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J. Differential Equations 245 (2008) 1433–1453

Journal of Differential Equations

www.elsevier.com/locate/jde

Semiclassical and relaxation limits of bipolar quantum hydrodynamic model for semiconductors

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Received 9 September 2006

Available online 16 July 2008

Abstract

The global in-time semiclassical and relaxation limits of the bipolar quantum hydrodynamic model for semiconductors are investigated in R^3. We prove that the unique strong solution exists and converges globally in time to the strong solution of classical bipolar hydrodynamical equation in the process of semiclassical limit and that of the classical drift–diffusion system under the combined relaxation and semiclassical limits.

Keywords: Quantum hydrodynamics; Semiclassical limit; Relaxation limit

1. Introduction

Recently, the quantum hydrodynamic (QHD) model for semiconductors is derived and studied in the modelings and simulations of semiconductor devices (like MOSFET and RTD) in ultra-small size (say nano-size), where the effects of quantum mechanics, such as particle tunneling through potential barriers and built-up in quantum well, are taken into granted and dominate the transportation of electron and/or hole under the self-consistent electric field.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.06.019

The basic observation concerning the quantum hydrodynamics is that the energy density consists of one additional new quantum correction term of the order $O(\hbar)$ introduced first by Wigner [31] in 1932, and that the stress tensor contains also an additional quantum correction part [2,3] related to the quantum Bohm potential (or internal self-potential) [4]

$$Q(\rho) = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}, \tag{1.1}$$

with observable $\rho > 0$ the density, m mass, and $\hbar$ the Planck constant. The quantum potential Q was introduced by de Broglie and explored by Bohm to make a hidden variable theory and is responsible for producing the quantum behavior, so that all quantum features are related to its special properties. Such possible relation was also implied in the original idea initialized by Madelung [26] in 1927 to derive quantum fluid-type equations, in terms of Madelung's transformation applied to wave function of Schrödinger equation of pure state. In fact, based on this idea, one is able to derive quantum fluid-type equations from the (nonlinear) Schrödinger equation of pure state [10,17].

The moment method is employed recently to derive quantum hydrodynamic equations for semiconductor device at nano-size based on the Wigner–Boltzmann (or quantum Liouville) equation [28]

$$W_t + \xi \cdot \nabla_x W + \frac{q}{m}\mathbb{P}[\Phi]W = [W_t]_c \tag{1.2}$$

where $W = W(x, \xi, t)$, $(x, \xi, t) \in R^3 \times R^3 \times R_+$ is the distribution function, and $\mathbb{P}$ the pseudo-differential operator defined by

$$\mathbb{P}[\Phi]W = \frac{im}{(2\pi)^N}\iint \frac{\Phi(x + \frac{\hbar}{2m}\eta) - \Phi(x - \frac{\hbar}{2m}\eta)}{\hbar} e^{i\eta\cdot(\xi-\xi')} W(x, \xi', t)\, d\eta\, d\xi'.$$

The electrostatic potential $\Phi = \Phi(x, t)$ is self-consistent through Poisson equation

$$\lambda_0 \Delta\Phi = q\left(\int W\, d\xi - \mathcal{C}\right),$$

with $\lambda_0 > 0$ the permittivity characteristic of device, q the elementary charge, and $\mathcal{C} = \mathcal{C}(x) > 0$ the given doping profile [28], and $[W_t]_c$ refers to the collision term. In fact, applying moment method to the Wigner–Boltzmann equation (1.2) near the "momentum-shifted quantum Maxwellian" [31] together with appropriate closure assumption [8,11], one can obtain the quantum hydrodynamic equation [8]. For more derivation and related topics on the modeling of quantum models, one refers to [8,10,28] and the references therein.

In the present paper, we consider the bipolar quantum hydrodynamic model of semiconductors (for carriers of two type)

$$\partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0, \tag{1.3}$$

$$\partial_t(\rho_i u_i) + \nabla \cdot (\rho_i u_i \otimes u_i) + \nabla P_i(\rho_i) = q_i \rho_i E + \frac{\varepsilon^2}{2}\rho_i \nabla\left(\frac{\Delta\sqrt{\rho_i}}{\sqrt{\rho_i}}\right) - \frac{\rho_i u_i}{\tau_i}, \tag{1.4}$$

$$\lambda^2 \nabla \cdot E = \rho_a - \rho_b - \mathcal{C}, \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty, \tag{1.5}$$

where $(x,t) \in R^3 \times R^+$ and the index $i = a, b$ and $q_a = 1$, $q_b = -1$. The observable $\rho_a > 0$, $\rho_b > 0$, u_a, u_b and E are the densities, velocities and electric field, respectively. $P_a(.)$, $P_b(.)$ are the pressure–density functions. The parameters $\varepsilon > 0$, $\tau_a = \tau_b = \tau > 0$, and $\lambda > 0$ are the scaled Planck constant, momentum relaxation time, and Debye length respectively. $\mathcal{C} = \mathcal{C}(x)$ is doping profile.

In the real simulations of semiconductor devices, the size of the device is rather small (in nano-size, for instance). This in turn makes the scaled parameters τ, ε, λ rather smaller due to different situations under consideration [28,29]. In general, the scaled parameters ε, τ, λ are expressed as

$$\varepsilon^2 = \frac{\hbar^2}{2m\kappa_B T_0 L^2}, \qquad \lambda^2 = \frac{\lambda_0 \kappa T_0}{Nq^2L^2}, \qquad \tau^2 = \frac{\kappa_B T_0 \tau_0^2}{mL^2}$$

where we recall that the physical parameters are the elementary charge q, the Boltzmann constant k_B, the elective electron mass m, the reduced Planck constant $\hbar$, the permittivity λ_0, the ambient temperature T_0, and the characteristic device length L and density N. The typical values of the parameters for semiconductors are given in [28]. Therefore, one of the both mathematically and physically important problems is to justify the asymptotic approximation (or behavior) of the macroscopic observable of the quantum hydrodynamical model subject to the small parameters mentioned above.

In the present paper, we investigate the asymptotical analysis with respect to the scaled small parameters of bipolar time-dependent quantum hydrodynamical model. To begin with, let us present a complete description about the small-scale asymptotics of the QHD model. We first consider the semiclassical limit. Let $\varepsilon \to 0$ formally in (1.3)–(1.5), we get the well-known bipolar hydrodynamic (HD) model [1,9]

$$\partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0, \tag{1.6}$$

$$\partial_t(\rho_i u_i) + \nabla \cdot (\rho_i u_i \otimes u_i) + \nabla P_i(\rho_i) = q_i \rho_i E - \frac{\rho_i u_i}{\tau_i}, \tag{1.7}$$

$$\lambda^2 \nabla \cdot E = \rho_a - \rho_b - \mathcal{C}, \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty. \tag{1.8}$$

This limiting process shows the semiclassical approximation of bipolar quantum hydrodynamical model in terms of bipolar hydrodynamical model for small Planck constant, and describes the relation from quantum mechanics to the classical Newtonian mechanics.

The semiclassical limits of the stationary unipolar quantum hydrodynamical model (carrier of one type) are well studied recently. In one-dimensional bounded domain, the semiclassical limit of the thermal equilibrium solutions [5] and the isentropic subsonic solutions [12] are analyzed respectively due to different boundary conditions. This limit is also investigated for a stationary unipolar viscous quantum hydrodynamical system [7] for a special class of viscosity in one-dimensional interval subject to the boundary condition of density and quantum Fermi potential, where the communication between vanishing viscosity and semiclassical limit is also investigated in subsonic regime. For bipolar stationary quantum hydrodynamical model, the semiclassical limits are investigated in multi-dimensional bounded domain for isothermal solutions in thermal equilibrium state [22,30], by recovering the minimizer of limiting functional of a quantized energy functional corresponding the original system, and in multi-dimensional unbounded domain for stationary isentropic system [34]. A rigorous analysis is also made for the bipolar viscous quantum hydrodynamical system [22].

However, all those analyses for stationary problems cannot apply to the time-dependent case because that unlike the case of stationary problem, the maximum principle usually does not apply to the time-dependent case and it is not clear how to derive enough the a priori estimates with respect to time (derivatives) so as to pass into the semiclassical limit. Although such process of semiclassical limit has been investigated recently for nonlinear Schrödinger equation [6,23] for potential flow in terms of Friedrich–Kato–Lax's theory and is concerned with the finite (short) time theory, the frame work does not apply here to general multi-dimensional rotational (non-potential) flow and is not fit for global in-time theory. We should do the semiclassical limit for QHD model in a different way in order to present the global in-time semiclassical limit for general non-potential flow.

Next, we turn to the analysis of relaxation limit. To this end, let us introduce the diffusion scaling as [20,27]

$$x \to x, \quad t \to \frac{t}{\tau}, \quad \big(\rho_i^\tau, u_i^\tau, E^\tau\big)(x,t) = \left(\rho_i, \frac{u_i}{\tau}, E\right)\left(x, \frac{t}{\tau}\right). \tag{1.9}$$

Then (1.3)–(1.5) can be rewritten as

$$\partial_t \rho_i^\tau + \nabla \cdot \big(\rho_i^\tau u_i^\tau\big) = 0, \tag{1.10}$$

$$\tau^2 \partial_t \big(\rho_i^\tau u_i^\tau\big) + \tau^2 \nabla \cdot \big(\rho_i^\tau u_i^\tau \otimes u_i^\tau\big) + \nabla P_i\big(\rho_i^\tau\big) = q_i \rho_i^\tau E^\tau + \frac{\varepsilon^2}{2} \rho_i^\tau \nabla\left(\frac{\Delta\sqrt{\rho_i^\tau}}{\sqrt{\rho_i^\tau}}\right) - \rho_i^\tau u_i^\tau, \tag{1.11}$$

$$\lambda^2 \nabla \cdot E^\tau = \rho_a^\tau - \rho_b^\tau - \mathcal{C}(x), \quad \nabla \times E^\tau = 0, \quad E^\tau(x) \to 0, \quad |x| \to +\infty. \tag{1.12}$$

Also formally, let $\tau \to 0$ in (1.10)–(1.12), the quantum drift–diffusion (QDD) model is obtained

$$\partial_t \rho_i + \nabla\left[q_i \rho_i E - \nabla P_i(\rho_i) + \frac{\varepsilon^2}{2} \rho_i \nabla\left(\frac{\Delta\sqrt{\rho_i}}{\sqrt{\rho_i}}\right)\right] = 0, \tag{1.13}$$

$$\lambda^2 \nabla \cdot E = \rho_a - \rho_b - \mathcal{C}(x), \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty. \tag{1.14}$$

This limiting process provides a singular approximation of quantum hydrodynamical model via parabolic quantum drift–diffusion model for small momentum relaxation time. Note that although there are many results obtained for classical hydrodynamic model [1,21,27], few is known for the relaxation limit for the quantum hydrodynamical model due to the less of enough information to control the nonlinear third-order dispersion term. Although the relaxation limit of the stationary solutions are investigated in one-dimensional bounded domain for unipolar case [12], and in multi-dimensional bounded domain for bipolar case [22], like the situation of semiclassical analysis, all these studies seems not enough in the resolution of the time-dependent problems. Note that, the singular relaxation time limit presented above is not mathematically rigorous, the first rigorous analysis result about relaxation time limit of QHD model has been obtained recently in [20], where the QHD system is proven to be approximated by a quantum drift–diffusion model (QDD), a nonlinear parabolic equation, for small relaxation time. However, this analysis depends strongly on the effects of the nonlinear dispersion. That is, the scaled Planck constant is required to be fixed in order to help getting enough control to pass into the relaxation limit, which is therefore not enough to prove the relaxation limit for possibly arbitrary small Planck constant ε. Thus, it is natural for us to consider the relaxation limit of quantum hydrodynamical model for any small Planck constant ε and furthermore the combined relaxation and semiclassical limit. In fact, we can show in the present paper that one can derive the following limiting drift–diffusion (DD) model

$$\partial_t \rho_i + \nabla\big[q_i \rho_i E - \nabla P_i(\rho_i)\big] = 0, \tag{1.15}$$

$$\lambda^2 \nabla \cdot E = \rho_a - \rho_b - \mathcal{C}(x), \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty, \tag{1.16}$$

by setting $\tau \to 0$ and $\varepsilon \to 0$ in (1.10)–(1.12) for strong solutions. Note here that although we only deal with the combined relaxation and semiclassical limits for the quantum hydrodynamical model (1.10)–(1.12), we claim that the analysis made here does not require any (communication) restriction between ε and τ. That is, one can fix any of the two parameters ε and τ and let the other tend to zero.

We shall also mention the asymptotical analysis about the zero-Debye length limit for QHD model. This process is quite well understood for both stationary problems [12,30] for one and multi-dimension bounded domain respectively and the time-dependent problem for multi-dimension [24]. We omit the corresponding analysis here.

The rest part of the paper is arranged as follows. The main results related to semiclassical limit and relaxation time limit are presented in Section 2, the proofs are established in Section 3.

Notations. C or c always denote the generic positive constants. $L^2(R^3)$ is the space of square integral functions on R^3 with the norm $\|\cdot\|$ or $\|\cdot\|_{L^2(R^3)}$. $H^k(R^3)$ with integer $k \geqslant 1$ denotes the usual Sobolev space of function f satisfying $\partial_x^i f \in L^2(R^3)$ $(0 \leqslant i \leqslant k)$ with norm

$$\|f\|_k = \sqrt{\sum_{0 \leqslant |\alpha| \leqslant k} \big\|D^\alpha f\big\|^2},$$

here and after $\alpha \in N^3$, $D^\alpha = \partial_{x_1}^{s_1} \partial_{x_2}^{s_2} \partial_{x_3}^{s_3}$ for $|\alpha| = s_1 + s_2 + s_3$. Especially $\|\cdot\|_0 = \|\cdot\|$. Let $\mathcal{B}$ be a Banach space, $C^k([0,t];\mathcal{B})$ denotes the space of $\mathcal{B}$-valued k-times continuously differentiable functions on $[0,t]$. We can extend the above norm to the vector-valued function $u = (u_1, u_2, u_3)$ with $|D^\alpha u|^2 = \sum_{r=1}^3 |D^\alpha u_r|^2$ and

$$\big\|D^k u\big\|^2 = \int_{R^3} \left(\sum_{r=1}^{3} \sum_{|\alpha|=k} \big(D^\alpha u_r\big)^2 \right) dx,$$

and $\|u\|_k = \|u\|_{H^k(R^3)} = \sum_{i=0}^k \|D^i u\|$, $\|f\|_{L^\infty([0,T];\mathcal{B})} = \sup_{0 \leqslant t \leqslant T} \|f(t)\|_{\mathcal{B}}$. We also use the space $\mathcal{H}^k(R^3) = \{f \in L^6(R^3),\ Df \in H^{k-1}(R^3)\}$, $k \geqslant 1$. Sometimes we use $\|(.,.,\ldots)\|_{H^k(R^3)}$ or $\|(.,.,\ldots)\|_k$ to denote the norm of the space $H^k(R^3) \times H^k(R^3) \times \cdots \times H^k(R^3)$ and the $\mathcal{H}^k(R^3)$ as well.

2. Main results and preliminary

2.1. Main results

We consider the initial value problem for the quantum system (1.3)–(1.5) with the following initial data

$$(\rho_i, u_i)(x,0) = (\rho_{i_0}, u_{i_0})(x), \quad \rho_{i_0}(x) \to \rho_i^*, \quad u_{i_0}(x) \to 0, \quad |x| \to +\infty, \tag{2.1}$$

with $i = a, b$. From now on, we set the scaled Debye length to be one $\lambda = 1$ for simplicity.

First of all, we have the global existence and uniqueness theory of the IVP problem for the quantum system (1.3)–(1.5) and (2.1).

Theorem 2.1 *(Global existence). Let the parameters $\varepsilon > 0$, $\tau > 0$ be fixed. Assume $P_a, P_b \in C^5(0,+\infty)$ and $\mathcal{C}(x) = c^*$ is a constant satisfying for two positive constants ρ_a^*, ρ_b^* that*

$$\rho_a^* - \rho_b^* - c^* = 0, \qquad P_a'(\rho_a^*), P_b'(\rho_b^*) > 0. \tag{2.2}$$

Suppose $\rho_{a_0} > 0$, $\rho_{b_0} > 0$ and $(\sqrt{\rho_{a_0}} - \sqrt{\rho_a^*}, \sqrt{\rho_{b_0}} - \sqrt{\rho_b^*}, u_{a_0}, u_{b_0}) \in (H^6(R^3))^2 \times (\mathcal{H}^5(R^3))^2$. Then, there is $\Lambda_1 > 0$ so that if $\Lambda_0 := \|(\sqrt{\rho_{a_0}} - \sqrt{\rho_a^*}, \sqrt{\rho_{b_0}} - \sqrt{\rho_b^*}, u_{a_0}, u_{b_0})\|_{H^6 \times \mathcal{H}^5(R^3)} \leqslant \Lambda_1$, the unique solution $(\rho_a^\varepsilon, \rho_b^\varepsilon, u_a^\varepsilon, u_b^\varepsilon, E^\varepsilon)$ of the IVP problem (1.3)–(1.5) and (2.1) exists globally in time with $\rho_a^\varepsilon, \rho_b^\varepsilon > 0$ and satisfies

$$\left(\rho_i^\varepsilon - \rho_i^*, E^\varepsilon\right) \in C^k\left(0,T; H^{6-2k}\left(R^3\right)\right), \qquad u_i^\varepsilon \in C^k\left(0,T; \mathcal{H}^{5-2k}\left(R^3\right)\right), \quad k = 0,1,2,$$

and

$$\left\|\left(\rho_a^\varepsilon - \rho_a^*, \rho_b^\varepsilon - \rho_b^*\right)\right\|_{L^\infty(R^3)} + \left\|E^\varepsilon\right\|_{L^\infty(R^3)} + \left\|\left(u_a^\varepsilon, u_b^\varepsilon\right)\right\|_{L^\infty(R^3)} \to 0,$$

as time tends to infinity.

Remark 2.2. Unlike the unipolar quantum hydrodynamical model [14–16,25], we cannot get the exponential convergence to the asymptotical equilibrium state for bipolar quantum model due to the coupling and cancellation of two carriers. It has been shown recently that usually the optimal decay rate is algebraic for charge density and exponential for electric field [33].

We then state semiclassical limit $\varepsilon \to 0_+$ of the global in-time solutions to the IVP (1.3)–(1.5) and (2.1) for any fixed momentum relaxation time $\tau > 0$.

Theorem 2.3 *(Global semiclassical limit). Let $\tau = 1$ and $(\rho_a^\varepsilon, \rho_b^\varepsilon, u_a^\varepsilon, u_b^\varepsilon, E^\varepsilon)$ be the solution of the IVP problem (1.3)–(1.5) and (2.1) given by Theorem 2.1. Then, there is $(\rho_a, u_a, \rho_b, u_b, E)$ with $\rho_a > 0$, $\rho_b > 0$ so that as the Planck constant $\varepsilon \to 0$, it holds*

$$\begin{aligned}
&\rho_i^\varepsilon \to \rho_i \quad \text{strongly in } C\left(0,T; C_b^3 \cap H_{\mathrm{loc}}^{5-s}\right),\\
&u_i^\varepsilon \to u_i \quad \text{strongly in } C\left(0,T; C_b^3 \cap \mathcal{H}_{\mathrm{loc}}^{5-s}\right),\\
&E^\varepsilon \to E \quad \text{strongly in } C\left(0,T; C_b^4 \cap \mathcal{H}_{\mathrm{loc}}^{6-s}\right), \quad s \in \left(0, \frac{1}{2}\right),
\end{aligned}$$

for any $T > 0$, $i = a, b$. Note here that (ρ_i, u_i, E) with $i = a, b$ is the global in-time solution of IVP problem of the bipolar hydrodynamic model (1.6)–(1.8) *and* (2.1).

Finally, we consider the combined semiclassical and relaxation limits for the quantum hydrodynamical model (1.3)–(1.5). To this end, we consider indeed the initial value problem for the re-scaled system (1.10)–(1.12) together with the following initial data

$$\left(\rho_i^\tau, u_i^\tau\right)(x,0) := \left(\rho_{i0}^\tau, u_{i0}^\tau\right) = \left(\rho_{i0}, \frac{u_{i0}}{\tau}\right)(x). \tag{2.3}$$

It is easy to verify that there is a unique global in-time strong solution $(\rho_i^{(\tau,\varepsilon)}, u_i^{(\tau,\varepsilon)}, E^{(\tau,\varepsilon)})$ with $i = a, b$ for the IVP problem (1.10)–(1.12) and (2.3) based on Theorem 2.1 and the diffusion scaling (1.9). What left is to establish the uniform estimates with respect to the parameters $\varepsilon > 0$, $\tau > 0$ in order to pass into the limits. We have

Theorem 2.4 *(Global relaxation and semiclassical limits). Let $(\rho_i^{(\tau,\varepsilon)}, u_i^{(\tau,\varepsilon)}, E^{(\tau,\varepsilon)})$ with $i = a, b$ be the unique global solution of the bipolar QHD equations* (1.10)–(1.12) *and* (2.3) *given by Theorem* 2.1*, then there exists (ρ_a, ρ_b, E) such that as $\varepsilon \to 0$ and $\tau \to 0$*

$$\begin{aligned}
\rho_i^{(\tau,\varepsilon)} &\to \rho_i \quad \textit{strongly in } C\left(0,T; C_b^2 \cap H_{\mathrm{loc}}^{4-s}\left(R^3\right)\right),\\
E^{(\tau,\varepsilon)} &\to E \quad \textit{strongly in } C\left(0,T; C_b^3 \cap \mathcal{H}_{\mathrm{loc}}^{5-s}\left(R^3\right)\right),\\
\tau^2\left|u_i^{(\tau,\varepsilon)}\right|^2 &\to 0 \quad \textit{strongly in } L^1\left(0,T; W_{\mathrm{loc}}^{3,3}\left(R^3\right)\right), \quad s \in \left(0, \frac{1}{2}\right),
\end{aligned}$$

and (ρ_a, ρ_b, E) is the strong solution of the IVP problem of bipolar drift–diffusion system (1.15)–(1.16) *with initial data $(\rho_a, \rho_b)(x,0) = (\rho_{a0}, \rho_{b0})$.*

Remark 2.5. Although we only state the combined relaxation and semiclassical limits for the quantum hydrodynamical model (1.10)–(1.12) here, we should mention that the analysis made here does not require any (communication) restriction between ε and τ. That is, one can fix any of the two parameters ε and τ and let the other tend to zero. Moreover, our analysis for the bipolar model (1.10)–(1.12) can be applied to justify the semiclassical limit and relaxation limit for the unipolar model [14,15,20].

Remark 2.6. Although we have only taken the steady state of constant solution in the profile in the above theorems, our analysis in the present paper can be applied for general subsonic steady state.

2.2. Some lemmas

Lemma 2.7. *Let $f \in H^s(R^3)$, $s \geqslant \frac{3}{2}$. There is a unique solution of the divergence equation*

$$\nabla \cdot u = f, \quad \nabla \times u = 0, \quad u(x) \to 0, \quad |x| \to +\infty,$$

satisfying

$$\|u\|_{L^6(R^3)} \leqslant C\|f\|_{L^2(R^3)}, \qquad \|Du\|_{H^s(R^3)} \leqslant C\|f\|_{H^s(R^3)}.$$

Lemma 2.8. *Let $f \in H^s(R^3)$, $s \geqslant \frac{3}{2}$ with $\nabla \cdot f = 0$. There is a unique solution u of the vorticity equation*

$$\nabla \times u = f, \quad \nabla \cdot u = 0, \quad u(x) \to 0, \quad |x| \to +\infty,$$

satisfying

$$\|u\|_{L^6(R^3)} \leqslant C\|f\|_{L^2(R^3)}, \qquad \|Du\|_{H^s(R^3)} \leqslant C\|f\|_{H^s(R^3)}.$$

We will also use the Moser type calculus lemmas.

Lemma 2.9. *Let $f, g \in H^s(R^3) \cap L^\infty(R^3)$, then it holds*

$$\left\|D^\alpha(fg)\right\| \leqslant C\|g\|_{L^\infty} \cdot \left\|D^\alpha f\right\| + C\|f\|_{L^\infty} \cdot \left\|D^\alpha g\right\|$$

for $\alpha \in N^3$, $1 \leqslant |\alpha| \leqslant s$, $s \geqslant 0$ is an integer.

Lemma 2.10. *Let $f \in H^s(R^3)$ with $s \geqslant 0$ be an integer and function $F(\rho)$ smooth enough and $F(0) = 0$ then $F(f)(x) \in H^s(R^3)$ and*

$$\left\|F(f)\right\|_{H^s(R^3)} \leqslant C\|f\|_{H^s(R^3)}.$$

3. The proof of main results

The local in-time existence result of QHD model has been obtained in [15,25]. The framework used there is to study an extended problem derived based on a deposition of the original problem, which in turn implies the expected problem as a special case. The method employed in [15,25] can be applied to our bipolar model directly. The proof is straightforward, and we have

Lemma 3.1 *(Local existence). Let the parameters $\varepsilon > 0$, $\tau > 0$, $\lambda > 0$ be fixed. Assume that there are constants $\rho_a^*, \rho_b^* > 0$ and c^* satisfying $\rho_a^* - \rho_b^* - c^* = 0$, $\mathcal{C}(x) - c^* \in H^5(R^3)$, and $P_a, P_b \in C^5(0, +\infty)$. Assume $(\sqrt{\rho_{i_0}} - \sqrt{\rho_i^*}, u_{i_0}) \in H^6(R^3) \times \mathcal{H}^5(R^3)$ with $\rho_{i_0} > 0$, then there exists a finite time $T^* > 0$ such that the unique solution $(\rho_a, \rho_b, u_a, u_b, E)$ with $\rho_a > 0$, $\rho_b > 0$ of the problem* (1.3)–(1.5) *and* (2.3) *exists in $[0, T^*]$, and it satisfies*

$$\rho_i - \rho_i^* \in C^k\big(\big[0, T^*\big]; H^{6-2k}\big(R^3\big)\big), \qquad u_i \in C^k\big(\big[0, T^*\big]; \mathcal{H}^{5-2k}\big(R^3\big)\big), \quad k = 0, 1, 2,$$
$$E \in C^k\big(\big[0, T^*\big]; \mathcal{H}^{6-k}\big(R^3\big)\big), \quad k = 0, 1.$$

Here, we also mention the global existence theory for the quantum hydrodynamical model and bipolar hydrodynamical model. The well-posedness of steady state subsonic solutions has been proved also in [18,19,32]. Transient solutions are shown to exist either locally in time [13,14] or globally in time for data close to a steady state [15,16,19,25]. The bipolar hydrodynamic (HD) model of the global solutions has been studied in [9].

3.1. Reformulation of original problem

In this section, we study the global solutions and the asymptotic limits with the case $\mathcal{C}(x) = c^*$. Inspired by [20] we consider the problem when the initial data of $(\rho_i^\tau, u_i^\tau, E^\tau)$ is around the steady state $(\rho_i^*, 0, 0)$ and make use of energy estimates to analyze perturbation of the global in-time solutions. To this end, we employ the fourth-order wave equations for $\sqrt{\rho_i^\tau}$ and the equation of the vorticity of velocity u_i^τ. The Poisson equation is used to deal with the coupling of the two carriers and some technique is used to deal with the smallness both of ε and τ.

Since we are interested in not only the global existence theory but also the asymptotical analysis of strong solutions with respect to small parameters, we deal with the scaled IVP problem (1.10)–(1.12) and (2.3) directly. Because the scaled IVP problem (1.10)–(1.12) and (2.3) is equivalent to the original IVP problem (1.3)–(1.5) and (2.1) for strong (classical) solutions. For simplicity, we take $\lambda = 1$ and let $(.)_t$ denote $\partial_t(.)$ and omit the index ε, τ to simplify the presentation in the following argument. From (1.10)–(1.12) and (2.3) the equations for $\psi_i = \sqrt{\rho_i^\tau}$ with $u_i = u_i^\tau$ $(i = a, b)$ can be obtained

$$\tau^2\psi_{itt} + \psi_{it} + \frac{\varepsilon^2\Delta^2\psi_i}{4} + \frac{q_i}{2\psi_i}\nabla\cdot\big(\psi_i^2 E\big) - \frac{1}{2\psi_i}\nabla^2\big(\psi_i^2 u_i\otimes u_i\big)$$
$$- \frac{1}{2\psi_i}\Delta P_i\big(\psi_i^2\big) + \frac{\psi_{it}^2}{\psi_i} - \frac{\varepsilon^2|\Delta\psi_i|^2}{4\psi_i} = 0, \tag{3.1}$$

with the initial value

$$\psi_i(x,0) := \psi_{i0}(x) = \psi_{i_0}^\tau(x) = \sqrt{\rho_{i0}(x)},$$
$$\psi_{it}(x,0) := \psi_{i1}(x) = -\frac{1}{2}\psi_{i0}^\tau\nabla\cdot u_{i0}^\tau - u_{i0}^\tau\cdot\nabla\psi_{i0}^\tau.$$

Also from (1.10)–(1.12) and (2.3) with the fact $(u_i\cdot\nabla)u_i = \frac{1}{2}\nabla(|u_i|^2) - u_i\times(\nabla\times u_i)$, the equations for $u_i = u_i^\tau$ $(i = a, b)$

$$\tau^2 u_{it} + u_i + \frac{\tau^2}{2}\nabla\big(|u_i|^2\big) - \tau^2 u_i\times\phi_i + \frac{\nabla(\psi_i^2)}{\psi_i^2} = q_i E + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi_i}{\psi_i}\right) \tag{3.2}$$

where $\phi_i = \nabla\times u_i$ denotes the vorticity of u_i. Taking curl of (3.2), we have

$$\tau^2\phi_{it} + \phi_i + \tau^2(u_i\cdot\nabla)\phi_i + \tau^2\phi_i\nabla\cdot u_i - \tau^2(\phi_i\cdot\nabla)u_i = 0. \tag{3.3}$$

Introduce new variables $w_i = \psi_i - \sqrt{\rho_i^*}$ with $i = a, b$, then the system for $(w_a, w_b, \phi_a, \phi_b, E)$ is

$$\tau^2 w_{att} + w_{at} + \frac{\varepsilon^2\Delta^2 w_a}{4} + \frac{1}{2}\big(w_a + \sqrt{\rho_a^*}\big)\nabla\cdot E - P_a'\big(\rho_a^*\big)\Delta w_a = f_{a1}, \tag{3.4}$$

$$\tau^2 w_{btt} + w_{bt} + \frac{\varepsilon^2\Delta^2 w_b}{4} - \frac{1}{2}\big(w_b + \sqrt{\rho_b^*}\big)\nabla\cdot E - P_b'\big(\rho_b^*\big)\Delta w_b = f_{b1}, \tag{3.5}$$

$$\tau^2\phi_{at} + \phi_a = f_{a2}, \tag{3.6}$$

$$\tau^2\phi_{bt} + \phi_b = f_{b2}, \tag{3.7}$$

and

$$\nabla\cdot E = w_a^2 - w_b^2 + 2\sqrt{\rho_a^*}w_a - 2\sqrt{\rho_b^*}w_b, \quad \nabla\times E = 0, \tag{3.8}$$

where

$$\begin{aligned}
f_{i1} := f_{i1}(x,t) &= \frac{-\tau^2 w_{it}^2}{w_i + \sqrt{\rho_i^*}} - q_i\nabla w_i E + \big(P_i'\big(\big(w_i + \sqrt{\rho_i^*}\big)^2\big) - P_i'\big(\rho_i^*\big)\big)\Delta w_i \\
&\quad + 2\big(w_i + \sqrt{\rho_i^*}\big)P_i''\big(\big(w_i + \sqrt{\rho_i^*}\big)^2\big)|\nabla w_i|^2 + P_i'\big(\big(w_i + \sqrt{\rho_i^*}\big)^2\big)\frac{|\nabla w_i|^2}{w_i + \sqrt{\rho_i^*}} \\
&\quad + \frac{\varepsilon^2(\Delta w_i)^2}{4(w_i + \sqrt{\rho_i^*})} + \frac{\tau^2\nabla^2((w_i + \sqrt{\rho_i^*})^2 u_i\otimes u_i)}{2(w_i + \sqrt{\rho_i^*})},
\end{aligned} \tag{3.9}$$

$$f_{i2} := f_{i2}(x,t) = \tau^2\big((\phi_i\cdot\nabla)u_i - (u_i\cdot\nabla)\phi_i - \phi_i\nabla\cdot u_i\big), \quad i = a, b. \tag{3.10}$$

The last term in (3.9) can be decomposed by using Eq. (1.10) as

$$\begin{aligned}
&\frac{\tau^2\nabla^2((w_i + \sqrt{\rho_i^*})^2 u_i\otimes u_i)}{2(w_i + \sqrt{\rho_i^*})} \\
&= \tau^2\Bigg\{-w_{it}\nabla\cdot u_i - 2u_i\cdot\nabla w_{it} - \frac{w_{it}u_i\cdot\nabla w_i}{2(w_i + \sqrt{\rho_i^*})} + \nabla w_i\cdot\big((u_i\cdot\nabla)u_i\big) \\
&\quad + \frac{(w_i + \sqrt{\rho_i^*})}{2}\sum_{k,l=1}^{3}\big|\partial_k u_i^l\big|^2 - \frac{(w_i + \sqrt{\rho_i^*})}{2}|\phi_i|^2 - u_i\cdot\nabla(u_i\cdot\nabla w_i) \\
&\quad + \frac{1}{2(w_i + \sqrt{\rho_i^*})}(w_{it} + u_i\cdot\nabla w_i)(u_i\cdot\nabla w_i)\Bigg\}, \quad i = a, b.
\end{aligned} \tag{3.11}$$

The initial conditions for (3.4)–(3.7) are

$$\begin{aligned}
w_i(x,0) &:= w_{i_0}(x) = \psi_{i_0} - \sqrt{\rho_i^*}, \qquad \phi_i(x,0) := \phi_{i_0}(x) = \frac{1}{\tau}\nabla\times u_{i_0}(x), \\
w_{it}(x,0) &:= w_{i1}(x) = \frac{1}{\tau}\bigg(-u_{i_0}\cdot\nabla w_{i_0} - \frac{1}{2}\big(w_{i_0} + \sqrt{\rho_i^*}\big)\nabla\cdot u_{i_0}\bigg), \quad i = a, b.
\end{aligned}$$

We will also use the relation between $\nabla\cdot u_i$ and ∇w_i, w_{it}

$$2w_{it} + 2u_i\cdot\nabla w_i + \big(w_i + \sqrt{\rho_i^*}\big)\nabla\cdot u_i = 0, \quad i = a, b. \tag{3.12}$$

3.2. The a priori estimates

In this section, we will mainly study the reformulated equations (3.4)–(3.8) in order to obtain the a priori estimates of w_a, w_b, ϕ_a, ϕ_b, E.

Set the workspace as

$$X(T) = \left\{(w_a, w_b, u_a, u_b) \in L^\infty\left([0,T]; \left(H^6\left(R^3\right)\right)^2 \times \left(\mathcal{H}^5\left(R^3\right)\right)^2\right)\right\}$$

and assume the quantity

$$\begin{aligned}\delta_T = {} & \max_{0\leqslant t\leqslant T}\left\{\left\|(w_a, w_b)(.,t)\right\|_4^2 + \left\|\tau(\partial_t w_a, \partial_t w_b)(.,t)\right\|_3^2 + \left\|\tau(u_a, u_b)(.,t)\right\|_{\mathcal{H}^4}^2\right\} \\ & + \int_0^T \left\{\left\|(u_a, u_b)(.,t)\right\|_{\mathcal{H}^3}^2 + \left\|(w_a, w_b)(.,t)\right\|_5^2 + \left\|E(.,t)\right\|_{\mathcal{H}^2}^2\right\} dt \qquad (3.13)\end{aligned}$$

is small, then by Sobolev embedding theorem we know that the sufficiently small δ_T can assure the positivity of ψ_a, ψ_b as

$$\frac{\sqrt{\rho_a^*}}{2} \leqslant w_a + \sqrt{\rho_a^*} \leqslant \frac{3}{2}\sqrt{\rho_a^*}, \qquad \frac{\sqrt{\rho_b^*}}{2} \leqslant w_b + \sqrt{\rho_b^*} \leqslant \frac{3}{2}\sqrt{\rho_b^*}.$$

By Sobolev embedding theorem, from the assumption for δ_T, we also have

$$\left\|\left(D^\alpha w_a, D^\alpha w_b, \tau D^\beta w_{at}, \tau D^\beta w_{bt}\right)\right\|_{L^\infty(R^3\times[0,T])} \leqslant c\delta_T, \quad |\alpha| \leqslant 2,\ |\beta| \leqslant 1, \qquad (3.14)$$

$$\left\|\left(\tau D^\alpha u_a, \tau D^\alpha u_b\right)\right\|_{L^\infty(R^3\times[0,T])} \leqslant c\delta_T, \quad |\alpha| \leqslant 2, \qquad (3.15)$$

$$\int_0^T \left\|\left(D^\alpha u_a, D^\alpha u_b, \tau^2 u_{at}, \tau^2 u_{bt}\right)(.,t)\right\|_{L^\infty(R^3)}^2 dt \leqslant c\delta_T, \quad |\alpha| \leqslant 1. \qquad (3.16)$$

The last inequality (3.16) is obtained from the equations for u_a, u_b and Sobolev embedding theorem, the assumption for δ_T. The c or C denote the generic positive constant and does not necessarily be the same here and after. Using Lemma 2.7, from the Poisson equation (3.8) we have

$$\|E\|_{L^\infty([0,T];\mathcal{H}^5(R^3))} \leqslant c\delta_T, \qquad \left\|D^\alpha E\right\|_{L^\infty(R^3\times[0,T])} \leqslant c\delta_T, \quad |\alpha| \leqslant 3. \qquad (3.17)$$

Next, we will establish energy estimates to extend the solution to global one.

We have the main a priori estimate lemma.

Lemma 3.2. *Suppose $(w_a, w_b, u_a, u_b, E)(x,t)$ is local solution with $\delta_T \ll 1$, then it holds*

$$E_1(t) + \int_0^t E_2(s)\,ds \leqslant c\Lambda_0 \qquad (3.18)$$

for $t \in (0,T)$ and $c > 0$ is a constant independent of ε and τ. The Λ_0 is defined in Theorem 2.1*, and here*

$$\begin{aligned}
E_1(t) := \bigl\{ &\bigl\|(w_a, w_b)(.,t)\bigr\|_4^2 + \bigl(\tau + \varepsilon^2\bigr)\bigl\|\bigl(D^5 w_a, D^5 w_b\bigr)(.,t)\bigr\|^2 \\
&+ \tau\varepsilon^2\bigl\|\bigl(D^6 w_a, D^6 w_b\bigr)(.,t)\bigr\|^2 + \tau^2\bigl\|(w_{at}, w_{bt})(.,t)\bigr\|_3^2 \\
&+ \tau^3\bigl\|\bigl(D^4 w_{at}, D^4 w_{bt}\bigr)(.,t)\bigr\|^2 + \tau^2\bigl\|(u_a, u_b)(.,t)\bigr\|_{\mathcal{H}^4}^2 \\
&+ \tau^3\bigl\|\bigl(D^5 u_a, D^5 u_b\bigr)(.,t)\bigr\|^2 + \bigl\|E(.,t)\bigr\|_{\mathcal{H}^5}^2 \bigr\}, \\
E_2(t) := \bigl\{ &\bigl\|(\nabla w_a, \nabla w_b)(.,t)\bigr\|_4^2 + \varepsilon^2\bigl\|\bigl(D^6 w_a, D^6 w_b\bigr)(.,t)\bigr\|^2 \\
&+ \bigl\|(w_{at}, w_{bt})(.,t)\bigr\|_3^2 + \tau\bigl\|\bigl(D^4 w_{at}, D^4 w_{bt}\bigr)(.,t)\bigr\|^2 \\
&+ \bigl\|(u_a, u_b)(.,t)\bigr\|_{\mathcal{H}^4}^2 + \tau\bigl\|\bigl(D^5 u_a, D^5 u_b\bigr)(.,t)\bigr\|^2 + \bigl\|E(.,t)\bigr\|_{\mathcal{H}^5}^2 \bigr\}.
\end{aligned}$$

Proof. Step 1. The estimates for w_a, w_b. *Step 1.1. Basic estimates.* Assume $\tau < 1$ for simplicity. Multiplying (3.4) by $(w_a + 2w_{at})$ and (3.5) by $(w_b + 2w_{bt})$, integrating by parts the resulted equations over R^3, summing the resulted two equalities and noticing the facts from Eq. (3.8)

$$\begin{aligned}
&\int_{R^3} \left\{ \left(\frac{1}{2}\bigl(w_a + \sqrt{\rho_a^*}\bigr)\nabla\cdot E \right) w_a - \left(\frac{1}{2}\bigl(w_b + \sqrt{\rho_b^*}\bigr)\nabla\cdot E \right) w_b \right\} dx \\
&\quad = \frac{1}{4}\int_{R^3} |\nabla\cdot E|^2\,dx - \frac{1}{4}\int_{R^3} \nabla\bigl(w_a^2 - w_b^2\bigr)\cdot E\,dx,
\end{aligned}$$

and

$$\int_{R^3} \left\{ \frac{1}{2}\bigl(\bigl(w_a + \sqrt{\rho_a^*}\bigr)\nabla\cdot E\bigr)2w_{at} - \frac{1}{2}\bigl(\bigl(w_b + \sqrt{\rho_b^*}\bigr)\nabla\cdot E\bigr)2w_{bt} \right\} dx = \frac{1}{4}\frac{d}{dt}\int_{R^3} |\nabla\cdot E|^2\,dx,$$

then after a tedious but straightforward calculation we have

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3} \left\{ \tau^2 w_{at}^2 + \tau^2 w_a w_{at} + \frac{w_a^2}{2} + \tau^2 w_{bt}^2 + \tau^2 w_b w_{bt} + \frac{w_b^2}{2} + P_a'\bigl(\rho_a^*\bigr)|\nabla w_a|^2 \right. \\
&\quad \left. + P_b'\bigl(\rho_b^*\bigr)|\nabla w_b|^2 + \frac{\varepsilon^2}{4}\bigl(|\Delta w_a|^2 + |\Delta w_b|^2\bigr) + \frac{1}{4}|\nabla\cdot E|^2 \right\} dx \\
&\quad + \int_{R^3} \left\{ \bigl(2 - \tau^2\bigr)\bigl(w_{at}^2 + w_{bt}^2\bigr) + P_a'\bigl(\rho_a^*\bigr)|\nabla w_a|^2 + P_b'\bigl(\rho_b^*\bigr)|\nabla w_b|^2 \right. \\
&\quad \left. + \frac{\varepsilon^2}{4}\bigl(|\Delta w_a|^2 + |\Delta w_b|^2\bigr) + \frac{1}{4}|\nabla\cdot E|^2 \right\} dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \int_{R^3} (w_a \nabla w_a - w_b \nabla w_b) \cdot E \, dx \\
&\quad + \int_{R^3} \big\{ f_{a1}(x,t)(w_a + 2w_{at}) + f_{b1}(x,t)(w_a + 2w_{bt}) \big\} \, dx. \qquad (3.19)
\end{aligned}
$$

The right-hand side of (3.19) can be analyzed as follows. By Sobolev embedding theorem and Hölder inequality, Young's inequality

$$
\begin{aligned}
\int_{R^3} w_i \nabla w_i \cdot E \, dx &\leqslant \|w_i\|_{L^3} \|\nabla w_i\|_{L^2} \cdot \|E\|_{L^6} \\
&\leqslant c\big(\|w_i\|_{L^2} + \|\nabla w_i\|_{L^2}\big)\big(\|\nabla w_i\|_{L^2} \cdot \|E\|_{L^6}\big) \\
&\leqslant c(\delta_T)^{\frac{1}{2}} \big(\|\nabla w_i\|^2 + \|\nabla \cdot E\|^2\big), \qquad (3.20)
\end{aligned}
$$

$i = a, b$. Here we have used Lemma 2.7 to estimate $\|DE\|^2$ by $\|\nabla \cdot E\|^2$. Some other key terms of the right-hand side are analyzed as

$$
\int_{R^3} \big[P_i'\big(\big(w_i + \sqrt{\rho_i^*}\big)^2\big) - P_i'\big(\rho_i^*\big)\big] \Delta w_i \cdot (2w_{it}) \, dx \leqslant c(\delta_T)^{\frac{1}{2}} \big(\|\Delta w_i\|^2 + \|w_{it}\|^2\big), \qquad (3.21)
$$

$$
\int_{R^3} \tau^2 u_i \cdot \nabla w_{it} (2w_{it}) \, dx = -\int_{R^3} \tau^2 \nabla \cdot u_i (w_{it})^2 \, dx \leqslant c(\delta_T)^{\frac{1}{2}} \|w_{it}\|^2, \qquad (3.22)
$$

$$
\int_{R^3} \tau^2 u_i \nabla (u_i \cdot \nabla w_i)(2w_{it}) \, dx \leqslant c(\delta_T)^{\frac{1}{2}} \big\|(\nabla w_i, \Delta w_i, w_{it}, \phi_i)\big\|^2. \qquad (3.23)
$$

In (3.23) we have used $\|Du_i\|^2 \leqslant c(\|\nabla \cdot u_i\|^2 + \|\nabla \times u_i\|^2)$, and $\|\nabla w_i\|^2$, $\|w_{it}\|^2$ to estimate $\nabla \cdot u_i$ through Eq. (3.12). Then, by (3.19)–(3.23), using integration by parts, Hölder inequality, Young's inequality and the Moser type Lemmas 2.9, 2.10 to estimate the other terms of the right-hand side of (3.19), we can arrive at

$$
\begin{aligned}
&\frac{d}{dt} \int_{R^3} \bigg\{ \tau^2 w_{at}^2 + \tau^2 w_a w_{at} + \frac{w_a^2}{2} + \tau^2 w_{bt}^2 + \tau^2 w_b w_{bt} + \frac{w_b^2}{2} + P_a'\big(\rho_a^*\big)|\nabla w_a|^2 \\
&\qquad + P_b'\big(\rho_b^*\big)|\nabla w_b|^2 + \frac{\varepsilon^2}{4}\big(|\Delta w_a|^2 + |\Delta w_b|^2\big) + \frac{1}{4}|\nabla \cdot E|^2 \bigg\} \, dx \\
&\qquad + \int_{R^3} \bigg\{ \big(2 - \tau^2\big)\big(w_{at}^2 + w_{bt}^2\big) + P_a'\big(\rho_a^*\big)|\nabla w_a|^2 + P_b'\big(\rho_b^*\big)|\nabla w_b|^2 \\
&\qquad + \frac{\varepsilon^2}{4}\big(|\Delta w_a|^2 + |\Delta w_b|^2\big) + \frac{1}{4}|\nabla \cdot E|^2 \bigg\} \, dx \\
&\leqslant c(\delta_T)^{\frac{1}{2}} \big\|(\nabla w_a, \nabla w_b, w_{at}, w_{at}, \nabla \cdot E, \phi_a, \phi_b)\big\|^2 + c(\delta_T)^{\frac{1}{2}} \big\|(\Delta w_a, \Delta w_b)\big\|^2. \qquad (3.24)
\end{aligned}
$$

The right-hand side of estimate (3.24) will be used later in the closure of the a priori estimates.

Step 1.2. The higher-order estimates for w_a, w_b. Differentiate (3.4) and (3.5) with respect to x, then the functions $\widetilde{w_a} := D^\alpha w_a$, $\widetilde{w_b} := D^\alpha w_b$ and $\widetilde{E} := D^\alpha E$ $(1 < |\alpha| \leqslant 3)$[1] satisfy

$$\begin{aligned}
&\tau^2 \widetilde{w}_{i\,tt} + \widetilde{w}_{i\,t} + \frac{\varepsilon^2}{4}\Delta^2 \widetilde{w}_i + \frac{q_i}{2}\big(w_i + \sqrt{\rho_i^*}\big)\nabla\cdot\widetilde{E} - P_i'\big(\rho_i^*\big)\Delta\widetilde{w}_i \\
&\quad = D^\alpha f_{i1}(x,t) - D^\alpha\left(\frac{q_i}{2}\big(w_i + \sqrt{\rho_i^*}\big)\nabla\cdot E\right) + \frac{q_i}{2}\big(w_i + \sqrt{\rho_i^*}\big)\nabla\cdot\widetilde{E} \\
&\quad \overset{\mathrm{def}}{=} F_i(x,t) \quad (i = a, b,\ q_a = 1,\ q_b = -1).
\end{aligned} \tag{3.25}$$

Multiplying (3.25) for $i = a$ by $(\widetilde{w_a} + 2\widetilde{w_{at}})$, and (3.25) for $i = b$ by $(\widetilde{w_b} + 2\widetilde{w_{bt}})$, integrating by parts over R^3, summing the resulted equalities, also noticing the facts

$$\begin{aligned}
&\int_{R^3}\left\{\frac{1}{2}\big(w_a + \sqrt{\rho_a^*}\big)\nabla\cdot(\widetilde{E})(\widetilde{w_a} + 2\widetilde{w_{at}}) - \frac{1}{2}\big(w_b + \sqrt{\rho_b^*}\big)\nabla\cdot(\widetilde{E})(\widetilde{w_b} + 2\widetilde{w_{bt}})\right\}dx \\
&\quad = \frac{1}{4}\int_{R^3}|\nabla\cdot\widetilde{E}|^2\,dx + \frac{1}{4}\frac{d}{dt}\int_{R^3}|\nabla\cdot\widetilde{E}|^2\,dx - \frac{1}{4}\int_{R^3}\nabla\cdot\widetilde{E}D^\alpha\big(w_a^2 - w_b^2\big)\,dx \\
&\qquad - \frac{1}{2}\int_{R^3}\nabla\cdot\widetilde{E}D^\alpha\big(w_a^2 - w_b^2\big)_t\,dx + \int_{R^3}\frac{1}{2}w_a\nabla\cdot(\widetilde{E})(\widetilde{w_a} + 2\widetilde{w_{at}})\,dx \\
&\qquad - \int_{R^3}\frac{1}{2}w_b\nabla\cdot(\widetilde{E})(\widetilde{w_b} + 2\widetilde{w_{bt}})\,dx,
\end{aligned} \tag{3.26}$$

after a tedious but straightforward computation one can get

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3}\left\{\tau^2\widetilde{w_a}_t^2 + \tau^2\widetilde{w_a}\widetilde{w_{at}} + \frac{1}{2}\widetilde{w_a}^2 + \tau^2\widetilde{w_b}_t^2 + \tau^2\widetilde{w_b}\widetilde{w_{bt}} + \frac{1}{2}\widetilde{w_b}^2 + P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2\right. \\
&\qquad \left. + P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2 + \frac{\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2 + |\Delta\widetilde{w_b}|^2\big) + \frac{1}{4}|\nabla\cdot\widetilde{E}|^2\right\}dx \\
&\qquad + \int_{R^3}\left\{\big(2-\tau^2\big)\big(\widetilde{w_a}_t^2 + \widetilde{w_b}_t^2\big) + P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2 + P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2\right. \\
&\qquad \left. + \frac{\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2 + |\Delta\widetilde{w_b}|^2\big) + \frac{1}{4}|\nabla\cdot\widetilde{E}|^2\right\}dx \\
&\quad = \int_{R^3}\big\{F_a\cdot(\widetilde{w_a} + 2\widetilde{w_{at}}) + F_b\cdot(\widetilde{w_b} + 2\widetilde{w_{bt}})\big\}\,dx + \frac{1}{4}\int_{R^3}\nabla\cdot\widetilde{E}D^\alpha\big(w_a^2 - w_b^2\big)\,dx
\end{aligned}$$

[1] We can first assume the solution (w_a, w_b, u_a, u_b) has higher-order regularity so that we can take derivatives since the final a priori estimation will be still valid for these solutions by applying the Friedrich mollifier to (w_a, w_b, u_a, u_b).

$$+\frac{1}{2}\int_{R^3}\nabla\cdot\widetilde{E}D^\alpha\big(w_a^2-w_b^2\big)_t\,dx-\frac{1}{2}\int_{R^3}w_a\nabla\cdot\widetilde{E}(\widetilde{w_a}+2\widetilde{w_{at}})\,dx$$
$$+\frac{1}{2}\int_{R^3}w_b\nabla\cdot\widetilde{E}(\widetilde{w_b}+2\widetilde{w_{bt}})\,dx. \tag{3.27}$$

Similar with the analysis of basic estimates, using Moser type inequality Lemmas 2.9, 2.10 and the prior assumptions (3.13)–(3.17) and using Hölder inequality, Young's inequality to estimate the terms of the right-hand side of (3.27), we can arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3}\bigg\{\tau^2\widetilde{w_a}_t^2+\tau^2\widetilde{w_a}\widetilde{w_{at}}+\frac{1}{2}\widetilde{w_a}^2+\tau^2\widetilde{w_b}_t^2+\tau^2\widetilde{w_b}\widetilde{w_{bt}}+\frac{1}{2}\widetilde{w_b}^2+P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2\\
&\quad+P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2+\frac{\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2+|\Delta\widetilde{w_b}|^2\big)+\frac{1}{4}|\nabla\cdot\widetilde{E}|^2\bigg\}\,dx\\
&\quad+\int_{R^3}\bigg\{\big(2-\tau^2\big)\big(\widetilde{w_a}_t^2+\widetilde{w_b}_t^2\big)+P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2+P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2\\
&\quad+\frac{\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2+|\Delta\widetilde{w_b}|^2\big)+\frac{1}{4}|\nabla\cdot\widetilde{E}|^2\bigg\}\,dx\\
&\leqslant c\delta_T^{\frac{1}{2}}\big\|(\nabla w_a,\nabla w_b,w_{at},w_{bt},\phi_a,\phi_b,\nabla\cdot E)\big\|_3^2+c\delta_T^{\frac{1}{2}}\big\|\big(D^5w_a,D^5w_b\big)\big\|^2.
\end{aligned}\tag{3.28}$$

Note that we cannot deal with the last term in (3.28) by the energy of left-hand side now, so we have to do the highest-order estimates in different way in order to overcome the difficulty.

Step 1.3. The highest-order estimates for w_a, w_b. Taking $|\alpha|=4$, we can get the equations for $\widetilde{w_a}:=D^\alpha w_a$, $\widetilde{w_b}:=D^\alpha w_b$ and $\widetilde{E}:=D^\alpha E$. We also use the form of (3.25) for simplicity. This time, using $(\widetilde{w_a}+2\tau\widetilde{w_{at}})$ to multiply $(3.25)_{i=a}$ and $(\widetilde{w_b}+2\tau\widetilde{w_{bt}})$ to multiply $(3.25)_{i=b}$ but for $|\alpha|=4$. We can get as former

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3}\bigg\{\tau^3\widetilde{w_a}_t^2+\tau^2\widetilde{w_a}\widetilde{w_{at}}+\frac{1}{2}\widetilde{w_a}^2+\tau^3\widetilde{w_b}_t^2+\tau^2\widetilde{w_b}\widetilde{w_{bt}}+\frac{1}{2}\widetilde{w_b}^2+\tau P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2\\
&\quad+\tau P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2+\frac{\tau\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2+|\Delta\widetilde{w_b}|^2\big)+\frac{\tau}{4}|\nabla\cdot\widetilde{E}|^2\bigg\}\,dx\\
&\quad+\int_{R^3}\bigg\{\big(2\tau-\tau^2\big)\big(\widetilde{w_a}_t^2+\widetilde{w_b}_t^2\big)+P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2+P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2\\
&\quad+\frac{\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2+|\Delta\widetilde{w_b}|^2\big)+\frac{1}{4}|\nabla\cdot\widetilde{E}|^2\bigg\}\,dx\\
&=\int_{R^3}\big\{F_a\cdot(\widetilde{w_a}+2\tau\widetilde{w_{at}})+F_b\cdot(\widetilde{w_b}+2\tau\widetilde{w_{bt}})\big\}\,dx+\frac{1}{4}\int_{R^3}\nabla\cdot\widetilde{E}D^\alpha\big(w_a^2-w_b^2\big)\,dx
\end{aligned}$$

$$+\frac{1}{2}\int\limits_{R^3} \tau\nabla\cdot\widetilde{E}D^\alpha\big(w_a^2-w_b^2\big)_t\,dx-\frac{1}{2}\int\limits_{R^3} w_a\nabla\cdot\widetilde{E}(\widetilde{w_a}+2\tau\widetilde{w}_{at})\,dx$$
$$+\frac{1}{2}\int\limits_{R^3} w_b\nabla\cdot\widetilde{E}(\widetilde{w_b}+2\tau\widetilde{w}_{bt})\,dx. \tag{3.29}$$

In the right-hand side of (3.29), the terms multiplied by $2\tau\widetilde{w}_{at}, 2\tau\widetilde{w}_{bt}$ need a special analysis. Taking $i=a$ for example, the key terms are analyzed as

$$\int\limits_{R^3}\big[P_a'\big(\big(w_a+\sqrt{\rho_a^*}\big)^2\big)-P_a'\big(\rho_a^*\big)\big]\Delta\widetilde{w_a}\cdot 2\tau\widetilde{w}_{at}\,dx$$
$$\leqslant-\frac{d}{dt}\int\limits_{R^3}\tau\big[P_a'\big(\big(w_a+\sqrt{\rho_a^*}\big)^2\big)-P_a'\big(\rho_a^*\big)\big]|\nabla\widetilde{w_a}|^2\,dx+c\delta_T^{\frac{1}{2}}\|\nabla\widetilde{w_a}\|^2$$
$$+c\delta_T^{\frac{1}{2}}\tau\|\widetilde{w}_{at}\|^2, \tag{3.30}$$

$$\int\limits_{R^3}\tau^2u_a\nabla\widetilde{w}_{at}\cdot 2\tau\widetilde{w}_{at}\,dx=-\int\limits_{R^3}\tau^3\nabla\cdot u_a|\widetilde{w}_{at}|^2\,dx\leqslant c\delta_T^{\frac{1}{2}}\tau\|\widetilde{w}_{at}\|^2, \tag{3.31}$$

and

$$\int\limits_{R^3}\tau^2u_a\nabla(u_a\cdot\nabla\widetilde{w_a})\cdot 2\tau\widetilde{w}_{at}\,dx$$
$$\leqslant-\frac{d}{dt}\int\limits_{R^3}\tau(\tau u_a\cdot\nabla\widetilde{w_a})^2\,dx+\int\limits_{R^3}2\tau^3(u_a\cdot\nabla\widetilde{w_a})u_{at}\nabla\widetilde{w_a}\,dx$$
$$+c\delta_T^{\frac{1}{2}}\|\nabla\widetilde{w_a}\|^2+c\delta_T^{\frac{1}{2}}\tau\|\widetilde{w}_{at}\|^2. \tag{3.32}$$

The other terms in the right-hand side of (3.29) can be analyzed just use Moser type Lemmas 2.9, 2.10, the assumptions (3.13)–(3.17) and the Sobolev embedding theorem, the Hölder inequality, Young's inequality. In a word, these estimates with the above estimates (3.29)–(3.32) will lead to

$$\frac{d}{dt}\int\limits_{R^3}\bigg\{\tau^3\widetilde{w}_{at}^2+\tau^2\widetilde{w}_a\widetilde{w}_{at}+\frac{1}{2}\widetilde{w}_a^{\,2}+\tau^3\widetilde{w}_{bt}^2+\tau^2\widetilde{w}_b\widetilde{w}_{bt}+\frac{1}{2}\widetilde{w}_b^{\,2}+\tau P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2$$
$$+\tau P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2+\frac{\tau\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2+|\Delta\widetilde{w_b}|^2\big)+\frac{\tau}{4}|\nabla\cdot\widetilde{E}|^2\bigg\}\,dx$$
$$+\frac{d}{dt}\int\limits_{R^3}\tau\big[P_a'\big(\big(w_a+\sqrt{\rho_a^*}\big)^2\big)-P_a'\big(\rho_a^*\big)\big]|\nabla\widetilde{w_a}|^2\,dx+\frac{d}{dt}\int\limits_{R^3}\tau(\tau u_a\cdot\nabla\widetilde{w_a})^2\,dx$$
$$+\frac{d}{dt}\int\limits_{R^3}\tau\big[P_b'\big(\big(w_b+\sqrt{\rho_b^*}\big)^2\big)-P_b'\big(\rho_b^*\big)\big]|\nabla\widetilde{w_b}|^2\,dx+\frac{d}{dt}\int\limits_{R^3}\tau(\tau u_b\cdot\nabla\widetilde{w_b})^2\,dx$$

$$
\begin{aligned}
&+\int\limits_{R^3}\Big\{\big(2\tau-\tau^2\big)\big(\widetilde{w_a}_t^2+\widetilde{w_b}_t^2\big)+P_a'\big(\rho_a^*\big)|\nabla\widetilde{w_a}|^2+P_b'\big(\rho_b^*\big)|\nabla\widetilde{w_b}|^2\\
&+\frac{\varepsilon^2}{4}\big(|\Delta\widetilde{w_a}|^2+|\Delta\widetilde{w_b}|^2\big)+\frac{1}{4}|\nabla\cdot\widetilde{E}|^2+2\tau^3(u_a\cdot\nabla\widetilde{w_a})u_{at}\nabla\widetilde{w_a}\Big\}\,dx\\
&\leqslant c\delta_T^{\frac{1}{2}}\big\|(\nabla w_a,\nabla w_b)\big\|_4^2+c\delta_T^{\frac{1}{2}}\varepsilon^2\big\|\big(D^6w_a,D^6w_b\big)\big\|^2+c\delta_T^{\frac{1}{2}}\|\nabla\cdot E\|_4^2\\
&+c\delta_T^{\frac{1}{2}}\tau\big\|\big(D^4w_{at},D^4w_{bt}\big)\big\|^2+c\delta_T^{\frac{1}{2}}\big\|(\phi_a,\phi_b)\big\|_4^2.
\end{aligned}
\tag{3.33}
$$

Note that the right-hand side of the estimates (3.24), (3.28), (3.33) will be treated later in terms of the estimates of ϕ_a, ϕ_b.

Step 2. The estimates for $\boldsymbol{\phi_a}$, $\boldsymbol{\phi_b}$. Differentiating Eqs. (3.6) and (3.7) for ϕ_a, ϕ_b with respect to x, then $\widetilde{\phi_a} = D^\alpha \phi_a, \widetilde{\phi_b} = D^\alpha \phi_b$ ($|\alpha| \leqslant 4$) will satisfy (taking ϕ_a for example)

$$
\tau^2\widetilde{\phi_a}_t+\widetilde{\phi_a}=D^\alpha f_{a2},
\tag{3.34}
$$

recall f_{a2} in (3.10) for $i = a$. Taking inner product between $2\widetilde{\phi_a}$ and (3.34), integrating over R^3, we obtain

$$
\tau^2\frac{d}{dt}\int\limits_{R^3}|\widetilde{\phi_a}|^2\,dx+2\int\limits_{R^3}|\widetilde{\phi_a}|^2\,dx=\int\limits_{R^3}D^\alpha f_{a2}\cdot 2\widetilde{\phi_a}\,dx.
\tag{3.35}
$$

The terms in right-hand side of (3.35) can be estimated using Moser type Lemmas 2.9, 2.10, Young's inequality and the assumptions (3.13)–(3.17) and the inequality $\|Du\| \leqslant c(\|\nabla\cdot u\| + \|\nabla \times u\|)$ and also the presentation of $\nabla \cdot u_a$ by $w_{at}, \nabla w_a$ through Eq. (3.12) for $i = a$. Then we deduce

$$
\tau^2\frac{d}{dt}\int\limits_{R^3}|\widetilde{\phi_a}|^2\,dx+2\int\limits_{R^3}|\widetilde{\phi_a}|^2\,dx\leqslant c\delta_T^{\frac{1}{2}}\|\phi_a\|_4^2+c\delta_T^{\frac{1}{2}}\tau\|w_{at}\|_4^2+c\delta_T^{\frac{1}{2}}\|\nabla w_a\|_4^2.
\tag{3.36}
$$

Step 3. The closure of energy estimates. The assumption $\delta_T \ll 1$ and the combination of the estimates (3.24), (3.28) for all $|\alpha| \leqslant 3$, and (3.33) for all $|\alpha| = 4$, (3.36) for all $|\alpha| \leqslant 4$ can give us

$$
\frac{d}{dt}H_1(t)+H_2(t)\leqslant\sum_{i=a,b}\tau\big\|u_i(.,t)\big\|_{L^\infty(R^3)}\cdot\big\|\tau^2u_{it}(.,t)\big\|_{L^\infty(R^3)}\cdot\big\|D^5w_i\big\|^2
\tag{3.37}
$$

where $H_1(t)$, $H_2(t)$ are two terms satisfying

$$
0<c_1E_1(t)<H_1(t)<c_2E_1(t),\qquad 0<c_3E_2(t)<H_2(t)<c_4E_2(t)
$$

for $t \in [0, T]$, and c_1, c_2, c_3, c_4 are positive constants independent of ε, τ, the $E_1(t)$, $E_2(t)$ are the terms defined in Lemma 3.2. From (3.37) we can write

$$
\frac{d}{dt}H_1(t)+H_2(t)\leqslant cg(t)H_1(t),\quad t\in[0,T],
\tag{3.38}
$$

with

$$g(t)=\sum_{i=a,b}\left\|u_i(.,t)\right\|_{L^\infty(R^3)}\cdot 2\left\|\tau^2 u_{it}(.,t)\right\|_{L^\infty(R^3)}.$$

The assumption (3.13) then (3.16) with the Gronwall inequality applying to (3.38) makes us know

$$H_1(t)\leqslant ce^{\int_0^t g(s)\,ds}H_1(0)\leqslant ce^{c\delta_T}H_1(0)\leqslant CH_1(0) \tag{3.39}$$

for $t\in[0,T]$ provided $\delta_T\ll 1$. Integrating (3.38) on $[0,t]$ and using (3.39), we derive

$$\int_0^t H_2(s)\,ds\leqslant H_1(0)+H_1(t)+C\delta_T H_1(0)\leqslant C'H_1(0). \tag{3.40}$$

The above constants c, C and C' denote the positive constant independent of the parameters $\varepsilon>0$, $\tau>0$.

It follows from (3.39), (3.40) and the equivalence between $H_1(t)$ and $E_1(t)$, and between $H_2(t)$ and $E_2(t)$ the conclusion stated in Lemma 3.2. Thus the proof of Lemma 3.2 is completed. □

3.3. The global existence and asymptotical limits

The proof of global existence (Theorem 2.1). Theorem 2.1 is a direct conclusion of the combination of the local existence theory Lemma 3.1 and global a priori estimates Lemma 3.2 in terms of the variable transformation presented above and the standard continuity argument, we omit the details. □

The proof of semiclassical limit (Theorem 2.3). Starting from Lemma 3.2, using a continuity argument, one can easily prove the existence of the global in-time solutions of the original problem (1.10)–(1.12) and (2.3) with any small ε and τ provided the $\Lambda_1>0$ then $\Lambda_0>0$ small enough.

Let $(\psi_a^\varepsilon,\psi_b^\varepsilon,u_a^\varepsilon,u_b^\varepsilon,E^\varepsilon)$ be the solution of (1.10)–(1.12) and (2.3), then from Lemma 3.2 and the Poisson equation (3.8) the uniform estimates to ε hold

$$\begin{aligned}&\sum_{k=0}^{1}\left\|\left(\partial_t^k\left(\psi_a^\varepsilon-\sqrt{\rho_a^*}\right),\partial_t^k\left(\psi_b^\varepsilon-\sqrt{\rho_b^*}\right)\right)(.,t)\right\|_{5-i}^2+\sum_{k=0}^{1}\left\|\left(\partial_t^k u_a^\varepsilon,\partial_t^k u_b^\varepsilon\right)(.,t)\right\|_{\mathcal{H}^{5-2i}}^2\\&\quad+\left\|E^\varepsilon(.,t)\right\|_{\mathcal{H}^6}^2\leqslant c\Lambda_0,\end{aligned} \tag{3.41}$$

$$\int_0^t\left\{\left\|\left(\left(\psi_a^\varepsilon-\sqrt{\rho_a^*}\right),\left(\psi_b^\varepsilon-\sqrt{\rho_b^*}\right)\right)(.,s)\right\|_5^2+\left\|\left(\partial_t\psi_a^\varepsilon,\partial_t\psi_b^\varepsilon\right)(.,s)\right\|_4^2\right\}ds\leqslant c\Lambda_0 t, \tag{3.42}$$

$$\int_0^t\left\{\sum_{k=0}^{1}\left\|\left(\partial_t^k u_a^\varepsilon,\partial_t^k u_b^\varepsilon\right)(.,s)\right\|_{\mathcal{H}^{5-2i}}^2+\sum_{k=0}^{1}\left\|\left(\partial_t^k E^\varepsilon\right)(.,s)\right\|_{\mathcal{H}^{6-i}}^2\right\}ds\leqslant c\Lambda_0 \tag{3.43}$$

for any $t>0$. The right-hand sides of the above inequalities are independent of ε. Thus these uniform estimates and Aubin's lemma imply the existence of subsequence denoted also by $(\psi_a^\varepsilon, \psi_b^\varepsilon, u_a^\varepsilon, u_b^\varepsilon, E^\varepsilon)$ such that

$$\psi_a^\varepsilon \to \psi_a, \qquad \psi_b^\varepsilon \to \psi_b \quad \text{in } C\big(0,t; C_b^3 \cap H_{\mathrm{loc}}^{5-s}\big(R^3\big)\big), \tag{3.44}$$

$$u_a^\varepsilon \to u_a, \qquad u_b^\varepsilon \to u_b \quad \text{in } C\big(0,t; C_b^3 \cap \mathcal{H}_{\mathrm{loc}}^{5-s}\big(R^3\big)\big), \tag{3.45}$$

$$E^\varepsilon \to E \quad \text{in } C\big(0,t; C_b^4 \cap \mathcal{H}_{\mathrm{loc}}^{6-s}\big(R^3\big)\big), \tag{3.46}$$

with $s \in (0, \frac{1}{2})$, as $\varepsilon \to 0$. We also have

$$\frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi_i^\varepsilon}{\psi_i^\varepsilon}\right) \to 0 \quad \text{in } L^2\big(0,t; H_{\mathrm{loc}}^3\big(R^3\big)\big),$$

as $\varepsilon \to 0$. Thus (3.41)–(3.46) allow the ε pass to the zero, and the limiting solutions satisfy

$$\begin{gathered}
2\psi_a\partial_t\psi_a + \nabla\cdot\big(\psi_a^2 u_a\big) = 0,\\
\tau^2\partial_t\big(\psi_a^2 u_a\big) + \tau^2\nabla\big(\psi_a^2 u_a \otimes u_a\big) + \nabla P_a\big(\psi_a^2\big) + \psi_a^2 u_a - \psi_a^2 E = 0,\\
2\psi_b\partial_t\psi_b + \nabla\cdot\big(\psi_b^2 u_b\big) = 0,\\
\tau^2\partial_t\big(\psi_b^2 u_b\big) + \tau^2\nabla\big(\psi_b^2 u_b \otimes u_b\big) + \nabla P_b\big({\psi_b}^2\big) + {\psi_b}^2 u_b + \psi_b^2 E = 0,\\
\lambda^2\nabla\cdot E = \psi_a^2 - \psi_b^2 - \mathcal{C}, \quad \nabla\times E = 0.
\end{gathered}$$

Let $\rho_a = (\psi_a)^2$, $\rho_b = (\psi_b)^2$. It is easily to verify that (ρ_a, ρ_b, E) solves the bipolar hydrodynamic model (1.6)–(1.8). The convergence of the bipolar quantum QHD model to bipolar hydrodynamic model is established, and the proof of Theorem 2.3 is complete. □

The proof of combined semiclassical and relaxation limits (Theorem 2.4). Since the estimates established for the solutions in Lemma 3.2 hold uniformly for any small ε and τ, thus we can study the combined limits as both ε and τ tends to zero freely.

Let $(\psi_a^{(\tau,\varepsilon)}, \psi_b^{(\tau,\varepsilon)}, u_a^{(\tau,\varepsilon)}, u_b^{(\tau,\varepsilon)}, E^{(\tau,\varepsilon)})$ be the global solution derived in Theorem 2.1, by the estimates (3.18) in Lemma 3.2, we have the uniform estimates about ε and τ as

$$\big\|\big(\psi_a^{(\tau,\varepsilon)} - \sqrt{\rho_a^*}, \psi_b^{(\tau,\varepsilon)} - \sqrt{\rho_b^*}\big)(.,t)\big\|_4^2 + \big\|\big(\tau u_a^{(\tau,\varepsilon)}, \tau u_b^{(\tau,\varepsilon)}\big)(.,t)\big\|_{\mathcal{H}^4}^2 \leqslant c\Lambda_0, \tag{3.47}$$

$$\big\|\big(\tau\partial_t\psi_a^{(\tau,\varepsilon)}, \tau\partial_t\psi_b^{(\tau,\varepsilon)}\big)(.,t)\big\|_3^2 + \big\|E^{(\tau,\varepsilon)}(.,t)\big\|_{\mathcal{H}^5}^2 \leqslant c\Lambda_0, \tag{3.48}$$

and

$$\int_0^t \big(\big\|\big(\psi_a^{(\tau,\varepsilon)} - \sqrt{\rho_a^*}, \psi_b^{(\tau,\varepsilon)} - \sqrt{\rho_b^*}\big)(.,s)\big\|_5^2 + \big\|\big(\partial_t\psi_a^{(\tau,\varepsilon)}, \partial_t\psi_b^{(\tau,\varepsilon)}\big)(.,s)\big\|_3^2\big)\,ds \leqslant c\Lambda_0 t, \tag{3.49}$$

$$\int_0^t \big(\big\|\big(u_a^{(\tau,\varepsilon)}, u_b^{(\tau,\varepsilon)}\big)(.,s)\big\|_{\mathcal{H}^4}^2 + \big\|E^{(\tau,\varepsilon)}(.,s)\big\|_{\mathcal{H}^5}^2\big)\,ds \leqslant c\Lambda_0 \tag{3.50}$$

for any $t>0$.

Also use Aubin's lemma with the above uniform estimates, we can get the subsequence (not relabeled) and functions denoted also by ψ_a, ψ_b, u_a, u_b, E such that as $\varepsilon, \tau \to 0$ r

$$\psi_a^{(\tau,\varepsilon)} \to \psi_a, \qquad \psi_b^{(\tau,\varepsilon)} \to \psi_b \quad \text{in } C\big(0, t; C_b^2 \cap H_{\mathrm{loc}}^{4-s}\big(R^3\big)\big), \tag{3.51}$$

$$u_a^{(\tau,\varepsilon)} \rightharpoonup u_a, \qquad u_b^{(\tau,\varepsilon)} \rightharpoonup u_b \quad \text{weakly in } L^2\big(0, t; \mathcal{H}^4\big(R^3\big)\big), \tag{3.52}$$

$$E^{(\tau,\varepsilon)} \to E \quad \text{in } C\big(0, t; C_b^3 \cap \mathcal{H}_{\mathrm{loc}}^{5-s}\big(R^3\big)\big) \tag{3.53}$$

for any $t > 0$ and $s \in (0, \frac{1}{2})$.

From (3.47), (3.48) we know ψ_a, ψ_b are positive in $(0, t) \times R^3$, and also

$$\tau^2\big|u_a^{(\tau,\varepsilon)}\big|^2 \to 0, \qquad \tau^2\big|u_b^{(\tau,\varepsilon)}\big|^2 \to 0 \quad \text{in } L^1\big(0, t; W_{\mathrm{loc}}^{3,3}\big(R^3\big)\big), \text{ as } \tau, \varepsilon \to 0. \tag{3.54}$$

Thus the above converging results allow the solutions to pass to the limit $\tau, \varepsilon \to 0$ from the bipolar QHD model to the bipolar drift–diffusion (DD) model:

$$\begin{aligned}
&2\psi_a \partial_t \psi_a - \nabla \cdot \big[\nabla P_a\big((\psi_a)^2\big) - (\psi_a)^2 E\big] = 0,\\
&2\psi_b \partial_t \psi_b - \nabla \cdot \big[\nabla P_b\big((\psi_b)^2\big) + (\psi_b)^2 E\big] = 0,\\
&\lambda^2 \nabla \cdot E = (\psi_a)^2 - (\psi_b)^2 - \mathcal{C}, \quad \nabla \times E = 0,
\end{aligned}$$

which is equivalent to the bipolar DD model (1.15)–(1.16) in Section 1 for strong solution. Namely, $(\rho_a = (\psi_a)^2, \rho_b = (\psi_b)^2, E)$ solves the bipolar drift–diffusion model (1.15)–(1.16). The proof of Theorem 2.4 is completed. $\square$

Acknowledgments

The authors acknowledge the partial support by the National Science Foundation of China (No. 10571102), the Key Research Project on Science and Technology of the Ministry of Education of China (No. 104072), the grant NNSFC (No. 10431060), Beijing Nova program, the Institute of Mathematics and Interdisciplinary Science at CNU, the NCET support of the Ministry of Education of China, the Huo Ying Dong Foundation (No. 111033).

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