JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **178**, 102–115 (1993)

Multiple Positive Solutions for a Class of Semipositone Neumann Two Point Boundary Value Problems

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Submitted by Thomas S. Angell

Received January 8, 1991

We consider the two point Neumann boundary value problem

$$-u'' = (x) = \lambda f(u(x)); \qquad x \in (0, 1)$$
$$u'(0) = 0 = u'(1)$$

where λ is a positive parameter, $f \in C^2[0, \infty)$, $f'(u) > 0$ for $u > 0$, and for some $\beta > 0$, $f(u) < 0$ for $u \in [0, \beta)$ (semipositone) and $f(u) > 0$ for $u > \beta$. We discuss existence and multiplicity results for positive solutions. In particular, we prove that if the set $S = (\pi^2 n^2/f'(\beta), \theta^2/-2F(\beta))$, where $n \in N$, $F(u) = \int_0^u f(s)\, ds$ and θ is the unique positive zero of F, is *nonempty*, then there exist at least $2n+1$ positive solutions for each $\lambda \in S$. Furthermore, if $f'' > 0$ on $[0, \beta)$ and $f'' < 0$ on (β, ∞), then we prove that there are *exactly* $2n+1$ positive solutions for each $\lambda \in S$. We also discuss examples to which our results apply. © 1993 Academic Press, Inc.

1. INTRODUCTION

Consider the nonlinear Dirichlet boundary value problem

$$-u''(x) = \lambda f(u(x)); \qquad x \in (0, 1) \tag{1.1}$$

$$u(0) = 0 = u(1) \tag{1.2}$$

where $\lambda > 0$ is a constant, $f \in C^2[0, \infty)$, $f(0) < 0$ (semipositone), $f' \geqslant 0$ on $(0, \infty)$. Recently, existence, uniqueness, and multiplicity results have been established for the semipositone problem (1.1)–(1.2) (see Castro and Shivaji [3], Khamayseh [6]). See also [1, 2, 4, 5] for higher dimensional results for the Dirichlet case. However, to date, all the results for the semipositone problem have been obtained in the Dirichlet case. The case of the semipositone problem with Neumann boundary condition

$$u'(0) = 0 = u'(1) \tag{1.3}$$

* Supported in part by NSF Grant DMS-8905936.

0022-247X/93 $5.00

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has yet to be analyzed. In this paper we give information about the number of positive solutions of (1.1) and (1.3) for a certain range of λ. We leave the higher dimensional case for future study.

Our major results are:

THEOREM 1.1. *Let* $f \in C^2[0, \infty)$, $f(0) < 0$, $f'(u) > 0$ *for* $u > 0$, $\lim_{u \to +\infty} f(u) > 0$, β *the unique positive zero of* f *and* $\theta(>\beta)$ *the unique positive zero of* $F(s) = \int_0^s f(t)\,dt$. *Further, let* $S = (\pi^2/f'(\beta), \theta^2/-2F(\beta))$ *be nonempty. Then for* $\lambda \in S$, *there exist at least three positive solutions of* (1.1) *and* (1.3).

THEOREM 1.2. *Let* f *satisfy the same hypotheses as in Theorem* 1.1. *Let* $n \in \mathbb{N}$ *and assume further that* $S_n = (\pi^2 n^2/f'(\beta), \theta^2/-2F(\beta))$ *is nonempty. Then for* $\lambda \in S_n$, *there exist at least* $2n+1$ *positive solutions of* (1.1) *and* (1.3).

THEOREM 1.3. *Assume the hypotheses of Theorem* 1.2 *and further assume* $f'' > 0$ *on* $(0, \beta)$ *and* $f'' < 0$ *on* (β, ∞). *Then for* $\lambda \in S_n$ *there exist exactly* $2n+1$ *positive solutions of* (1.1) *and* (1.3).

Remark 1.1. Note that if $f'(\beta) \to \infty$, then $\pi^2 n^2/f'(\beta) \to 0$. Moreover, observe that $-F(\beta) = -\int_0^\beta f(u)\,du < -f(0)\,\beta$. (See Fig. 1.1) But $\theta > \beta$ and if $\beta > 1$, then $\theta^2/-2F(\beta) > \beta/-2f(0)$. Thus $\theta^2/-2F(\beta)$ is bounded away from zero no matter what $f'(\beta)$ is. Hence given an $n \in \mathbb{N}$, it is clear geometrically that there are large classes of functions (see Fig. 1.2) for which S_n is nonempty.

Now to illustrate Theorems 1.1 and 1.2, consider the simple example $f(u) = e^{au} - e^a$, $a > 0$. Here $\beta = 1$. Then $\theta^2/-2F(\beta) > \beta/-2f(0) = 1/2(e^a - 1)$,

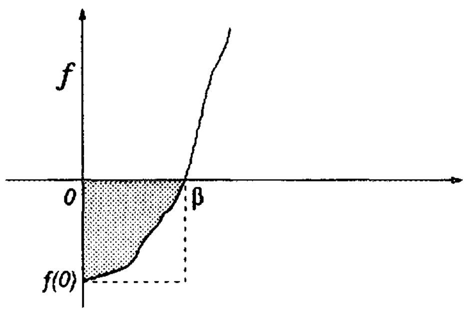

FIGURE 1.1

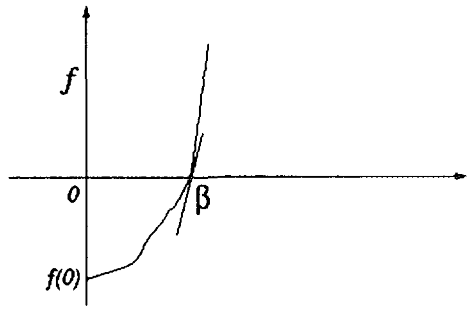

FIGURE 1.2

$\pi^2n^2/f'(\beta) = \pi^2n^2/ae^a$ and $1/2(e^a - 1) > \pi^2n^2/ae^a$ for a large. Thus given $n \in \mathbb{N}$, $\exists a_0$ such that for all $a \geq a_0$ S_n is nonempty and the boundary value problem

$$-u''(x) = \lambda(e^{au(x)} - e^a); \qquad x \in (0, 1)$$
$$u'(0) = 0 = u'(1)$$

has at least $2n+1$ positive solutions for all $\lambda \in S_n$.

Next to illustrate Theorem 1.3 consider $f(u) = \tanh[A(u - \beta)]$. Then $f'' > 0$ on $(0, \beta)$, $f'' < 0$ on (β, ∞), $f'(\beta) = A$ and $f(0) = -\tanh A\beta < 0$ for $A > 0$, $\beta > 0$. Thus given $n \in \mathbb{N}$, $\beta/2 \tanh A\beta > \pi^2n^2/A$ for A sufficiently large. Hence there exists A_0 such that for $A \geq A_0$ S_n is nonempty and the boundary value problem

$$-u''(x) = \lambda \tanh[A(u(x) - \beta)]; \qquad x \in (0, 1)$$
$$u'(0) = 0 = u'(1)$$

has exactly $2n+1$ positive solutions for all $\lambda \in S_n$.

The method we use to prove our results is the extension of the Quadrature Technique introduced by Laetsch in [7] for the Dirichlet case. See also [2, 3, 6] for various useful extensions of the Quadrature Technique in the Dirichlet case. We discuss the extension in the next section and in Section 3 we prove Theorems (1.1)-(1.3).

We conclude this introduction with the note that no two positive solutions of (1.1) and (1.3) are ordered when $f(u)/u$ is increasing. In fact this result is true in higher dimension with more general boundary conditions such as:

$$Bu(x) = 0; \qquad x \in \partial\Omega$$

where Ω is a bounded region in $\mathbb{R}^n$, $Bu(x) = \alpha h(x)\, u(x) + (1-\alpha)\, \partial u/\partial \nu$, $\partial u/\partial \nu$ is the outward normal derivative, $\alpha \in [0, 1]$, $h: \partial\Omega \to \mathbb{R}$, $h = 1$ when $\alpha = 1$, and $\partial\Omega$ is a smooth boundary. For details see Miciano [8].

2. Quadrature Technique for Neumann Boundary Value Problems

Consider the Neumann boundary value problem

$$-u''(x) = \lambda f(u(x)); \qquad x \in (0, 1) \tag{2.1}$$

$$u'(0) = 0 = u'(1) \tag{2.2}$$

where $\lambda > 0$ is a constant, $f \in C^1$, $f' > 0$ on $(0, \infty)$, $f < 0$ on $[0, \beta)$ and $f > 0$ on (β, ∞). Since f is autonomous the following lemmas hold:

Lemma 2.1. *If $u(x)$ is a solution of (2.1)–(2.2), then $u(1-x)$ is also a solution of (2.1)–(2.2).*

Lemma 2.2. *If $u(x)$ is any solution of (2.1)–(2.2), then $u(x)$ is symmetric about any point $x_0 \in (0, 1]$ such that $u'(x_0) = 0$ (i.e., $u(x_0 - z) = u(x_0 + z)$ for all $z \in [0, \min\{x_0, 1 - x_0\}]$).*

Proof. Define $w_1(z) = u(x_0 - z)$ and $w_2(z) = u(x_0 + z)$. Then both w_1 and w_2 satisfy the initial value problem $-w''(z) = \lambda f(z)$, $w(0) = u(x_0)$, $w'(0) = 0$. Hence the result.

Remark 2.1. Any zero of f is a solution of (2.1)–(2.2).

Now consider positive solutions $u(x)$ of the form shown in Fig. 2.1. Here $u(0) = \alpha$, $u(1) = \gamma$, $0 \leqslant \alpha < \beta < \gamma$, and $u'' > 0$ on $(0, t_0)$ and $u'' < 0$ on $(t_0, 1)$, where $t_0 \in (0, 1)$ is such that $u(t_0) = \beta$. To study positive solutions $v(x)$ of the form shown in Fig. 2.2, that is, positive solutions with $n-1$ interior critical points at k/n; $k = 1, 2, ..., n-1$, it suffices by Lemma 2.2, to study only solutions $v_n(x)$ of the form in Fig. 2.2 on the interval $[0, 1/n]$ instead of $[0, 1]$. Notice that solutions of the form $v_n(x)$ are easily obtained from solutions $u(x)$ of the form in Fig. 2.1 on $[0, 1]$, by setting $v_n(x) = u(nx)$ for $x \in [0, 1/n]$. Moreover, recall Lemma 2.1, that is, if $u(x)$ is a solution then $u(1-x)$ is also a solution. Thus the study of solutions of the form in Fig. 2.1 on $[0, 1]$ will provide information on all types of positive solutions.

We now apply the quadrature technique to (2.1)–(2.2). First multiply (2.1) by $u'(x)$ to obtain

$$u''(x)\, u'(x) + \lambda f(u(x))\, u'(x) = 0 \tag{2.3}$$

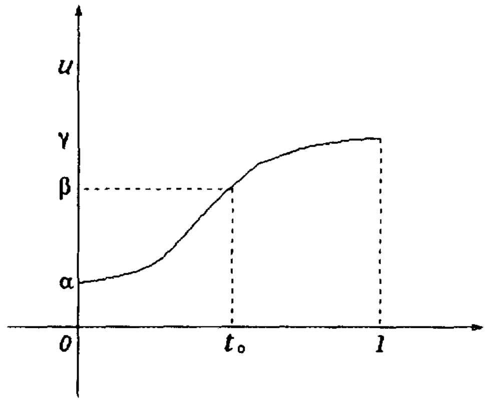

FIGURE 2.1

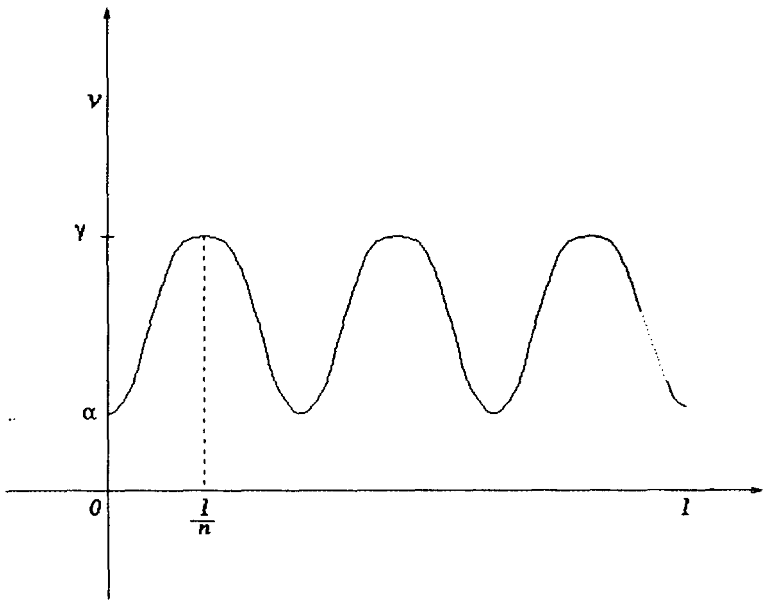

FIGURE 2.2

which is equivalent to

$$\frac{d}{dx}\left[\frac{(u'(x))^2}{2}+\lambda F(u(x))\right]=0; \qquad x\in[0,1] \tag{2.4}$$

where $F(u)=\int_0^u f(s)\,ds$. Integrating (2.4), we have

$$\frac{[u'(x)]^2}{2}+\lambda F(u(x))=C \tag{2.5}$$

where C is a constant. Applying the boundary conditions and the assumption that $u(0)=\alpha$ and $u(1)$, we see that $\lambda F(\alpha)=C=\lambda F(\gamma)$. Hence for each $\alpha\in[0,\beta)$ such that $u(0)=\alpha$, $\gamma(\alpha)\in(\beta,\theta]$ is the unique solution of $F(\alpha)=F(\gamma)$. Here θ is the positive zero of F. (See Fig. 2.3.)
Thus if $u(0)=\alpha$ (2.5) becomes

$$u'(x)=\sqrt{2\lambda[F(\alpha)-F(u(x))]}; \qquad x\in[0,1]. \tag{2.6}$$

Integrating (2.6) on $[0,x]$ and applying the boundary conditions, we obtain

$$\frac{1}{\sqrt{2}}\int_\alpha^{u(x)}\frac{du}{\sqrt{F(\alpha)-F(u)}}=\sqrt{\lambda}\,x; \qquad x\in[0,1]. \tag{2.7}$$

Substituting $x=1$ in (2.7), we have

$$\sqrt{\lambda}=\frac{1}{\sqrt{2}}\int_\alpha^{\gamma(\alpha)}\frac{ds}{\sqrt{F(\alpha)-F(s)}}\equiv G(\alpha) \quad \text{(say)}. \tag{2.8}$$

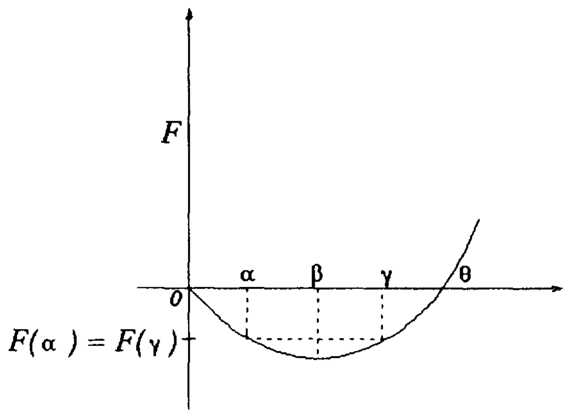

FIGURE 2.3

In fact the following result is true:

THEOREM 2.1. *Let* $f \in C^1[0, \infty)$ *with* $f(0) < 0$, $\lim_{u \to \infty} f(u) > 0$, $f' > 0$ *on* $(0, \infty)$, β, *denote the unique positive zero of* f *and* $\theta(>\beta)$, *denote the unique positive zero of* $F(s) = \int_0^s f(t)\, dt$. *Given* $\lambda > 0$ *if* $\exists \alpha \in S = [0, \beta)$ *such that* $G(\alpha) = \sqrt{\lambda}$ *then* (2.1)–(2.2) *has a unique positive solution* $u(x)$ *satisfying* $u(0) = \alpha$, $u(1) = \gamma$, *where* $\gamma(\alpha)$ *is such that* $F(\gamma) = F(\alpha)$ *and* $u' > 0$ *on* $(0, 1)$. *Furthermore* $G(\alpha)$ *is a continuous and differentiable function in* S. *Its derivative is given by*

$$\frac{dG(\alpha)}{d\alpha} = -\frac{1}{2\sqrt{2}} \int_0^1 \frac{H(\alpha) - H(s(\beta - \alpha) + \alpha)}{[F(\alpha) - F(s(\beta - \alpha) + \alpha)]^{3/2}}\, ds + \left(\frac{d\gamma}{d\alpha}\right) \frac{1}{2\sqrt{2}} \int_0^1 \frac{H(\gamma) - H(s(\beta - \gamma) + \gamma)}{[F(\alpha) - F(s(\beta - \alpha) + \alpha)]^{3/2}}\, ds \tag{2.10}$$

where $H(s) = 2F(s) + (\beta - s) f(s)$. (*For proof, see Miciano* [9].)

3. PROOFS OF THEOREMS 1.1–1.3

Proof of Theorem 1.1

In order to prove Theorem 1.1, we recall $G(\alpha)$ and prove the following inequalities:

$$\text{(a)} \quad \frac{\theta^2}{-2F(\beta)} \leqslant [G(0)]^2 \leqslant \frac{2\theta^2}{-F(\beta)}$$

$$\text{(b)} \quad \frac{4}{f'(\beta)} \leqslant [\lim_{\alpha \to \beta^-} G(\alpha)]^2 \leqslant \frac{16}{f'(\beta)}.$$

Then since the only possible bifurcation points on the curve of solution (λ, β), are $\pi^2 n^2 / f'(\beta)$; $n = 0, 1, \ldots$, from (b) it follows that $[\lim_{\alpha \to \beta^-} G(\alpha)]^2 = \pi^2 n^2 / f'(\beta)$. Hence by our hypothesis, that is, $\pi^2 / f'(\beta) < \theta^2 / -2F(\beta)$, the range of $[G(\alpha)]^2$ contains $S = (\pi^2 / f'(\beta), \theta^2 / -2F(\beta))$ and hence from Theorem 2.1, Lemma 2.1 and Remark 2.1, Theorem 1.1 follows. (See Fig. 3.1.)

Proof of (a). Consider $G(\alpha) = 1/\sqrt{2} \int_\alpha^{\gamma(\alpha)} ds / \sqrt{F(\alpha) - F(s)}$.
Then

$$\lim_{\alpha \to 0^+} G(\alpha) = \frac{1}{\sqrt{2}} \int_0^\theta \frac{ds}{\sqrt{-F(s)}} = G(0). \tag{3.1}$$

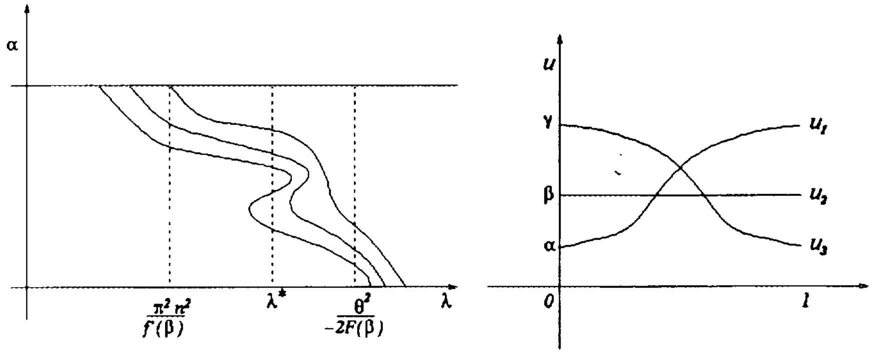

FIGURE 3.1

Since $F''(s) = f'(s) > 0$ for $s > 0$, we have

$$-F(s) \geqslant \begin{cases} \dfrac{-F(\beta)}{\beta}\, s, & 0 < s \leqslant \beta \\ -\left[\dfrac{F(\beta)}{\theta - \beta}\right](\theta - s), & \beta < s < \theta. \end{cases}$$

(See Fig. 3.2.)
Thus

$$\frac{1}{\sqrt{-F(s)}} \leqslant \begin{cases} \sqrt{\dfrac{\beta}{-F(\beta)\, s}}, & 0 < s \leqslant \beta \\ \sqrt{\dfrac{\theta - \beta}{-F(\beta)(\theta - s)}}, & \beta < s < \theta. \end{cases}$$

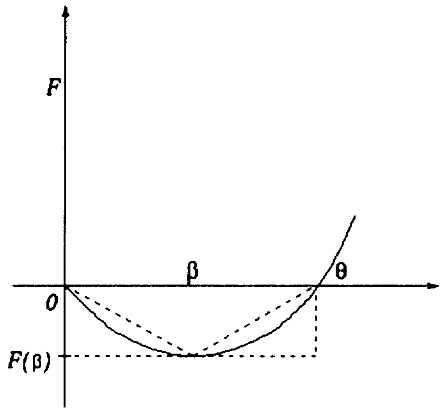

FIGURE 3.2

Rewriting $G(0)$ in (3.1) we have

$$G(0)=\frac{1}{\sqrt{2}}\int_0^\beta \frac{ds}{\sqrt{-F(s)}}+\frac{1}{\sqrt{2}}\int_\beta^\theta \frac{ds}{\sqrt{-F(s)}}$$

$$\leqslant\frac{1}{\sqrt{2}}\left[\sqrt{\frac{\beta}{-F(\beta)}}\,2\sqrt{\beta}+\sqrt{\frac{\theta-\beta}{-F(\beta)}}\,2\sqrt{\theta-\beta}\right]$$

$$=\frac{\sqrt{2}\,\theta}{\sqrt{-F(\beta)}}. \tag{3.2}$$

Also from Figure 3.2 we see that

$$-F(s)\leqslant -F(\beta)\ \forall s\in[0,\theta].$$

Hence

$$G(0)=\frac{1}{\sqrt{2}}\int_0^\theta \frac{ds}{\sqrt{-F(s)}}\geqslant\frac{1}{\sqrt{2}}\int_0^\theta \frac{ds}{\sqrt{-F(\beta)}}$$

$$=\frac{1}{\sqrt{2}}\frac{\theta}{\sqrt{-F(\beta)}}. \tag{3.3}$$

Then from (3.2) and (3.3)

$$\frac{\theta^2}{-2F(\beta)}\leqslant[G(0)]^2\leqslant\frac{2\theta^2}{-F(\beta)}.$$

Proof of (b). First observe that

$$G(\alpha)=\frac{1}{\sqrt{2}}\int_\alpha^{\gamma(\alpha)}\frac{ds}{\sqrt{F(\alpha)-F(s)}}\leqslant\frac{1}{\sqrt{2}}\int_\alpha^{\gamma(\alpha)}\frac{ds}{\sqrt{Z(s)}},$$

where $Z(s)$ is defined as

$$Z(s)=\begin{cases}\left[\dfrac{F(\alpha)-F(\beta)}{\beta-\alpha}\right](s-\alpha), & \alpha<s<\beta\\[2ex] \left[\dfrac{F(\gamma)-F(\beta)}{\gamma-\beta}\right](\gamma-s), & \beta<s<\gamma.\end{cases}$$

(See Fig. 3.3)

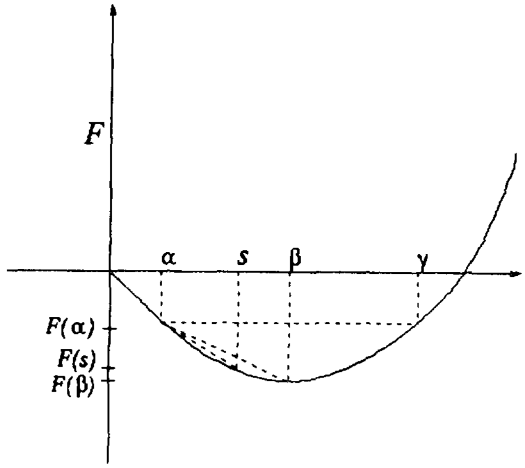

FIGURE 3.3

Furthermore,

$$\begin{aligned}\int_\alpha^\gamma \frac{ds}{\sqrt{Z(s)}} &= \sqrt{\frac{\beta-\alpha}{F(\alpha)-F(\beta)}}\int_\alpha^\beta \frac{ds}{\sqrt{s-\alpha}} + \sqrt{\frac{\gamma-\beta}{F(\gamma)-F(\beta)}}\int_\beta^\gamma \frac{ds}{\sqrt{\gamma-s}} \\ &= \sqrt{\frac{\beta-\alpha}{F(\alpha)-F(\beta)}}\,(2\sqrt{s-\alpha})\Big|_\alpha^\beta + \sqrt{\frac{\gamma-\beta}{F(\gamma)-F(\beta)}}\,(-2\sqrt{\gamma-s})\Big|_\beta^\gamma \\ &= \frac{2(\beta-\alpha)}{\sqrt{F(\alpha)-F(\beta)}} + \frac{2(\gamma-\beta)}{\sqrt{F(\gamma)-F(\beta)}}. \end{aligned} \tag{3.4}$$

Now as $\alpha \to \beta^-$, $\gamma \to \beta^+$

$$\lim_{\alpha\to\beta^-} \frac{4(\beta-\alpha)^2}{F(\alpha)-F(\beta)} = \frac{8}{f'(\beta)} \quad \text{and} \quad \lim_{\gamma\to\beta^+} \frac{4(\gamma-\beta)^2}{F(\gamma)-F(\beta)} = \frac{8}{f'(\beta)}. \tag{3.5}$$

Hence (3.4) and (3.5) imply that

$$\begin{aligned}\lim_{\alpha\to\beta^-} G(\alpha) &\leqslant \lim_{\alpha\to\beta^-} \frac{1}{\sqrt{2}}\int_\alpha^\gamma \frac{ds}{\sqrt{Z(s)}} \\ &= \frac{1}{\sqrt{2}}\left[\frac{2\sqrt{2}}{\sqrt{f'(\beta)}} + \frac{2\sqrt{2}}{\sqrt{f'(\beta)}}\right] \\ &= \frac{4}{\sqrt{f'(\beta)}}. \end{aligned} \tag{3.6}$$

Next

$$\begin{aligned}G(\alpha)&=\frac{1}{\sqrt{2}}\int_\alpha^\gamma \frac{ds}{\sqrt{F(\alpha)-F(s)}}\\&=\frac{1}{\sqrt{2}}\int_\alpha^\beta \frac{ds}{\sqrt{F(\alpha)-F(s)}}+\frac{1}{\sqrt{2}}\int_\beta^\gamma \frac{ds}{\sqrt{F(\gamma)-F(s)}}\\&\geqslant\frac{1}{\sqrt{2}}\int_\alpha^\beta \frac{ds}{\sqrt{F(\alpha)-F(\beta)}}+\frac{1}{\sqrt{2}}\int_\beta^\gamma \frac{ds}{\sqrt{F(\gamma)-F(\beta)}}\\&=\frac{1}{\sqrt{2}}\frac{\beta-\alpha}{\sqrt{F(\alpha)-F(\beta)}}+\frac{1}{\sqrt{2}}\frac{\gamma-\beta}{\sqrt{F(\gamma)-F(\beta)}},\end{aligned}$$

and thus as $\alpha\to\beta^-$, using (3.5), we obtain

$$\lim_{\alpha\to\beta^-} G(\alpha)\geqslant\frac{2}{\sqrt{f'(\beta)}}. \tag{3.7}$$

Then from (3.6) and (3.7), we obtain

$$\frac{2}{\sqrt{f'(\beta)}}\leqslant \lim_{\alpha\to\beta^-} G(\alpha)\leqslant\frac{4}{\sqrt{f'(\beta)}}.$$

Hence

$$\frac{4}{f'(\beta)}\leqslant [\lim_{\alpha\to\beta^-} G(\alpha)]^2\leqslant\frac{16}{f'(\beta)}.$$

Proof of Theorem 1.2

Consider positive solutions with $n-1$ interior critical points as shown in Fig. 3.4. By Lemma 2.2, the analysis of these types of solutions is achieved by studying nondecreasing positive solutions on the interval $[0, 1/n]$. (See Figure 3.5.) But the existence of a nondecreasing solution, say $v(x)$ on $[0, 1/n]$ is equivalent to the existence of a nondecreasing solution $u(x)=v(x/n)$ on $[0, 1]$, since

$$-u''(x)=\frac{1}{n^2}v''\left(\frac{x}{n}\right)=\frac{1}{n^2}\lambda f\left(v\left(\frac{x}{n}\right)\right)=\frac{1}{n^2}\lambda f(u(x)),$$

and

$$u'(0)=\frac{1}{n}v'(0)=0,$$

$$u'(1)=\frac{1}{n}v'\left(\frac{1}{n}\right)=0.$$

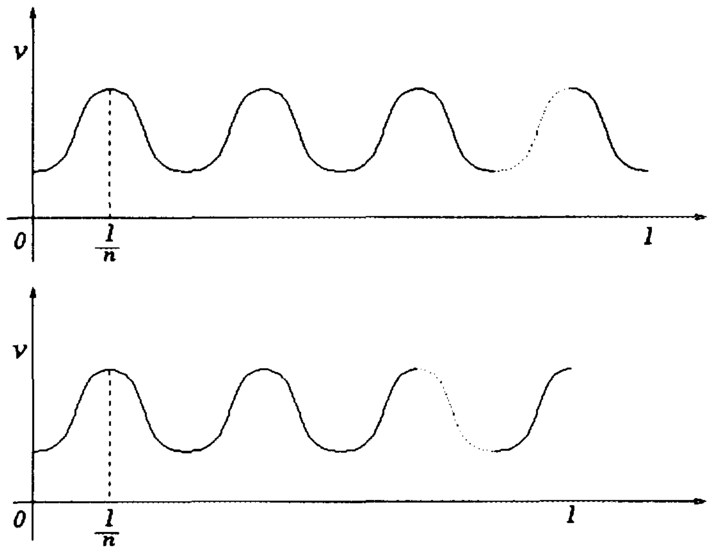

FIGURE 3.4

Hence from Theorem 2.1 a solution with $n-1$ interior critical points of the form shown in Fig. 3.5 exists only if λ/n^2 belongs to the range of $[G(\alpha)]^2$. In fact $u(1-x)$ is also a second solution (see Lemma 2.1) with $n-1$ critical interior points.

Now if $S_n=(16n^2/f'(\beta), \theta^2/-2F(\beta))$ is nonempty and if $\lambda \in S_n$, then $\lambda/m^2 \in (16/f'(\beta), \theta^2/-2F(\beta))$ for each $m=1, 2, ..., n$. Thus by Theorem 1.1 and Lemma 2.1, for each $m=1, 2, ..., n$, we obtain two solutions with m interior critical points. These solutions along with the solution $u\equiv\beta$, gives us at least $2n+1$ positive solutions for $\lambda\in S_n$. Hence Theorem 1.2 is proven. (See Fig. 3.6.)

Proof of Theorem 1.3

In order to prove Theorem 1.3, we recall $dG(\alpha)/d\alpha$ given by (2.10) and show $dG(\alpha)/d\alpha \leqslant 0$. Now

$$\frac{dG(\alpha)}{d\alpha}=-\frac{1}{2\sqrt{2}}\int_0^1 \frac{H(\alpha)-H(s(\beta-\alpha)+\alpha)}{[F(\alpha)-F(s(\beta-\alpha)+\alpha)]^{3/2}}\,ds$$
$$+\left(\frac{d\gamma}{d\alpha}\right)\frac{1}{2\sqrt{2}}\int_0^1 \frac{H(\gamma)-H(s(\beta-\gamma)+\gamma)}{[F(\alpha)-F(s(\beta-\alpha)+\alpha)]^{3/2}}\,ds$$

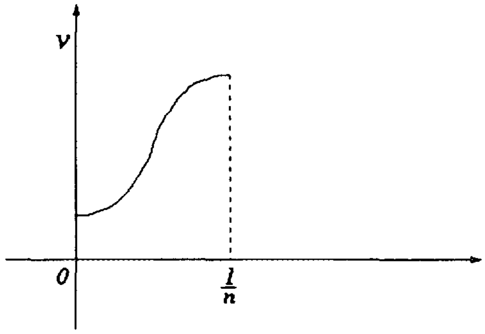

FIGURE 3.5

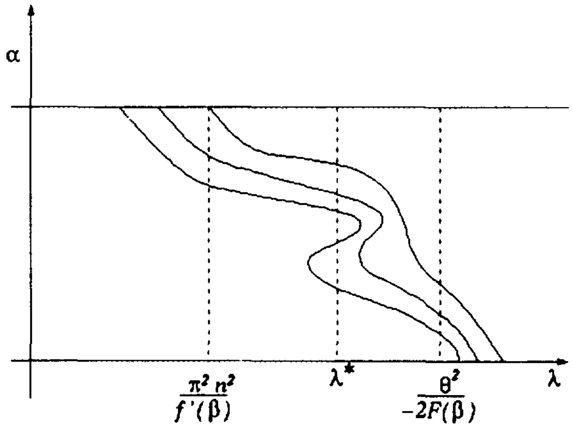

FIGURE 3.6

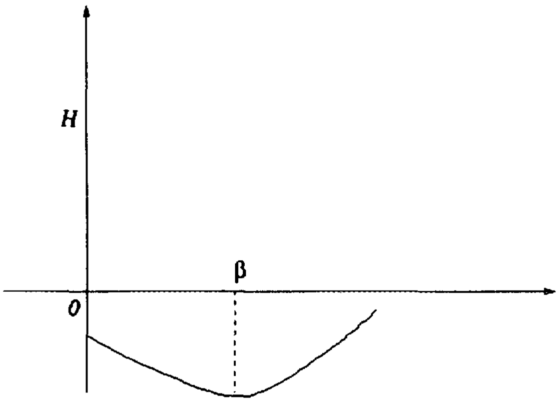

FIGURE 3.7

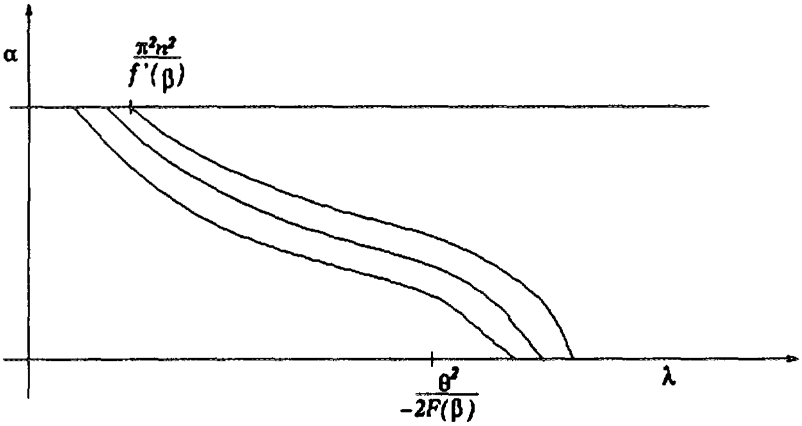

FIGURE 3.8

where $H(s) = 2F(s) + (\beta - s) f(s)$. Then $H'(s) = f(s) + (\beta - s) f'(s)$, and $H''(s) = (\beta - s) f''(s)$. But $f(0) < 0$, $f(\beta) = 0$ and here we assume that $f''(s) > 0$ for $s \in (0, \beta)$ and $f''(s) < 0$ for $s \in (\beta, \theta)$. Hence $H(0) < 0$, $H'(\beta) = 0$ and $H''(s) > 0$ for $s \neq \beta$. (See Fig. 3.7). Thus $H(\alpha) - H(s(\beta - \alpha) + \alpha) \geqslant 0$ and $H(\gamma) - H(s(\beta - \gamma) + \gamma) \geqslant 0$ $\forall s \in (0, 1)$. But we know that $d\gamma/d\alpha < 0$, so $dG/d\alpha \leqslant 0$. Hence from Theorem 1.2 we obtain exactly $2n + 1$ positive solutions. (See Fig. 3.8.)

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