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Entropy and temperatures of Nariai black hole

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ABSTRACT

The statistical entropy of the Nariai black hole in a thermal equilibrium is calculated by using the brick-wall method. Even if the temperature depends on the choice of the timelike Killing vector, the entropy can be written by the ordinary area law which agrees with the Wald entropy. We discuss some physical consequences of this result and the properties of the temperatures.

1. Introduction

It has been claimed that the entropy of a black hole is proportional to the surface area at the event horizon [1], and the Schwarzschild black hole has been studied through the quantum field theoretic calculation [2]. One of the convenient methods to get the entropy is to use the brick-wall method, which gives the statistical entropy satisfying the area law of the black hole [3]. Then, there have been extensive applications to various black holes [4–26]. In fact, there is another way to obtain the entropy which regards the entropy of black holes as the conserved Noether charge corresponding to the symmetry of time translation [27]. For the Einstein gravity, the Wald entropy is always given by the $A_H/(4G)$, where A_H and G are the surface area at the event horizon and the Newton's gravitational constant, respectively. Actually, there are many extensive studies for the entropy as the Noether charge in the general theory of gravity including the higher power of the curvature [28–33].

The fact that the cosmological constant seems to be positive in our universe deserves to study the Schwarzschild black hole on the de Sitter background, which can be easily realized in the form of the Schwarzschild–de Sitter (SdS) spacetime. It has the black hole horizon and the cosmological horizon, and the observer lives between them. In this spacetime, the temperature of the black hole is different from the temperature due to the cosmological horizon when $0 < M < 1/(3\sqrt{\Lambda})$, where M and Λ are the mass of the black hole and the cosmological constant, respectively. Therefore, we are in trouble to study the thermodynamics of the system since it is not in thermally equilibrium due to the different temperatures. Nevertheless, there are several studies for the entropy through the improved brick-wall method for the SdS black hole [34–36] and the Kerr–de Sitter black hole [37]. In order to avoid the difficulty due to the non-equilibrium state of the SdS black hole, they have considered two thin-layers near the black hole horizon and the cosmological horizon, and then calculated the entropy for each thin layer.

On the other hand, for the special limit of $M = 1/(3\sqrt{\Lambda})$ in the SdS spacetime, the two horizons are coincident in the Schwarzschild coordinate. However, the Nariai metric is obtained through the coordinate transformation to avoid the coordinate singularity, where the two horizons are still separated [38–41]. The Nariai spacetime is in thermally equilibrium since the black hole and the cosmological horizon give the same temperatures. Thus, we can treat the whole Nariai spacetime as one thermodynamical system. However, even in spite of thermodynamic equilibrium, there are few thermodynamic studies. Moreover, one can define two kinds of temperatures for the Nariai black hole: the Bousso–Hawking temperature and the Hawking temperature since there exist two different normalizations of timelike Killing vectors [39].

In this Letter, we would like to study the statistical entropy of the Nariai black hole by using the brick-wall method. In Section 2, we introduce the SdS spacetime and the Nariai spacetime, and define two kinds of temperatures based on the different normalizations of the Killing vectors. We will also apply the Wald formula to the Nariai black hole in order to get the entropy without resort to

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normalizations of the Killing vector. Since both of the surface gravity and the Noether potential are proportional to the normalization constant of the Killing vector, the Wald entropy is independent to the normalization of the Killing vector. In Section 3, the entropy will be calculated by using the brick-wall method. Although the energy and the temperature depend on the normalization of the timelike Killing vector, the normalization-independent statistical entropy can be obtained, which is compatible with the Wald entropy. Finally, summary and discussion are given in Section 4.

2. Temperatures and Wald entropy in Nariai black hole

Let us start with the four-dimensional Einstein–Hilbert action with the cosmological constant Λ, which is given by

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R-2\Lambda). \tag{1}$$

The equation of motion obtained from the action (1) becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{2}$$

The static and spherically symmetric solution of Eq. (2) is written as

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega_2^2, \tag{3}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2, \tag{4}$$

where Ω_2 is two-dimensional solid angle defined by $d\Omega_2^2 = d\theta^2 + \sin^2\theta\,d\phi^2$. Hereafter, we will consider only the Schwarzschild–de Sitter spacetime with $\Lambda > 0$. For $0 < M \leqslant 1/(3\sqrt{\Lambda})$, it has two horizons of the black hole horizon r_b and the cosmological horizon r_c. In this case, the metric function (4) can be neatly written as $f(r) = (1 - r_b/r)[1 - (\Lambda/3)(r^2 + r_b r + r_b^2)] = (r - r_b)(r_c - r)(r + r_b + r_c)/[r(r_b^2 + r_b r_c + r_c^2)]$. For $M = 0$, it has only the cosmological horizon with $r_c = \sqrt{3/\Lambda}$.

The symmetry of time translation in the SdS spacetime can be described by a timelike Killing vector, which is written as

$$\xi = \gamma_t \frac{\partial}{\partial t}, \tag{5}$$

where γ_t is a normalization constant. In the standard normalization, γ_t is obtained from the condition to satisfy $\xi^\mu \xi_\mu = -1$ at the asymptotically flat Minkowski spacetime. For instance, its value usually becomes $\gamma_t = 1$ for a Schwarzschild metric. In the SdS spacetime, there is no asymptotically flat region, so that we should consider the reference point r_g where the gravitational acceleration vanishes due to the balance between the forces of the black hole by the mass and the cosmological horizon by the cosmological constant. Thus, we can choose the normalization constant in Eq. (5) to satisfy $\xi^\mu \xi_\mu = -1$ at that reference point r_g, which yields

$$\gamma_t = \frac{1}{\sqrt{f(r_g)}}, \tag{6}$$

where the reference point can be found from $f'(r_g) = 0$ and is explicitly given by $r_g = (3M/\Lambda)^{1/3}$. Now, the surface gravities κ_b and κ_c on the black hole horizon and the cosmological horizon are written as

$$\kappa_{b,c} = \lim_{r \to r_{b,c}} \sqrt{\frac{\xi^\mu \nabla_\mu \xi_\nu \xi^\rho \nabla_\rho \xi^\nu}{-\xi^2}}, \tag{7}$$

respectively. Then, the temperatures along with the normalization (6) are calculated as

$$T_{\rm BH}^{b,c} = \frac{\kappa_{b,c}}{2\pi} = \frac{f'(r_{b,c})}{4\pi\sqrt{f(r_g)}}, \tag{8}$$

which are called the Bousso–Hawking temperatures [39]. This temperature can be also obtained from $\tilde{\xi} = \partial/\partial\tilde{t}$ when the time is rescaled as $\tilde{t} = t\sqrt{f(r_g)}$, where $\xi^\mu \xi_\mu = -1$ is satisfied at $r = r_g$.

On the other hand, in the Euclidean geometry, the Hawking temperature agrees with the inverse of the period of the Euclidean time to avoid a conical singularity at the horizon. Setting the Euclidean time τ to $\tau = it$, the Euclidean line element of Eq. (3) is written as

$$ds_E^2 = f(r)\,d\tau^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega_2^2. \tag{9}$$

From Eq. (9), the Hawking temperatures for the black hole horizon and the cosmological horizon become

$$T_{\rm H}^{b,c} = \beta_{\rm H}^{-1} = \frac{f'(r_{b,c})}{4\pi}, \tag{10}$$

respectively, which agree with the temperatures obtained from the Killing vector (5) with the normalization constant $\gamma_t = 1$. Note that the Hawking temperature (10) is definitely different from the Bousso–Hawking temperature (8). For the scaled Euclidean time given by $\tilde{\tau} = i\tilde{t}$, the Bousso–Hawking temperatures are obtained.

Similar argument can be done for the Nariai black hole by taking the limit of $M = 1/(3\sqrt{\Lambda})$ in Eq. (4) so that the two horizons are coincident in the Schwarzschild coordinate. In this degenerate case with $r_b = r_c$, the metric (3) should be transformed to an appropriate coordinate system because it has the coordinate singularity and becomes inappropriate. Near the degenerate case, the mass can be written as [38–40]

$$9M^2\Lambda = 1 - 3\epsilon^2, \quad 0 \leqslant \epsilon \ll 1, \tag{11}$$

where the degenerate case can be obtained by taking $\epsilon = 0$. One can define the new time and the radial coordinate ψ and χ by

$$t = \frac{1}{\epsilon\sqrt{\Lambda}}\psi, \qquad r = \frac{1}{\sqrt{\Lambda}}\left(1 - \epsilon\cos\chi - \frac{1}{6}\epsilon^2\right). \tag{12}$$

In terms of the new coordinates (12), the line element (3) is written in the form of

$$ds^2 = \frac{1}{\Lambda}\left[-\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi\,d\psi^2 + \left(1 - \frac{2}{3}\epsilon\cos\chi\right)d\chi^2 + (1 - 2\epsilon\cos\chi)\,d\Omega_2^2\right], \tag{13}$$

up to the first order in ϵ. For the case of $\epsilon = 0$, Eq. (13) is called the Nariai metric, which is given by

$$ds^2 = \frac{1}{\Lambda}\left(-\sin^2\chi\,d\psi^2 + d\chi^2 + d\Omega_2^2\right). \tag{14}$$

In this coordinate system, the back hole horizon and the cosmological horizon correspond to $\chi = 0$ and $\chi = \pi$, respectively, where the proper distance between the two horizons is given by $\pi/\sqrt{\Lambda}$ which is not zero. From now on, we will study this Nariai black hole which is actually real geometry to describe thermal equilibrium since the horizon temperature is the same with the cosmological temperature. However, there are two kinds of temperatures depending on the definitions of the normalization of the Killing vector.

With Eqs. (11) and (12), the Killing vector (5) becomes

$$\xi = \sqrt{\Lambda}\frac{\partial}{\partial \psi}, \tag{15}$$

to the leading order in ϵ. Using Eq. (7), the Bousso–Hawking temperature is calculated as

$$T_{\rm BH}^{b,c} = \frac{\sqrt{\Lambda}}{2\pi}. \tag{16}$$

This can be also obtained from the Killing vector $\tilde{\xi} = \partial/\partial\tilde{\psi}$ at the coordinate system with the rescaled time $\tilde{\psi} = \psi/\sqrt{\Lambda}$. As expected, the temperature of the black hole horizon is the same with that of the cosmological horizon. On the other hand, one can also get the Hawking temperature from the Euclidean metric (14) by setting the Euclidean time as $\psi_E = i\psi$. Then, the Euclidean Nariai metric can be written as

$$ds_E^2 = \frac{1}{\Lambda}\left(\sin^2\chi\, d\psi_E^2 + d\chi^2 + d\Omega_2^2\right). \tag{17}$$

In order to avoid a conical singularity at the two horizons, the period of the Euclidean time for the black hole horizon or the cosmological horizon are chosen as 2π, respectively. Then the Hawking temperatures are given by

$$T_H^{b,c} = \beta_H^{-1} = \frac{1}{2\pi}, \tag{18}$$

which corresponds to the surface gravity obtained from the Killing vector $\partial/\partial\psi$ using Eq. (7). It is interesting to note that the Hawking temperature is constant as long as the Nariai condition $M = 1/(3\sqrt{\Lambda})$ is met. Moreover, it can be easily checked that the Bousso–Hawking temperature (16) is obtained from the condition to avoid a conical singularity at the horizons for the scaled Euclidean time $\tilde{\psi}_E = \psi_E/\sqrt{\Lambda}$.

Before we get down to the brick-wall calculations for statistical entropy, we will identify the form of the entropy in terms of the Wald formula which comes from the Noether current associated with the above mentioned Killing vectors. At first sight, it depends on the Killing vectors since it is related to the symmetry. In our case, there are two kinds of Killing vectors which give two distinct temperatures so that it is necessary to justify whether the entropy depends on normalizations of the Killing vector or not. Since both of the surface gravity and the Noether potential are proportional to the normalization constant of the Killing vector, they are canceled out and the Wald entropy eventually turns out to be independent of the normalization of the Killing vector. It means that the entropy by the brick-wall method may be also independent of the normalization of the Killing vectors and it will be explicitly shown later how the statistical entropy is also independent of the Killing vector.

In order to find the Wald entropy of the Nariai spacetime, one should consider a diffeomorphism invariance with the Killing vector ξ^μ which is associated with the conservation law of $\nabla_\mu J^\mu = 0$ [27–32], for which the Noether potential $J^{\mu\nu}$ can be defined by $J^\mu = \nabla_\nu J^{\mu\nu}$. If a Lagrangian is written in the form of $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma})$, then the Noether potential is given by [31,32]

$$J^{\mu\nu} = -2\Theta^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma + 4\nabla_\rho\Theta^{\mu\nu\rho\sigma}\xi_\sigma, \tag{19}$$

where

$$\Theta^{\mu\nu\rho\sigma} = \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}. \tag{20}$$

For a timelike Killing vector, the Wald entropy [27] is expressed by

$$S = \frac{2\pi}{\kappa}\int_\Sigma d^2x\sqrt{h}\,\epsilon_{\mu\nu}J^{\mu\nu}, \tag{21}$$

where κ and $h_{\mu\nu}$ are the surface gravity and the induced metric on the hypersurface Σ of a horizon, respectively. And $\epsilon_{\mu\nu}$ is defined by

$$\epsilon_{\mu\nu} = \frac{1}{2}(n_\mu u_\nu - n_\nu u_\mu), \tag{22}$$

where n^μ is the outward unit normal vector of Σ. The proper velocity u^μ of a fiducial observer moving along the orbit of ξ^μ is given by $u^\mu = \alpha^{-1}\xi^\mu$ with $\alpha \equiv \sqrt{-\xi^\mu\xi_\mu}$. Note that the Wald entropy is independent to the normalization of the Killing vector since both of the Noether potential $J^{\mu\nu}$ in Eq. (19) and the surface gravity in Eq. (7) are proportional to the normalization constant of the Killing vector and so the constant canceled out in the formula of the Wald entropy (21).

For the Nariai metric (14), the Killing vector is given by

$$\xi = \gamma\frac{\partial}{\partial\psi}, \tag{23}$$

where γ is a normalization constant, which will be not specified in this section. From the norm of the Killing vector, we obtain $\alpha = \gamma\sin\chi/\sqrt{\Lambda}$ and $u_\mu = \xi_\mu/\alpha = -\delta_\mu^\psi\sin\chi/\sqrt{\Lambda}$. The outward unit normal vectors of the black hole horizon and the cosmological horizon are calculated as $n_\mu = (1/\sqrt{\Lambda})\delta_\mu^\chi$ and $n_\mu = -(1/\sqrt{\Lambda})\delta_\mu^\chi$, respectively. Then, the nonzero components of Eq. (22) are $\epsilon_{\psi\chi} = -\epsilon_{\chi\psi} = \pm\sin\chi/(2\Lambda)$, where the upper sign and the lower sign correspond to the black hole horizon and the cosmological horizon, respectively. Now, for the action (1), we obtain

$$\Theta^{\mu\nu\rho\sigma} = \frac{1}{32\pi G}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right), \tag{24}$$

which leads to

$$\epsilon_{\mu\nu}J^{\mu\nu} = \pm\frac{\gamma}{8\pi G}\cos\chi. \tag{25}$$

Inserting Eq. (23) into Eq. (7), we can obtain $\kappa_{b,c} = \gamma$. Then, from Eq. (21), the Wald entropy is given by

$$\begin{aligned} S &= \frac{1}{4G}\left(\int_{\Sigma_{\chi=0}} d^2x\sqrt{h}\cos\chi - \int_{\Sigma_{\chi=\pi}} d^2x\sqrt{h}\cos\chi\right) \\ &= \frac{A_b + A_c}{4G}, \end{aligned} \tag{26}$$

where A_b and A_c are the areas of the black hole horizon and the cosmological horizon, respectively. The total area given by the two horizons becomes $A = 8\pi/\Lambda$ since $A_b = A_c = 4\pi/\Lambda$. Eventually, the entropy (26) can be rewritten as

$$S = \frac{A}{4G}, \tag{27}$$

which also agrees with the Bekenstein–Hawking entropy. After all, we obtained the Wald entropy expressed by the expected area law, which is independent of the normalization of the Killing vector.

3. Entropy from brick-wall method

In the original work of the brick-wall model [3], a scalar field outside of the horizon was considered and the number of states was counted by using the WKB approximation for the wave equation. The cutoff parameter was introduced to handle the UV divergence near the horizon. Then, the free energy and the entropy were calculated at a given temperature. This method has been used to find the statistical entropy of various black holes. It has been shown that the cutoff occurs independently of the strength of the source for any D dimensions with $D > 3$, in agreement with the four-dimensional case, and it has been discussed why the cutoff depends on the strength of the source in the $D = 2$ [4]. When the brick-wall method is applied to an extremal Reissner–Nordstrom black hole, the entropy does not vanish and in fact a stronger divergence than usual [6]. In connection with quantum corrections, the entropy is written as the term proportional to the area of horizon and the logarithmic term by the area [18] and the finite entropy is obtained for the Schwarzschild–anti-de Sitter and Reissner–Nordstrom–anti-de Sitter black holes through a renormalization of the coupling constants in the one-loop effective gravitational Lagrangian [10]. In spite of the limited application to the time-dependent case, the brick-wall method was also applied to Vaidya black hole as the simplest nonstatic case assuming local equilibrium [9]. On the other hand, especially in two dimensions, it has been demonstrated that the quantum thermodynamical entropy of a black hole coincides with its statistical–mechanical entropy calculated by the brick-wall method and regularized by the Pauli–Villars scheme [7]. There are also studies for the entropies of black holes based on the generalized uncertainty principle or the modified dispersion relation using the brick-wall method [11, 13–17,22,25,26]. Furthermore, the understanding of black hole entropy by the brick-wall method has been contrasted with the understanding based on AdS/CFT correspondence [23]. So we will apply this convenient method to the Nariai black hole in what follows.

In the Nariai black hole governed by the line element (14), the black hole temperature is the same with the cosmological temperature as seen from Eqs. (16) and (18), which imply that the net flux is in fact zero. Thus, the thermal equilibrium can be realized in this special configuration, which is different from the non-equilibrium SdS black hole. In order to calculate the statistical entropy in this thermal background [3], we will consider a quantum scalar field in a box surrounded by the two horizons. The Klein–Gordon equation for the scalar field is written as

$$(\Box - m^2)\Phi = 0, \tag{28}$$

where m is the mass of the scalar field. By using the WKB approximation with $\Phi \sim \exp[-i\omega\psi + iS(\chi,\theta,\phi)]$ under the Nariai metric (14), the square module of the momentum is obtained as

$$\begin{aligned} k^2 &= g^{\mu\nu}k_\mu k_\nu \\ &= \Lambda\left(-\frac{\omega^2}{\sin^2\chi} + k_\chi^2 + k_\theta^2 + \frac{k_\phi^2}{\sin^2\theta}\right) = -m^2, \end{aligned} \tag{29}$$

where $k_\chi = \partial S/\partial\chi$, $k_\theta = \partial S/\partial\theta$, and $k_\phi = \partial S/\partial\phi$. Then, the number of quantum states with the energy less than ω is given by the volume of a phase space per its unit volume:

$$\begin{aligned} n(\omega) &= \frac{V_p}{(2\pi\hbar)^3} \\ &= \frac{1}{(2\pi)^3}\int_{V_p} d\chi\, d\theta\, d\phi\, dk_\chi\, dk_\theta\, dk_\phi, \end{aligned} \tag{30}$$

where V_p denotes the volume of the phase space satisfying $k^2 + m^2 \leqslant 0$ and $\hbar$ was set to one in second line of Eq. (30). In Eq. (30), the integral $\int dk_\chi\, dk_\theta\, dk_\phi$ is the volume of an ellipsoid satisfying $k_\chi^2/a^2 + k_\theta^2/b^2 + k_\phi^2/c^2 \leqslant 1$, which is obtained from Eq. (29), where $a^2 = b^2 = \omega^2/\sin^2\chi - m^2/\Lambda$ and $c^2 = a^2\sin^2\theta$. Since the volume of the ellipsoid is calculated as

$$\begin{aligned} \int dk_\chi\, dk_\theta\, dk_\phi &= \frac{4\pi}{3}abc \\ &= \frac{4\pi}{3}\sin\theta\left(\omega^2 - \frac{m^2}{\Lambda}\sin^2\chi\right)^{3/2}, \end{aligned} \tag{31}$$

Eq. (30) becomes

$$n(\omega) = \frac{2}{3\pi}\int \frac{d\chi}{\sin^3\chi}\left(\omega^2 - \frac{m^2}{\Lambda}\sin^2\chi\right)^{3/2}, \tag{32}$$

by integrating out with respect to θ and ϕ. For simplicity, we take the massless limit of $m^2 = 0$. As seen from (32), the number of states diverges at the horizons of $\chi = 0, \pi$, so that we need the UV cutoff at $\chi = h_b$ and $\chi = \pi - h_c$. The UV cutoff parameters h_b and h_c are assumed to be very small. Then, the free energy is given by

$$\begin{aligned} F &= -\int d\omega\, \frac{n(\omega)}{e^{\beta\omega} - 1} \\ &= -\frac{2}{3\pi}\int_{h_b}^{\pi - h_c} \frac{d\chi}{\sin^3\chi}\int_0^\infty d\omega\, \frac{\omega^3}{e^{\beta\omega} - 1} \\ &= -\frac{\pi^3}{45\beta^4}\left[-\frac{\cos\chi}{\sin^2\chi} + \ln\left(\tan\frac{\chi}{2}\right)\right]_{h_b}^{\pi - h_c} \\ &= -\frac{\pi^3}{45\beta^4}\left[\frac{1}{h_b^2} - \ln h_b + \frac{1}{h_c^2} - \ln h_c + O\left(h_b^0, h_c^0\right)\right]. \end{aligned} \tag{33}$$

Then, the entropy becomes

$$\begin{aligned} S &= \beta^2\frac{\partial F}{\partial\beta} \\ &= \frac{4\pi^3}{45\beta^3}\left[\frac{1}{h_b^2} - \ln h_b + \frac{1}{h_c^2} - \ln h_c + O\left(h_b^0, h_c^0\right)\right]. \end{aligned} \tag{34}$$

The proper lengths for the UV parameters are given by

$$\bar{h}_b = \int_0^{h_b} d\chi\, \sqrt{g_{\chi\chi}} = \frac{h_b}{\sqrt{\Lambda}}, \tag{35}$$

$$\bar{h}_c = \int_0^{\pi - h_c} d\chi\, \sqrt{g_{\chi\chi}} = \frac{h_c}{\sqrt{\Lambda}}, \tag{36}$$

which leads to $h_{b,c} = \sqrt{\Lambda}\,\bar{h}_{b,c}$. Then, Eq. (34) is written as

$$S = \frac{4\pi^3}{45\beta^3}\left(\frac{1}{\Lambda\bar{h}_b^2} + \frac{1}{\Lambda\bar{h}_c^2}\right), \tag{37}$$

within the leading order of $\bar{h}_{b,c}$.

When we perform the WKB approximation with the line element (14), the coordinate ψ plays a role of the time. The corresponding Killing vector is given by $\xi = \partial/\partial\psi$ and β in Eq. (37) should be taken as the inverse of the Hawking temperature (18). Then, the entropy is obtained as

$$S = \frac{\ell_P^2}{90\pi\bar{h}_b^2}\frac{c^3 A_b}{4G\hbar} + \frac{\ell_P^2}{90\pi\bar{h}_c^2}\frac{c^3 A_c}{4G\hbar}, \tag{38}$$

where $\ell_P \equiv \sqrt{G\hbar/c^3}$ is the Plank length. If the cutoff is chosen as $\bar{h}_{b,c} = \ell_P/\sqrt{90\pi}$ like the case of the Schwarzschild black hole [3], the entropy (38) is remarkably written as

$$S = \frac{c^3 A}{4G\hbar}, \tag{39}$$

where the total area is defined by $A = A_b + A_c$ for convenience. Then, it agrees with one quarter of the horizon area of the Bekenstein–Hawking entropy.

From the viewpoint of the renormalization [42], the total entropy can be written as the sum of the Wald entropy (27) and the quantum correction of Eq. (38). If we consider the bare gravitational coupling constant in the classical entropy, the divergent part can be easily absorbed in the gravitational constant.

4. Discussion

By using the brick-wall method for the Nariai spacetime, we obtained the Bekenstein–Hawking entropy which is proportional to the area of the horizon. In the brick-wall method, β was not the inverse of the Bousso–Hawking temperature but the inverse of the Hawking temperature. The reason why is that the time is chosen as ψ and the standard form of the corresponding Killing vector is given by $\partial/\partial\psi$. If we consider the scaled time $\tilde{\psi} = \psi/\sqrt{\Lambda}$, the Killing vector is given by $\xi = \partial/\partial\tilde{\psi} = \sqrt{\Lambda}\partial/\partial\psi$, which yields the Bousso–Hawking temperature (16). Then, the WKB approximation in the brick-wall method should be performed for the scalar field in the form of $\Phi \sim \exp[-i\tilde{\omega}\tilde{\psi} + iS(\chi,\theta,\phi)] = \exp[-i\tilde{\omega}\psi/\sqrt{\Lambda} + iS(\chi,\theta,\phi)]$. This indicates that the energy in Eq. (29) becomes $\tilde{\omega} = \omega\sqrt{\Lambda}$ and we can easily show that $\beta_H\omega = \beta_{\rm BH}\tilde{\omega}$. In the calculation of the free energy (33), ω in the integrand should be replaced by $\tilde{\omega}/\sqrt{\Lambda}$ and the integration should be performed for $\tilde{\omega}$. Then, we can obtain the same entropy with Eq. (38) based on the Bousso–Hawking temperature. Therefore, the entropy is always written as the area law of the Wald entropy, whereas the temperature and the energy depend on the choice of the time, that is, the normalization of the timelike Killing vector.

The final comment is in order. As for the Bousso–Hawking temperature, it can be regarded as a Tolman temperature [43]. It was defined at the vanishing surface gravity where it is the counterpart of the asymptotically Minkowski space in the asymptotically flat black holes. The Bousso–Hawking temperature can be derived from the definition of the Tolman temperature of $T_{\rm loc} = T_H/\sqrt{g_{\psi\psi}} = \sqrt{\Lambda}/(2\pi\sin\chi)$ where $T_H = 1/(2\pi)$. If we move the observer, for instance, to the black hole horizon of $\chi = 0$ or to the cosmological horizon of $\chi = \pi$, the temperature goes to infinity. In particular, at the middle point of $\chi = \frac{\pi}{2}$, it produces the Bousso–Hawking temperature. So the Bousso–Hawking normalization of Killing vector is compatible with the Tolman temperature. So, we can identify the Bousso–Hawking temperature with the Tolman temperature at the reference point.

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