} : x^2y^2 + y^2z^2 + x^2z^2 + a(x^3y + y^3z + xz^3) + b(x^3z + xy^3 + yz^3) = 0$$

of curves with an automorphism of order 3. Here one finds the example $\#X_{-7,8}(\mathbb{F}_{41}) = 78$.

In case $q = 59$ one can use the family

$$Y_{a,b} : (3x^2 + y^2)^2 + ax(x^2 - y^2)z + bz^4 = 0,$$

which are curves with a noncyclic group of automorphisms (of order 6) generated by $(x, y, z) \mapsto (x, -y, z)$ and $(x, y, z) \mapsto (-x - y, -3x + y, 2z)$. One finds for example $\#Y_{4,6}(\mathbb{F}_{59}) = 102$.

For $q = 64$ we quote [G-V], which states that a genus 3 curve over \mathbb{F}_{64} containing 113 rational points is given in [Wi]. Similarly, for the case $q = 81$ the reference [Wi] provides, according to [G-V], an example with 136 rational points.

It should be remarked here that the families C_λ , $D_{a,b}$, $X_{a,b}$, and $Y_{a,b}$ came up in a rather natural way when searching for curves: namely, in many cases we looked for a maximal curve, which means one with $q + 1 + 3m$ rational points. In such a case, if E is an elliptic curve over \mathbb{F}_q containing $q + 1 + m$ rational points, the Jacobian of the curve we try to find is isogenous to $E \times E \times E$. As is explained in the Appendix of [Lau-Se], the principal polarization on this Jacobian can be interpreted as a rank 3 unimodular indecomposable hermitian module over $\mathbb{Z}[(-m + \sqrt{m^2 - 4q})/2]$. The possible group(s) of isometries of such modules yield the possible automorphism groups of the curves we try to find. It turned out that in most cases this gave sufficiently many restrictions on the curves to be actually able to find them.

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(Received January 2003)