On a Conjecture about the Exponent Set of Primitive Matrices

Shao Jia-yu

Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706

Submitted by Richard A. Brualdi

ABSTRACT

M. Lewin and Y. Vitek conjecture [7] that every integer \( \leq \lceil (n^2 - 2n + 2)/2 \rceil + 1 \) is an exponent of some \( n \times n \) primitive matrix. In this paper, we prove three results related to Lewin and Vitek's conjecture: (1) Every integer \( \leq \lceil (n^2 - 2n + 2)/4 \rceil + 1 \) is an exponent of some \( n \times n \) primitive matrix. (2) The conjecture is true when \( n \) is sufficiently large. (3) We give a counterexample to show that the conjecture is not true in the case when \( n = 11 \).

1. INTRODUCTION

A \( n \times n \) nonnegative square matrix \( A = (a_{ij}) \) is primitive if \( A^k > 0 \) for some positive integer \( k \); the least such \( k \) is called the exponent of \( A \) and denoted by \( \gamma(A) \). The associated digraph of \( A \), denoted by \( G(A) \), is the digraph with vertex set \( V(G(A)) = \{1, 2, \ldots, n\} \) such that there is an arc from \( i \) to \( j \) in graph \( G(A) \) iff \( a_{ij} > 0 \). In this paper, every graph is a digraph and every matrix is nonnegative.

A digraph \( G \) is primitive if there exists an integer \( k > 0 \) such that for all ordered pairs of vertices \( i, j \in V(G) \) (not necessarily distinct), there is a walk from \( i \) to \( j \) with length \( k \). (By a walk we mean a direct path with possibly repeated vertices and arcs.) The least such \( k \) is called the exponent of the graph \( G \), denoted by \( \gamma(G) \). The following two results are well known:

(1) A matrix \( A \) is primitive iff its associated digraph \( G(A) \) is primitive, and in this case, we have \( \gamma(A) = \gamma(G(A)) \).
(2) A digraph $G$ is primitive iff $G$ is strongly connected and \( \gcd(r_1, r_2, \ldots, r_\lambda) = 1 \) where $L(G) = \{ r_1, r_2, \ldots, r_\lambda \}$ is the set of distinct lengths of the elementary circuits of $G$.

In 1959, Wielandt [10] first stated the exact general upper bound for $\gamma(A)$, that is: $\gamma(A) \leq (n - 1)^2 + 1$ for all $n \times n$ primitive matrices. In [3], Dulanje and Mendelsohn revealed the so-called "gaps" in the exponent set of $n \times n$ primitive matrices. Each gap is a set $S$ of consecutive integers below $W_n = (n - 1)^2 + 1 = n^2 - 2n + 2$ such that no $n \times n$ matrix $A$ has an exponent in $S$. In 1981, Lewin and Vitek [7] found the general method for determining all gaps between $[\frac{1}{2} W_n] + 1$ and $W_n$, where $[x]$ denotes the greatest integer $\leq x$. They also conjectured that there are no gaps below $[\frac{3}{2} W_n] + 1$.

In this paper we will prove the following three results related to Lewin and Vitek's conjecture:

1. There are no gaps below $[\frac{1}{4} W_n] + 1$ for all $n$.
2. Lewin and Vitek's conjecture is true for all sufficiently large $n$, namely, if $n$ is sufficiently large, then there are no gaps below $[\frac{1}{2} W_n] + 1$.
3. The conjecture is not true for $n = 11$.

Because of the connection between the exponent of a matrix and the exponent of a graph stated above, we will use graph theory as a major tool to prove our results.

2. SOME BASIC RESULTS ABOUT $\gamma(G)$

Let $G$ be a primitive graph, and $L(G) = \{ r_1, r_2, \ldots, r_\lambda \}$ be the set of distinct lengths of the elementary circuits of $G$, where $r_1 > r_2 > \cdots > r_\lambda$ and $\gcd(r_1, r_2, \ldots, r_\lambda) = 1$. Let $i, j$ be any ordered pair of vertices of $G$. The relative distance $d_{L(G)}(i, j)$ from $i$ to $j$ is the length of the shortest walk from $i$ to $j$ which meets at least one circuit of each length $r_i$ for $i = 1, 2, \ldots, \lambda$.

**Definition 2.1.** The exponent from vertex $i$ to vertex $j$, denoted by $\gamma(i, j)$, is the least integer $\gamma$ such that there exists a walk of length $m$ from $i$ to $j$ for all $m \geq \gamma$.

It is easy to see the following:

**Proposition 2.1.** If $G$ is a primitive graph, then

$$\gamma(G) = \max_{i, j \in V(G)} \gamma(i, j).$$
The proof of this proposition is obvious.

Now let \( r_1, r_2, \ldots, r_\lambda \) be a set of distinct positive integers with \( \gcd(r_1, r_2, \ldots, r_\lambda) = 1 \). Then we define \( \phi(r_1, r_2, \ldots, r_\lambda) \) to be the least integer \( m \) such that every integer \( k \geq m \) can be expressed in the form \( k = a_1 r_1 + a_2 r_2 + \cdots + a_\lambda r_\lambda \) where \( a_1, a_2, \ldots, a_\lambda \) are nonnegative integers. A result due to Schur shows that \( \phi(r_1, r_2) \) is well defined if \( \gcd(r_1, r_2) = 1 \). It is also well known that in the case when \( \lambda = 2 \) we have \( \phi(r_1, r_2) = (r_1 - 1)(r_2 - 1) \).

Roberts has shown [8] that if \( a_j = a_0 + jd, \ j = 0, 1, 2, \ldots, S, \ a_0 \geq 2 \), then

\[
\phi(a_0, a_1, \ldots, a_S) = \left[ \frac{a_0 - 2}{S} + 1 \right] a_0 + (d - 1)(a_0 - 1), \tag{2.1}
\]

where \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \). The proof of this result has been simplified by Bateman [1]. Brauer and Seelbinder [2], Johnson [5], Lewin [6], and Vitek [9] have discussed the function \( \phi \) and get various upper bounds for \( \phi(r_1, \ldots, r_\lambda) \).

We will use the following two basic upper bounds for \( \gamma(G) \) and \( \gamma(i, j) \) in the proof of our main results:

**Theorem 2.1.** Let \( G \) be a primitive graph and \( s \) be the length of the shortest elementary circuit of \( G \). Then

\[
\gamma(G) \leq n + s(n - 2).
\]

*Proof.* See [3].

**Theorem 2.2.** Let \( G \) be a primitive graph. Then for all ordered pairs \( i, j \in V(G) \) we have

\[
\gamma(i, j) \leq d_{L(G)}(i, j) + \phi(r_1, r_2, \ldots, r_\lambda),
\]

where \( L(G) = \{ r_1, r_2, \ldots, r_\lambda \} \) is the circuit length set of \( G \).

*Proof.* Let \( P \) be a walk from \( i \) to \( j \) with length \( d_{L(G)}(i, j) \) which meets some elementary circuit \( C_j \) of length \( r_j \) for all \( j = 1, 2, \ldots, \lambda \). For every set of nonnegative integers \( a_1, a_2, \ldots, a_\lambda \), add \( C_j \) to \( P \) \( a_j \) times for \( j = 1, 2, \ldots, \lambda \).
We get a new walk

$$P' = P \cup \left( C_1 \cup \cdots \cup C_1 \right) \cup \left( C_2 \cup \cdots \cup C_2 \right) \cup \cdots \cup \left( C_\lambda \cup \cdots \cup C_\lambda \right),$$

which is also a walk from $i$ to $j$. So every integer of the form $d_{L(G)}(i, j) + a_1 r_1 + \cdots + a_\lambda r_\lambda$ ($a_i$ nonnegative integers) is the length of some walk from $i$ to $j$.

Now if $m > d_{L(G)}(i, j) + \phi(r_1, \ldots, r_\lambda)$, then $m$ can be expressed in this form by the definition of $\phi(r_1, \ldots, r_\lambda)$. So we get

$$\gamma(i, j) \leq d_{L(G)}(i, j) + \phi(r_1, \ldots, r_\lambda).$$

3. A GENERAL INDUCTIVE LEMMA

Let $E_n = \{ m \in \mathbb{Z}^+ | m = \gamma(A) \text{ for some } n \times n \text{ primitive matrix } A \}$ be the set of all exponents of $n \times n \text{ primitive }$ matrices. We have the following

**Lemma.** $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \cdots$

**Proof.** It will suffice to show that $E_n \subseteq E_{n+1}$ holds for all $n$. We will use the matrix version to prove this lemma. (We can also prove this by using the graph theory version.)

If $A$ and $B$ are two $n \times n$ nonnegative matrices. We write $A \sim B$ if they have the same zero pattern, i.e., $A \sim B$ iff $a_{ij} > 0 \iff b_{ij} > 0$. It is clear that if $A \sim B \sim A$.

Let $M_n$ be the set of all $n \times n$ nonnegative matrices. Construct a map

$$\varphi: M_n \to M_{n+1}$$

by defining $\varphi(A)$ to be the unique $(n+1) \times (n+1)$ matrix satisfying the following two conditions:

1. The upper left $n \times n$ principal submatrix of $\varphi(A)$ is $A$.
2. The last two rows of $\varphi(A)$ are equal and the last two columns of $\varphi(A)$ are equal.
It is easy to check the following properties of the map $\varphi$:

(i) $A > 0 \iff \varphi(A) > 0$.

(ii) $\varphi(A)\varphi(B) = \varphi(AB + c_n(A)r_n'(B))$ where $c_n(A)$ is the last column of $A$ and $r_n'(B)$ is the last row of $B$.

Now it is clear that $AB \geq c_n(A)r_n'(B)$. So $AB \sim AB + c_n(A)r_n'(B)$ and $\varphi(AB + c_n(A)r_n'(B)) \sim \varphi(AB)$. Combining this with property (ii), we get

$$\varphi(A)\varphi(B) \sim \varphi(AB);$$

in particular, $\varphi(A)^k \sim \varphi(A^k)$. So $A^k > 0 \iff \varphi(A)^k > 0$. This means that if $A \in M_n$ is primitive then $\varphi(A) \in M_{n+1}$ is also primitive and they have the same exponent. This proves $E_n \subseteq E_{n+1}$ and hence completes the proof of the lemma.

This lemma tells us that if for a given integer $m$, we want to show that $m = \gamma(G)$ for some primitive graph $G$ with $n$ vertices, then we only need to show that $m = \gamma(G')$ for some primitive graph $G'$ with $\leq n$ vertices. In most cases this will enable us to simplify the construction of graphs.

4. SOME EXPONENT SETS

In this section we will show that certain families of integers are contained in $E_n$ by constructing several special primitive graphs and computing their exponents. The main result is Theorem 4.1.

For the sake of simplicity, we use $[a, \ldots, b]$ to denote the set of all integers between $a$ and $b$, namely $[a, \ldots, b] = \{m \in \mathbb{Z} | a \leq m \leq b\}$.

**Definition 4.1.** Let $m \geq r_1 > r_2 > \cdots > r_\lambda$ be positive integers. The graph $G_m(r_1, \ldots, r_\lambda)$ is defined to be the graph having the vertex set $V(G_m(r_1, \ldots, r_\lambda)) = \{v_1, \ldots, v_m\}$ and the following set of arcs:

(i) $(v_i, v_{i+1})$ for $i = 1, 2, \ldots, m-1$.

(ii) If $p \geq 0$, then $(v_{i+p}, v_i)$ is an arc of $G_m(r_1, \ldots, r_\lambda)$ iff $p + 1 = r_j$ for some $j$, $1 \leq j \leq \lambda$.

This graph is a generalization of the "minimal Frobenius graph" in [4]. Figure 1 is an example where $m = 9$, $r_1 = 4$, $r_2 = 3$ ($\lambda = 2$).
**Proposition 4.1.** The graph $G_m(r_1, \ldots, r_\lambda)$ satisfies the following properties:

(i) If $\epsilon > 0$, there is only one elementary path from $v_i$ to $v_{i+\epsilon}$.
(ii) $L(G) = \{ r_1, r_2, \ldots, r_\lambda \}$, where $L(G)$ is the circuit length set of $G = G_m(r_1, \ldots, r_\lambda)$.
(iii) Every vertex $v$ meets at least one circuit of each length $r_i$ for $i = 1, 2, \ldots, \lambda$.

In [4], Heap and Lynn had proved these properties for the special case when $m = r_1$ (i.e. for the minimal Frobenius graph). Their proof also works for general $G_m(r_1, \ldots, r_\lambda)$, but the proof given here for (i) is somewhat different.

**Proof of Proposition 4.1.** (i): Let $P$ be any walk from $v_i$ to $v_{i+\epsilon}$. Suppose $P$ contains some “backward arc” $(v_{i+p}, v_i)$. We claim that the vertex $v_{i+p}$ must be repeated if $t + p < i + \epsilon$. This is because $G - \{ v_{i+p} \}$ is not strongly connected, and every walk from $v_i$ to $v_{i+\epsilon}$ must pass through $v_{i+p}$. In the case $t + p > i + \epsilon$ similar reasoning shows that the vertex $v_{i+p}$ must be repeated. So if $P$ is an elementary path from $v_i$ to $v_{i+\epsilon}$, then $P$ can not contain any “backward arc.” So, the only elementary path from $v_i$ to $v_{i+\epsilon}$ is the path $v_i \rightarrow v_{i+1} \rightarrow \cdots \rightarrow v_{i+\epsilon}$.

Properties (ii) and (iii) follow easily from (i).

**Lemma 4.1.** Let $n \geq r_1 > \cdots > r_\lambda$ be a set of positive integers with $\text{gcd}(r_1, \ldots, r_\lambda) = 1$. Then

$$[\phi(r_1, \ldots, r_\lambda) + r_1 - 1, \ldots, \phi(r_1, \ldots, r_\lambda) + r_1 + \min(r_1 - 2, n - r_2 - 1)] \subseteq E_n.$$  

**Proof.** For any integer $\epsilon$ with $-1 \leq \epsilon \leq \min(r_1 - 2, n - r_2 - 1)$ we wish to construct a graph $G$ with $\gamma(G) = \phi(r_1, \ldots, r_\lambda) + r_1 + \epsilon$ and $|V(G)| \leq n$. Then this will imply $\phi(r_1, \ldots, r_\lambda) + r_1 + \epsilon \in E_{|V(G)|} \subseteq E_n$ by the inductive lemma in Section 3.
Let $M = \max(r_2, r_1 - \varepsilon - 1)$; let $G$ be the graph obtained by adding a new path

$$V_{r_1 - \varepsilon} \rightarrow U_{r_1 - \varepsilon} \rightarrow \cdots \rightarrow U_{r_1} \rightarrow v_1$$

of length $\varepsilon + 2$ to the graph $G_M(r_2, r_3, \ldots, r_{\lambda})$. See Figure 2.

Since $-1 \leq \varepsilon \leq \min(r_1 - 2, n - r_2 - 1)$, we have $1 \leq r_1 - \varepsilon - 1 \leq M$ and $r_1 - \varepsilon - 1 \leq r_1$, so $G$ is well defined. Also $|V(G)| = M + \varepsilon + 1 = \max(r_2 + \varepsilon + 1, r_1) \leq n$. It is clear that $G$ is strongly connected and $L(G) = \{r_1, \ldots, r_{\lambda}\}$ by the properties of the graph $G_m(r_2, \ldots, r_{\lambda})$. But $\gcd(r_1, \ldots, r_{\lambda}) = 1$. So $G$ is primitive and the only thing left is to check $\gamma(G) = \phi(r_1, \ldots, r_{\lambda}) + r_1 + \varepsilon$.

Consider the ordered vertex pair $(U_{r_1 - \varepsilon}, v_1)$ [in the case $\varepsilon = -1$, consider the pair $(v_1, v_1)$]. Let $P = U_{r_1 - \varepsilon} \cdots U_{r_1}$ be the unique elementary path from $U_{r_1 - \varepsilon}$ to $U_{r_1}$. By Proposition 4.1 about the properties of the graph $G_M(r_2, \ldots, r_{\lambda})$, we know that if $1 \leq j \leq r_1 - \varepsilon - 1$, then the vertex $v_j$ meets at least one circuit of each length $\gamma_i$ ($i = 1, 2, \ldots, \lambda$). So it is easy to see that an integer $m$ is the length of some walk from $U_{r_1 - \varepsilon}$ to $U_{r_1}$ iff $m = \varepsilon$ or $m = a_1 r_1 + \cdots + a_{\lambda} r_{\lambda}$ for some positive integer $a_1$ and nonnegative integers $a_2, a_3, \ldots, a_{\lambda}$. It follows that $\gamma(U_{r_1 - \varepsilon}, U_{r_1}) = \phi(r_1, \ldots, r_{\lambda}) + r_1 + \varepsilon$.

On the other hand, for any ordered pair of vertices $(x, y) \in V(G)$ we have

$$d_{L(G)}(x, y) \leq d(x, v_j) + d(v_j, y),$$

where $v_j$ is the vertex in $\{v_1, \ldots, v_{r_1 - \varepsilon - 1}\}$ which is closest to $x$ among $\{v_1, \ldots, v_{r_1 - \varepsilon - 1}\}$. It is not difficult to see that $d(x, v_j) \leq \varepsilon + 1$ [in the case when $M = r_2 > r_1 - \varepsilon - 1$ and $x = v_t$ for some $t$ with $r_1 - \varepsilon \leq t \leq r_2$, then $d(x, v_j) \leq d(x, v_t) \leq (r_2 - t) + 1 \leq r_2 - (r_1 - \varepsilon) + 1 \leq \varepsilon + 1$]. Also it is easy to see that $d(v_j, y) \leq d(v_1, u_{r_1}) = r_1 - 1$. So $d_{L(G)}(x, y) \leq \varepsilon + 1 + r_1 - 1 = r_1 + \varepsilon$.
for all \( x, y \in V(G) \), and

\[
\gamma(x, y) \leq d_{L(G)}(x, y) + \phi(r_1, \ldots, r_\lambda) \leq \phi(r_1, \ldots, r_\lambda) + r_1 + \varepsilon;
\]

hence \( \gamma(G) = \max_{i,j} \gamma(i, j) = \gamma(U_{r_1-r}, U_{r_1}) = \phi(r_1, \ldots, r_\lambda) + r_1 + \varepsilon. \)

\[\]

**Lemma 4.2.** If \( n > r_1 + r_2 \) and \( r_1 > r_2 > \cdots > r_\lambda \) are positive integers with \( \gcd(r_1, \ldots, r_\lambda) = 1 \), then

\[
\phi(r_1, \ldots, r_\lambda) + 2r_1 - 1 \in E_n.
\]

**Proof.** Let \( G \) be the graph with vertex set \( V(G) = \{ v_1, \ldots, v_{r_1 + r_2} \} \) and the following set of arcs:

(i) \((v_1, v_2), (v_2, v_3), \ldots, (v_{r_1 + r_2 - 1}, v_{r_1 + r_2}), (v_{r_1 + r_2}, v_1)\). These arcs form a Hamiltonian circuit of \( G \).

(ii) \((v_{r_2}, v_1), (v_{r_2}, v_1), \ldots, (v_{r_\lambda}, v_1)\).

(iii) \((v_1, v_{r_2 + 2}), (v_{r_2 + 2}, v_{r_2 + 1}), (v_{r_2 + 1}, v_{r_2 - 1}), (v_{r_2}, v_r)\). (See Figure 3.)

\(G\) is strongly connected, since \( G \) contains a Hamiltonian circuit. We can also check that \( L(G) = \{ r_1 + r_2, r_1, r_2, \ldots, r_\lambda \} \) and the vertex \( v_1 \) meets at least one circuit of each length \( r_1 + r_2, r_1, r_2, \ldots, r_\lambda \). Also we see that \( |V(G)| = r_1 + r_2 < n \) and note that

\[
\phi(r_1 + r_2, r_1, r_2, \ldots, r_\lambda) = \phi(r_1, r_2, \ldots, r_\lambda),
\]

since \( r_1 + r_2 \) is already a nonnegative integral combination of \( r_1 \) and \( r_2 \).

By the same argument as in Lemma 4.1, we see that

\[
\gamma(v_{r_2 + 1}, v_{r_1 + r_2}) = \phi(r_1, r_2, \ldots, r_\lambda) + 2r_1 - 1,
\]

and for any ordered pair of vertices \( x, y \in V(G) \), we have

\[
d_{L(G)}(x, y) \leq d(x, v_1) + d(v_1, y) \leq r_1 + (r_1 - 1) = 2r_1 - 1
\]

[since it is easy to check \( d(x, v_1) \leq r_1 \) for all \( x \in V(G) \) and \( d(v_1, y) \leq (r_1 - 1) \) for all \( y \in V(G) \)]. So

\[
\gamma(x, y) \leq d_{L(G)}(x, y) + \phi(r_1 + r_2, r_1, \ldots, r_\lambda) < \phi(r_1, \ldots, r_\lambda) + 2r_1 - 1;
\]
hence

$$\gamma(G) = \max_{i,j} \gamma(i,j) = \gamma(v_{r_2+1}, v_{r_1-r_2}) = \phi(r_1, \ldots, r_\lambda) + 2r_1 - 1. \quad \blacksquare$$

**Remark.** The above proof of Lemma 4.2 only works in the case $r_2 \geq 2$. If $r_2 = 1$, then $\phi(r_1, \ldots, r_\lambda) = 0$ and $n \geq r_1 + r_2 = r_1 + 1 \Rightarrow r_1 \leq n - 1$, so $\phi(r_1, \ldots, r_\lambda) + 2r_1 - 1 \leq 2n - 3$. In this case we can use the following:

**Proposition 4.2.** $[1, \ldots, 2n - 2] \subseteq E_n$.

**Proof.** Take $G$ to be the graph with $V(G) = \{1, 2, \ldots, n\}$ and the arc set as follows:

$$E(G) = \{(1,2), (2,3), \ldots, (n-1,n), (n,1); (n,n), (n-1,n-1), \ldots, (i,i)\}$$

for $2 \leq i \leq n$.

Then $\gamma(G) = n + i - 2$. So $[n, \ldots, 2n - 2] \subseteq E_n$. Now use the inductive lemma in Section 3 to get $[1, \ldots, 2n - 2] \subseteq E_n$.

**Lemma 4.3.** If $n \geq r_1 + r_2 + 1$, and $r_1 > r_2 > \cdots > r_\lambda$ are positive integers with $\gcd(r_1, \ldots, r_\lambda) = 1$, then

$$[\phi(r_1, \ldots, r_\lambda) + 2r_1, \ldots, \phi(r_1, \ldots, r_\lambda) + n + r_1 - r_2 - 1] \subseteq E_n.$$

**Fig. 3.**
Proof. In the case \( r_2 = 1 \) we can use Proposition 4.2 again, so we may assume \( r_2 \geq 2 \). Hence \( r_1 \geq 3 \).

Let \( \alpha \) be any integer satisfying \( r_1 + 1 \leq \alpha \leq n - r_2 \). We want to construct a primitive graph \( G \) with \( |V(G)| \leq n \) and \( \gamma(G) = \phi(r_1, \ldots, r_\lambda) + r_1 + \alpha - 1 \).

We can construct \( G \) by the following four steps (see Figure 4):

(i) Take \( H = G_{r_2}(r_2, \ldots, r_\lambda) \) with \( V(H) = \{ u_1, \ldots, u_{r_2} \} \).
(ii) Take \( F = G_{\alpha}(r_1) \) with \( V(F) = \{ v_1, \ldots, v_{\alpha} \} \).
(iii) Choose a vertex \( v_i \) in \( F \) so that the path \( v_i v_{i+1} \cdots v_{i+r_1-2} \) (of length \( r_1 - 2 \)) is "almost in the middle" of \( F \), i.e., choose \( l \) so that
\[
\alpha - (l + r_1 - 2) = l - 1 \quad \text{or} \quad \alpha - (l + r_1 - 2) - (l - 1) + 1
\]
according to the parity of \( \alpha \) and \( r_1 \).
(iv) Connect \( F \) and \( H \) by adding two arcs: \( (u_{r_2}, v_i) \) and \( (v_{l+r_1-2}, u_{r_2}) \).

This is our desired graph \( G \) (see Figure 4).

It is easy to see that \( G \) is strongly connected and \( |V(G)| = r_2 + \alpha \leq n \).

Also we see that \( L(G) = \{ r_1, r_2, \ldots, r_\lambda \} \) and the vertex \( u_{r_2} \) meets at least one circuit of each length \( r_i(i = 1, 2, \ldots, \lambda) \).

By the same argument as in Lemma 4.1 and Lemma 4.2, we get
\[
\gamma(v_1, v_{\alpha}) = \phi(r_1, \ldots, r_\lambda) + r_1 + \alpha - 1;
\]
also \( d_{L(G)}(x, y) \leq d(x, u_{r_2}) + d(u_{r_2}, y) \). So \( \gamma(x, y) \leq d_{L(G)}(x, y) + \phi(r_1, \ldots, r_\lambda) \leq d(x, u_{r_2}) + d(u_{r_2}, y) + \phi(r_1, \ldots, r_\lambda) \) holds for all \( x, y \in V(G) \).

We claim that the following two properties are true:

(i) \( d(x, u_{r_2}) \leq d(v_1, u_{r_2}) = l + r_1 - 2 \quad \forall x \in V(G), \)
(ii) \( d(u_{r_2}, y) \leq d(u_{r_2}, v_{\alpha}) = \alpha - l + 1 \quad \forall y \in V(G). \)

Check for (i): Only need to check for \( x = v_i \) where \( i \in [l + r_1 - 1, \ldots, \alpha] \).

![Figure 4](image-url)
Note that $\alpha \geq r_1 + 1$ and also $0 \leq [\alpha - (l + r_1 - 2)] - (l - 1) \leq 1$ ("almost in the middle") imply $l \geq 2$. Now suppose $i - l = c(r_1 - 1) + d$ where $0 \leq d \leq r_1 - 2$. Then $d(v_i, u_{r_2}) \leq c + r_1 - 1$. We claim $c \leq l - 1$. If $c = 1$, then $c \leq l - 1$, since $l \geq 2$. If $c \geq 2$, then $c \leq 2(c - 1) = c(r_1 - 1)(c - 1) = c(r_1 - 1) - (r_1 - 1) = i - l - d - (r_1 - 1) \leq \alpha - l - (r_1 - 1) \leq l - 1$. So in any case we have $c \leq l - 1$. Hence $d(v_i, u_{r_2}) \leq c + r_1 - 1 \leq l - 1 + r_1 - 1 - l + r_1 - 2$.

Check for (ii): Only need to check for $y = v_j$, where $j \in [1, \ldots, l - 1]$. Since $l \geq 2$, we can write

$$l = a(r_1 - 1) + b, \quad \text{where } 2 \leq b \leq r_1.$$  

Then $d(u_{r_2}, v_j) \leq 1 + a + r_1 - 1 = a + r_1$. But $\alpha - (l + r_1 - 2) \geq l - 1$. So we have

$$l - b \leq l - 2 \leq \alpha - (l + r_1 - 2) - 1 = \alpha - l - r_1 + 1$$

and $a \leq a(r_1 - 1) = l - b \leq \alpha - l - r_1 + 1$ and $a + r_1 \leq \alpha - l + 1$. So $d(u_{r_2}, v_j) \leq a + r_1 \leq \alpha - l + 1$, proving (ii).

Now $\gamma(x, y) \leq d(x, u_{r_2}) + d(u_{r_2}, y) + \phi(r_1, \ldots, r_\lambda) \leq (l + r_1 - 2) + (\alpha - l + 1) + \phi(r_1, \ldots, r_\lambda) = \phi(r_1, \ldots, r_\lambda) + r_1 + \alpha - 1$ for all $x, y \in V(G)$. So $\gamma(G) = \max_{i, j} \gamma(i, j) = \phi(r_1, \ldots, r_\lambda) + r_1 + \alpha - 1$. This completes the proof of Lemma 4.3. $\blacksquare$

Now we are ready to prove:

**Theorem 4.1.** If $n > r_1 > r_2 > \cdots > r_\lambda$ is a set of positive integers where $\gcd(r_1, \ldots, r_\lambda) = 1$ then

$$[\phi(r_1, \ldots, r_\lambda) + r_1 - 1, \ldots, \phi(r_1, \ldots, r_\lambda) + n + r_1 - r_2 - 1] \subseteq E_n.$$

**Proof.**

**Case 1:** $n \geq r_1 + r_2 - 1$. Then $\min(r_1 - 2, n - r_2 - 1) = n - r_2 - 1$, and the result follows from Lemma 4.1.

**Case 2:** $n = r_1 + r_2$. Then $n - r_2 - 1 = r_1 - 1$ and $\min(r_1 - 2, n - r_2 - 1) = r_1 - 2$. So using Lemma 4.1, we get $[\phi(r_1, \ldots, r_\lambda) + r_1 - 1, \ldots, \phi(r_1, \ldots, r_\lambda) + 2r_1 - 2] \subseteq E_n$, and using Lemma 4.2, we get $\phi(r_1, \ldots, r_\lambda) + 2r_1 - 1 \in E_n$. Combining these two results and noting $n + r_1 - r_2 - 1 = 2r_1 - 1$, we get the desired result.
Case 3: \( n \geq r_1 + r_2 + 1 \). Then \( \min(r_1 - 2, n - r_2 - 1) = r_1 - 2 \). Again

Lemma 4.1 \( \Rightarrow [\phi(r_1, \ldots, r_\lambda) + r_1 - 1, \ldots, \phi(r_1, \ldots, r_\lambda) + 2r_1 - 2] \subseteq E_n \), and

Lemma 4.2 \( \Rightarrow \phi(r_1, \ldots, r_\lambda) + 2r_1 - 1 \in E_n \), and

Lemma 4.3 \( \Rightarrow [\phi(r_1, \ldots, r_\lambda) + 2r_1, \ldots, \phi(r_1, \ldots, r_\lambda) + n + r_1 - r_\lambda - 1] \subseteq E_n \).

Combine these three cases to get Theorem 4.1.

5. NO GAPS BELOW \( \lceil \frac{1}{4}w_n \rceil + 1 \)

**Theorem 5.1.** \( [1, \ldots, \lceil \frac{1}{4}w_n \rceil, 1] \subseteq E_n \), where \( w_n = n^2 - 2n + 2 \) is the "Wielandt upper bowul."

**Proof.** We use Theorem 4.1 by suitably choosing the integers \( r_1, r_2, \ldots, r_\lambda \).

1. Choose \( r_1 = x, \ r_2 = x - 1 \) for any integer \( x \) with \( 2 \leq x \leq n \); then \( \phi(r_1, r_2) = (x - 1)(x - 2) \). So \( \phi(r_1, r_2) + r_1 - 1 = (x - 1)(x - 2) + x - 1, \phi(r_1, r_2) + n + r_1 - r_2 - 1 = (x - 1)(x - 2) + n \). Theorem 4.1 gives us

\[
[(x - 1)(x - 2) + x - 1, \ldots, (x - 1)(x - 2) + n] \subseteq E_n.
\]

2. Choose \( r_1 = 2x, \ r_2 = x, \ r_3 = x - 1 \) for any integer \( x \) with \( 2 \leq x \leq n/2 \); then \( \phi(r_1, r_2, r_3) = (x - 1)(x - 2) \) and Theorem 4.1 gives us

\[
[(x - 1)(x - 2) + 2x - 1, \ldots, (x - 1)(x - 2) + n + x - 1] \subseteq E_n.
\]

Now let \( x \) be any integer with \( 2 \leq x \leq n/2 \); then \( x - 1 \leq 2x - 1 \leq n \) \( n + x - 1, \) so the above two exponent sets overlap. Combining these two exponent sets, we get

\[
[(x - 1)(x - 2) + x - 1, \ldots, (x - 1)(x - 2) + n + x - 1] \subseteq E_n,
\]

or

\[
[(x - 1)^2, \ldots, (x - 1)^2 + n] \subseteq E_n
\]

for all integers \( x \) with \( 2 \leq x \leq n/2 \). Now \( x \leq n/2 \) also implies \( (x - 1)^2 \leq (x - 2)^2 + n \), so the intervals \( [(x - 1)^2, \ldots, (x - 2)^2 + n] \) and \( [(x - 1)^2, \ldots, (x - 1)^2 + n] \) overlap for all \( 2 \leq x \leq n/2 \). Take \( i = 2, 3, \ldots, \lfloor n/2 \rfloor \), and let \( I_i = [(i - 1)^2, \ldots, (i - 1)^2 + n] \); then the intervals \( I_2, I_3, I_4, \ldots, I_{\lfloor n/2 \rfloor} \) overlap one after
another. So we have

\[ I_2 \cup I_3 \cup \cdots \cup I_{[n/2]} = \left(2 - 1\right)^2, \ldots, \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)^2 + n \subseteq E_n; \]

but

\[
\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)^2 + n \geq \left(\frac{n - 1}{2} - 1\right)^2 + n = \frac{n^2 - 2n + 9}{4} \\
\geq \frac{n^2 - 2n + 2}{4} + 1 \geq \left\lfloor \frac{1}{2} w_n \right\rfloor + 1.
\]

This completes the proof of Theorem 5.1. \(\blacksquare\)

6. PROOF OF THE CONJECTURE FOR SUFFICIENTLY LARGE \(n\)

In this section we will prove that Lewin and Vitek’s conjecture is true for all sufficiently large \(n\), namely, if \(n\) is sufficiently large, then there are no gaps below \(\left\lfloor \frac{1}{2} w_n \right\rfloor + 1\).

**Lemma 6.1 (The equivalent form of the conjecture).**

(i) The following two statements are logically equivalent:

(6.1) \([1, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1] \subseteq E_n\) is true for all \(n\).

(6.2) \([\left\lfloor \frac{1}{2} w_n - 1 \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1] \subseteq E_n\) is true for all \(n \geq 2\).

(ii) The following two statements are also logically equivalent:

(6.3) \([1, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1] \subseteq E_n\) is true for all sufficiently large \(n\).

(6.4) \([\left\lfloor \frac{1}{2} w_{n-1} \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1] \subseteq E_n\) is true for all sufficiently large \(n\).

**Proof.** (i): (6.1) \(\Rightarrow\) (6.2) is trivial. Now suppose \([\left\lfloor \frac{1}{2} w_{i-1} \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_i \right\rfloor + 1] \subseteq E_i\) is true for all \(i\). Take \(i = 2, 3, \ldots, n\), and recall the inductive lemma in Section 3: \(E_2 \subseteq E_3 \subseteq \cdots \subseteq E_i \subseteq E_{i+1} \subseteq \cdots \subseteq E_n\). We get \([\left\lfloor \frac{1}{2} w_{i-1} \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_i \right\rfloor + 1] \subseteq E_n\) for all \(i = 2, \ldots, n\). Combining these, we get \([2, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1] \subseteq E_n\); also \(1 \in E_n\) is obvious. This proves (6.2) \(\Rightarrow\) (6.1).

(ii): (6.3) \(\Rightarrow\) (6.4) is trivial. Suppose \([\left\lfloor \frac{1}{2} w_{i-1} \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_i \right\rfloor + 1] \subseteq E_n\) is true for all \(i \geq N\), where \(N\) is some positive integer. Again using inductive lemma in Section 3, we get

\([\left\lfloor \frac{1}{2} w_{N-1} \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1] \subseteq E_n\)
for all $n \geq N$. Now $\lceil \frac{1}{2}w_{N-1} \rceil + 2$ is a fixed integer independent of $n$. Take $n$ sufficiently large, say $n \geq N_1$, such that $\lceil \frac{1}{4}w_n \rceil + 1 \geq \lceil \frac{1}{2}w_{N-1} \rceil + 2$. Then by Theorem 5.1 we have $[1, \ldots, \lceil \frac{1}{2}w_{N-1} \rceil + 2] \subset [1, \ldots, \lceil \frac{1}{4}w_n \rceil + 1] \subseteq E_n$ for all $n \geq N_1$. Combining these two results, we get $[1, \ldots, \lceil \frac{1}{2}w_n \rceil + 1] \subseteq E_n$ for all $n \geq \max(N, N_1)$. This completes the proof of Lemma 6.1.

**NOTE.** If $n$ is even, then

$$[[\frac{1}{2}w_{n-1}] + 2, \ldots, [\frac{1}{2}w_n] + 1] = \left[ \frac{n^2 - 4n + 8}{2}, \ldots, \frac{n^2 - 2n + 4}{2} \right],$$

and if $n$ is odd, then

$$[[\frac{1}{2}w_{n-1}] + 2, \ldots, [\frac{1}{2}w_n] + 1] = \left[ \frac{n^2 - 4n + 9}{2}, \ldots, \frac{n^2 - 2n + 3}{2} \right].$$

**Lemma 6.2.** Let $n$ be even with $n \geq 4$ and $n/2 + 1 < i \leq n - 1$. If $G$ is the graph with a Hamiltonian circuit $(12 \ldots n1)$ and two additional arcs $(1, 3)$ and $(i, i + 2)$, then $\gamma(G) = \phi(n - 2, n - 1, n) + i - 1$.

**Proof.** Figure 5 is the picture of $G$.

Note that $L(G) = \{n - 2, n - 1, n\}$, so $G$ is primitive. Also

$$\phi(n - 2, n - 1, n) = \left( \frac{n - 2}{2} \right)(n - 2) \quad \text{for even } n$$

by Robert's formula [see Section 2, Equation (2.1)]. Let $N(n - 2, n - 1, n)$

![Fig. 5.](image)
denote the set of all integers which are representable as a nonnegative linear combination of \( n - 2, n - 1, n \) with integer coefficients. The following two properties of \( N(n - 2, n - 1, n) \) are useful:

(i) \([\phi(n - 2, n - 1, n) - (n - 2), \ldots, \phi(n - 2, n - 1, n) - 2] \subseteq N(n - 2, n - 1, n)\).

We know that \( \phi = \phi(n - 2, n - 1, n) \in N = N(n - 2, n - 1, n) \) and \( \phi - 1 \notin N \) by the definition of \( \phi(n - 2, n - 1, n) \). Now

\[
\phi - (n - 2) = \left(\frac{n - 4}{2}\right) (n - 2);
\]

if \( 0 \leq a \leq n - 4 \), write \( a = 2k + c \), where \( c = 0 \) or 1 and \( k \geq 0 \). We can show now \( k + c \leq (n - 4)/2 \): If \( c = 0 \), then \( k + c = k = a/2 \leq (n - 4)/2 \). If \( c = 1 \), then \( a \) is odd, but \( n - 4 \) is even, so \( a + 1 \leq n - 4 \): Then \( k + c = k + 1 = (a - 1)/2 + 1 = (a + 1)/2 \leq (n - 4)/2 \). So

\[
\phi - (n - 2) + a = \left(\frac{n - 4}{2} - k - c\right) (n - 2) + k(n - 2) + c(n - 2) + 2k + c
\]

\[
\left(\frac{n - 4}{2} - k - c\right) (n - 2) + kn + c(n - 1)
\]

\( \in N(n - 2, n - 1, n) \) for all \( 0 \leq a \leq n - 4 \). This proves (i).

(ii) Let \( a, m \geq 0 \) be integers with \( m \geq \phi(n - 2, n - 1, n) - n + a + 1 \). Then either \( m \in a + N(n - 2, n - 1, n) \) or \( m \in (a - 1) + N(n - 2, n - 1, n) \).

To prove (ii), note that if \( m \neq \phi + a - 2 \), then \( m \geq \phi + n - a + 2 \Rightarrow m - (a - 1) \geq \phi - (n - 2) \), but \( m \neq \phi + a - 2 \) means \( m - (a - 1) \neq \phi - 1 \). So \( m - (a - 1) \in N(n - 2, n - 1, n) \) by property (i). If \( m = \phi + a - 2 \), then \( m - a = \phi - 2 \), so \( m - a \in N(n - 2, n - 1, n) \) by property (i). This proves property (ii).

Now we use properties (i) and (ii) to prove \( \gamma(G) = \phi(n - 2, n - 1, n) + i - 1 \). First note that if \( v \) is any vertex \( \neq 2, i + 1 \), then \( v \) belongs to every circuit. Hence any walk in \( G \) with length \( > 0 \) must meet every circuit of \( G \). Also note that any ordered pair of vertices \( x, y \) must be in one of the following two cases:

Case 1: \( x \neq y \) and there is only one elementary path from \( x \) to \( y \).

Case 2: There exist two walks from \( x \) to \( y \) with lengths \( a \) and \( a - 1 \), respectively, for some \( a \) with \( 1 \leq a \leq n \). (Note that the case \( x = y \) is already included in Case 2.)
If \( x, y \) are in Case 2, we see that any integer \( m \in a + N(n - 2, n - 1, n) \) or \( m \in (a - 1) + N(n - 2, n - 1, n) \) is the length of some walk from \( x \) to \( y \). So by property (ii) above we have \( \gamma(x, y) \leq \phi(n - 2, n - 1, n) - n + a + 1 \leq \phi(n - 2, n - 1, n) + 1 \). If \( x, y \) are in Case 1, we see that \( \gamma(x, y) = d(x, y) + \phi(n - 2, n - 1, n) \). But in Case 1, we have \( d(x, y) \leq d(2, i + 1) \) or \( d(x, y) \leq d(i + 1, 2) \) and \( d(2, i + 1) = i - 1, d(i + 1, 2) = n - i + 1 \). The hypothesis \( i > n/2 + 1 \) now means \( d(2, i + 1) \geq d(i + 1, 2) \). So

\[
\gamma(G) = \max_{x, y} \gamma(x, y) = \gamma(2, i + 1) = \phi(n - 2, n - 1, n) + i - 1.
\]

Corollary 6.1. If \( n \) is even, then

\[
\left[ \frac{n^2 - 3n + 4}{2}, \ldots, \frac{n^2 - 2n + 4}{2} \right] \subseteq E_n.
\]

Proof. The case \( n = 2 \) is trivial. So we assume \( n \geq 4 \). Lemma 6.2 gives us

\[
\left[ \phi(n - 2, n - 1, n) + \frac{n}{2}, \ldots, \phi(n - 2, n - 1, n) + n - 2 \right] \subseteq E_n,
\]

and Theorem 4.1 gives us

\[
\left[ \phi(n - 2, n - 1, n) + n - 1, \ldots, \phi(n - 2, n - 1, n) + n \right] \subseteq E_n.
\]

Combining these two results and noting that

\[
\phi(n - 2, n - 1, n) = \left( \frac{n - 2}{2} \right) (n - 2) = \frac{n^2 - 4n + 4}{2}
\]

gives the desired result.

Theorem 6.1. If \( n \) is even, then

\[
\left[ \left\lfloor \frac{1}{2} w_n \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1 \right] = \left[ \frac{n^2 - 4n + 8}{2}, \ldots, \frac{n^2 - 2n + 4}{2} \right] \subseteq E_n.
\]
Proof. Take \( r_1 = n - 1, \ r_2 = (n - 2)/2, \) and use Theorem 4.1. Note that \( \gcd(r_1, r_2) = 1 \) and

\[
\phi(r_1, r_2) = (r_1 - 1)(r_2 - 1) = (n - 2)\left(\frac{n - 4}{2}\right) = \frac{n^2 - 6n + 8}{2}.
\]

So Theorem 4.1 gives us

\[
\left[ \phi(r_1, r_2) + r_1 - 1, \ldots, \phi(r_1, r_2) + n + r_1 - r_2 - 1 \right]
\]

\[
= \left[ \frac{n^2 - 4n + 4}{2}, \frac{n^2 - 3n + 6}{2} \right] \subseteq E_n.
\]

Combining this with Corollary 6.1, we get Theorem 6.1.

For the case where \( n \) is odd, we need a few number theoretical results:

**Lemma 6.3.** Let \( P_1, P_2, \ldots, P_1, \ldots \) be the infinite sequence of all prime numbers of the form \( 4k + 3 \). Then the following inequality holds for all sufficiently large \( l \):

\[
(P_{l+1} + 2)^2 \leq 2P_3P_4 \cdots P_l - 1.
\]

**Proof.** Let \( \pi_3(x) \) be the number of primes \( \leq x \) which are of the form \( 4k + 3 \). By a theorem from analytic number theory, we know that

\[
\lim_{x \to \infty} \frac{\pi_3(x)}{\frac{1}{2}x \log x} = 1.
\]

Hence \( \pi_3(2x) > \pi_3(x) \) if \( x \) is sufficiently large. This means \( P_{l+1} < 2P_l \) if \( l \) is sufficiently large. So for all sufficiently large \( l \), we have \( P_{l+1}^2 \leq 4P_l^2 \leq 4P_l \cdot 2P_{l-1} = 8P_lP_{l-1} \). Hence

\[
(P_{l+1} + 2)^2 \leq 4P_{l+1}^2 \leq 32P_lP_{l-1} \leq 2P_3P_4 \cdots P_{l-1}P_l - 1.
\]

**Lemma 6.4.** Let \( L \) be the least integer such that \( (P_{l+1} + 2)^2 \leq 2P_3P_4 \cdots P_l - 1 \) holds for all \( l \geq L \), where \( P_1, P_2, \ldots, P_1, \ldots \) is the sequence of all prime numbers of the form \( 4k + 3 \) (\( L \geq 5 \)). If \( n > (P_3P_4 \cdots P_L + 1)/2, \)
then there exists a prime \( P \) (depending on \( n \)) satisfying the following properties:

(i) \( P \equiv 3 \pmod{4} \),
(ii) \( (P - 1)(P + 3)/4 \geq 19 \),
(iii) \( (P - 1)(P + 5)/4 \leq n - 3 \),
(iv) \( P \leq \frac{1}{3}(2n - 1) \),
(v) \( n \not\equiv (P + 1)/2 \pmod{P} \).

Proof. First note that (iii) always implies (iv), so we need only check (i), (ii), (iii), (v). By the choice of \( L \) we know that for all \( l \geq L \) we have \( P_{l + 1} + 2 \leq \sqrt[4]{2P_3P_4 \cdots P_l - 1} \) (Lemma 6.3). We will prove that for all \( n \) satisfying \( n > (P_3P_4 \cdots P_l + 1)/2 \), there correspondingly exists some prime number \( P \) satisfying (i)–(v). In the sequence \( \{P_3, P_4, \ldots, P_l, \ldots\} \) of all the prime numbers of the form \( 4k + 3 \) starting from 11, let \( P_{l + 1} \) be the first term such that

\[
\frac{P_{l + 1} + 1}{2} \not\equiv n \pmod{P_{l + 1}};
\]

take \( P = P_{l + 1} \). Then (i), (ii), (v) follow easily from the choice of \( P \). For (iii) we consider two cases:

Case 1: \( l \leq L \). Then \( P = P_{l + 1} \leq P_{L + 1} \). So \( P + 2 \leq P_{L - 1} + 2 \leq \sqrt[4]{2P_3P_4 \cdots P_L - 1} \leq \sqrt[4]{4n - 3} \), which implies \( (p - 1)(p + 5)/4 \leq n - 3 \). So (iii) follows.

Case 2: \( l > L \). Note that \( n \equiv (P_i + 1)/2 \pmod{P_i} \) for all \( i = 3, \ldots, l \). So \( n \equiv (P_3P_4 \cdots P_l + 1)/2 \pmod{P_i} \) for all \( i = 3, \ldots, l \). But \( P_3, P_4, \ldots, P_l \) are distinct primes, so \( n \equiv (P_3P_4 \cdots P_l + 1)/2 \pmod{P_3P_4 \cdots P_l} \). So

\[
n \geq \frac{P_3P_4 \cdots P_l + 1}{2}.
\]

Now \( l > L \), so by the definition of the integer \( L \), we have

\[
P_{l + 1} + 2 \leq \sqrt[4]{2P_3P_4 \cdots P_l - 1}.
\]

Then (iii) follows by a similar computation to that in Case 1. ■

Theorem 6.2. If \( n \) is odd and \( n > (P_3P_4 \cdots P_L + 1)/2 \), where \( L \) is the fixed integer defined in Lemma 6.4 and \( P_1, P_2, \ldots, P_l, \ldots \) is the sequence of
prime numbers of the form $4k + 3$ (as before), then we have

\[
\left\lfloor \frac{1}{2} w_{n-1} \right\rfloor + 2, \ldots, \left\lfloor \frac{1}{2} w_n \right\rfloor + 1 = \left\lfloor \frac{n^2 - 4n + 9}{2}, \ldots, \frac{n^2 - 2n + 3}{2} \right\rfloor \subseteq E_n.
\]

**Proof.** We divide $[(n^2 - 4n + 9)/2, \ldots, (n^2 - 2n + 3)/2]$ into several pieces.

(a) $[(n^2 - 4n + 3)/2, \ldots, (n^2 - 3n + 4)/2] \subseteq E_n$. For this we can take $r_1 = n - 2$ and $r_2 = (n - 1)/2$; then use Theorem 4.1.

(b) $[(n^2 - 3n - 4)/2, \ldots, (n^2 - 2n - 21)/2] \subseteq E_n$ for all odd numbers $n \geq 13$.

*Case 1:* $n \equiv 1 \pmod{4}$. Then $\gcd((n + 1)/2, n - 3) = 1$. We can take $r_1 = n - 3$ and $r_2 = (n + 1)/2$; then use Theorem 4.1, which gives $[(n^2 - 3n - 4)/2, \ldots, (n^2 - 2n - 5)/2] \subseteq E_n$, more exponents than in (b).

*Case 2:* $n \equiv 3 \pmod{4}$. Then $\gcd((n + 3)/2, n - 5) = 1$. We can take $r_1 = n - 5$ and $r_2 = (n + 3)/2$. Then $r_1 \geq r_2$, since $n \geq 13$. Then Theorem 4.1 gives $[(n^2 - 3n - 18)/2, \ldots, (n^2 - 2n - 21)/2] \subseteq E_n$, more exponents than in (b).

(c) $[(n^2 - 2n - 19)/2, \ldots, (n^2 - 2n + 3)/2] \subseteq E_n$ for all $n > (P_3 P_4 \cdots P_L + 1)/2$, where $L$ is the integer defined in Lemma 6.4. For this we take

\[
r_1 - n - \frac{P + 1}{2} \quad \text{and} \quad r_2 - n + \frac{(P - 1)/2}{2},
\]

where $P$ is the prime number satisfying properties (i)--(v) in Lemma 6.4. By (i) of Lemma 6.4, $P \equiv 3 \pmod{4}$, so $r_2$ is an integer, since $n$ is odd. By (iv) of Lemma 6.4, $p \leq \frac{1}{3}(2n - 1)$, so $r_1 \geq r_2$. Also we see that $2r_2 - r_1 = p$; either $\gcd(r_1, r_2) = p$ or $\gcd(r_1, r_2) = 1$. But if $\gcd(r_1, r_2) = P$ then $P \mid r_1$; hence $r_1 = n - (p + 1)/2 \equiv 0 \pmod{p}$. So $n \equiv (p + 1)/2 \pmod{p}$; this contradicts property (v) of Lemma 6.4. So $\gcd(r_1, r_2) = 1$. Now we use Theorem 4.1 to get

\[
\phi(r_1, r_2) + r_1 - 1 = \left(\frac{n + (p - 1)/2}{2}\right)\left(n - \frac{p + 3}{2}\right) = \frac{n^2 - 2n - (p - 1)(p + 3)/4}{2} \leq \frac{n^2 - 2n - 19}{2},
\]
since \((p - 1)(p + 3)/4 \geq 19\) by (ii) of Lemma 6.4. Also

\[
\phi(r_1, r_2) + n + r_1 - r_2 - 1 = \left( \frac{n + (p - 1)/2}{2} \right) \left( n - \frac{p + 5}{2} \right) + n
\]

\[
= \frac{n^2 - n - (p - 1)(p + 5)/4}{2}
\]

\[
= \frac{n^2 - 2n + 3 + \left[ n - 3 - (p - 1)(p + 5)/4 \right]}{2}
\]

\[
\geq \frac{n^2 - 2n + 3}{2}
\]

because \((p - 1)(p + 5)/4 \leq n - 3\) by (iii) of Lemma 6.4.

Combine (a), (b), (c) and we get Theorem 6.2.

Now we state our main theorem in this section:

**Theorem 6.3.** If \(n\) is sufficiently large, then there are no gaps below \(\left\lfloor \frac{n}{2} \right\rfloor + 1\).

**Proof.** Use Lemma 6.1, Theorem 6.1, and Theorem 6.2.

7. A COUNTEREXAMPLE WHEN \(n = 11\)

In this section we give a necessary and sufficient condition for an odd integer \(n \geq 11\) to satisfy \(\left\lfloor (n^2 - 4n + 9)/2 \right\rfloor, \ldots, \left\lfloor (n^2 - 2n + 3)/2 \right\rfloor \subseteq E_n\), and use this to find a counterexample to Lewin and Vitek's conjecture for \(n = 11\). In order to prove this, we need several upper bounds for \(d_{L(G)}(x, y)\) and \(\phi(r_1, \ldots, r_\lambda)\).

**Lemma 7.1.** Let \(G\) be a primitive graph with \(n\) vertices and \(L(G) = \{ r_1, \ldots, r_\lambda \}\), where \(r_1 > r_2 > \cdots > r_\lambda\). Let \(d(r_1, \ldots, r_\lambda) = \max_{x, y \in V(G)} d_{L(G)}(x, y)\). Then:

(i) If \(r_\lambda + r_{\lambda - 1} > n\), then \(d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i - 1} - r_i)\).

(ii) If \(r_1 + r_\lambda > n\), then \(d(r_1, \ldots, r_\lambda) \leq n + r_1 - 1\).

(iii) If \(3r_\lambda > n\), then \(d(r_1, \ldots, r_\lambda) \leq 2n + r_1 - 1\).
Proof. (i): For any \( x, y \in V(G) \), let \( P(x, y) \) be the shortest path from \( x \) to \( y \) with length \( d(x, y) \).

(a) If \( d(x, y) \geq n - r_\lambda \), then \( P(x, y) \) already meets every circuit of \( G \). So \( d_{L(G)}(x, y) = d(x, y) \leq n - 1 \).

(b) If \( d(x, y) < n - r_\lambda - 1 \), suppose \( x \) belongs to a circuit \( C_j \) of length \( r_j \). Then, since \( r_\lambda + r_{\lambda-1} > n \), \( C_j \) meets every other circuit with length \( \neq r_j \), so \( P(x, y) \cup C_j \) is a walk from \( x \) to \( y \) which meets at least one circuit of each length \( r_1, \ldots, r_\lambda \); so \( d_{L(G)}(x, y) \leq d(x, y) + r_j < n - r_\lambda - 1 + r_j < n - 1 \).

(c) If \( n - r_\lambda \leq d(x, y) < n - r_\lambda - 1 \), then there exists some index \( i \) such that

\[
 n - r_{i-1} \leq d(x, y) < n - r_i - 1.
\]

So \( P(x, y) \) meets a circuit \( C_{i-1} \) of length \( r_{i-1} \) and \( P(x, y) \cup C_{i-1} \) meets every circuit; hence we have \( d_{L(G)}(x, y) \leq n + r_{i-1} - r_i - 1 \).

Combine (a), (b), (c). We get \( d_{L(G)}(x, y) \leq n + \max_{2 \leq i \leq \lambda}(r_{i-1} - r_i) \) for all \( x, y \in V(G) \). This proves (i).

(ii): If \( d(x, y) \geq n - r_1 \), then \( P(x, y) \) meets some circuit \( C \) of length \( r_1 \). \( C \) meets every other circuit, since \( r_1 + r_\lambda > n \), so \( P(x, y) \cup C \) is a walk from \( x \) to \( y \) which meets every circuit, so in this case \( d_{L(G)}(x, y) \leq d(x, y) + r_1 \leq n + r_1 - 1 \).

Now if \( d(x, y) < n - r_1 - 1 \), let \( C_j \) be any circuit passing through \( x \); then \( r_1 + r_\lambda > n \) implies \( C_j \) meets some circuit \( C' \) of length \( r_1 \), and \( C' \) meets every other circuit. So \( P(x, y) \cup C_j \cup C \) is a walk from \( x \) to \( y \) which meets every circuit. So in this case

\[
d_{L(G)}(x, y) \leq d(x, y) + r_1 \leq n - r_1 - 1 + r_1 = n + r_1 - 1 \leq n + r_1 - 1.
\]

(iii): Let \( D \) be any circuit passing through \( x \); let \( r_j \) be the length of \( D \). If \( D \) meets every other circuit, then \( d_{L(G)}(x, y) \leq d(x, y) + r_j \). On the other hand, if there exists some other circuit \( F \) (with length \( r_i \) say) which is disjoint from \( D \), then \( D \cup F \) meets every other circuit, since \( 3r_\lambda > n \). Take a walk from \( x \) to some vertex \( z \) in \( F \) with length \( \leq n - r_\lambda \), add the circuits \( D \) at \( x \) and \( F \) at \( z \), and take a path from \( z \) to \( y \); we get

\[
d_{L(G)}(x, y) \leq d(x, z) + d(z, y) + r_j = n - r_i + n - 1 + r_i + r_j
\]

\[
 = 2n + r_j - 1 \leq 2n + r_1 - 1.
\]

This completes the proof of Lemma 7.1. \( \blacksquare \)
Let \( r_1 > r_2 > \cdots > r_\lambda \) be the set of positive integers with \( \gcd(r_1, \ldots, r_\lambda) = 1 \). We call \( \{r_1, \ldots, r_\lambda\} \) a reduced set if there exists some index \( i \) such that \( r_\lambda, \ldots, r_{i-1} \) are all multiples of \( r_i \) and \( r_i, \ldots, r_1 \) are all nonnegative integral combinations of \( r_\lambda \) and \( r_i \). It is clear that if \( \{r_1, \ldots, r_\lambda\} \) is a reduced set, then \( \phi(r_1, \ldots, r_\lambda) = \phi(r_i, r_\lambda) \). Vitek in [9] proved the following:

**Theorem.** If \( r_1 > r_2 > \cdots > r_\lambda \) with \( \gcd(r_1, r_2, \ldots, r_\lambda) = 1 \) and \( \lambda \geq 3 \), and if \( \{r_1, \ldots, r_\lambda\} \) is not a reduced set, then \( \phi(r_1, \ldots, r_\lambda) \leq \frac{1}{2} \lambda r_\lambda (r_1 - 2) \).

We also need an upper bound for \( \phi(r_1, \ldots, r_\lambda) \) in the following special cases:

**Lemma 7.2.** If \( n \) is an odd number and \( n \geq 11 \), then:

(i) If \( \{r_1, r_2, r_3\} = \{n, n - 2, n - 3\} \), then \( \phi(r_1, r_2, r_3) \leq \frac{n^2 - 6n + 13}{2} \).

(ii) If \( \{r_1, r_2, r_3\} = \{n, n - 1, n - 3\} \), then \( \phi(r_1, r_2, r_3) \leq \frac{n^2 - 6n + 13}{2} \).

Note here that

\[
\frac{n^2 - 6n + 13}{2} = \frac{n^2 - 5n + 6}{2} - \left( \frac{n - 3}{2} - 2 \right) = \left[ \frac{1}{2} r_3 \right] (r_1 - 2) - \left( \frac{n - 3}{2} - 2 \right).
\]

**Proof.** Since \( n \geq 11 \), \( \{r_1, r_2, r_3\} \) is not a reduced set. By the above Vitek’s theorem, we already know that \( \phi(r_1, r_2, r_3) \leq \left[ \frac{1}{2} (n - 3) \right] (n - 2) - (n^2 - 5n + 6)/2 \) (note that \( n \) is odd). So we only need to check for every integer \( a \) with \( 0 \leq a \leq (n - 3)/2 - 2 \), that \( (n - 3)(n - 2)/2 - a \) is a nonnegative integral combination of \( r_1, r_2, r_3 \).

(i): If \( \{r_1, r_2, r_3\} = \{n, n - 2, n - 3\} \), then \( \frac{1}{2} (n - 3)(n - 2) - a = \left( \frac{1}{2} (n - 3) - a \right) (n - 2) + a(n - 3) \).

(ii): If \( \{r_1, r_2, r_3\} = \{n, n - 1, n - 3\} \), first note that \( 2(n - 2) = (n - 1) + (n - 3) \) and \( 3(n - 2) = n + 2(n - 3) \), and if \( x \) is any integer \( \geq 2 \), then \( x = 2y + 3z \) for some nonnegative integers \( y \) and \( z \), since \( \phi(2, 3) = 2 \). So

\[
x(n - 2) = (2y + 3z)(n - 2) = y \cdot 2(n - 2) + z \cdot 3(n - 2) = y \cdot (n - 1 + n - 3) + z \{ n + 2(n - 3) \}
\]

is a nonnegative integral combination of \( n, n - 1, n - 3 \). Now \( \frac{1}{3} (n - 3)(n - 2) - a = \left( \frac{1}{3} (n - 3) - a \right) (n - 2) + a(n - 3) = t(n - 2) + a(n - 3) \), where \( t \)
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\[
= \frac{1}{2}(n - 3) - a \geq 2 \text{ by the choice of } a. \text{ So } t(n - 2) \text{ is a nonnegative integral combination of } n, n - 1, n - 3. \text{ This completes the proof of Lemma 7.2.} \]

**Theorem 7.1.** If \( n \) is an odd number \( \geq 11 \), and if \( G \) is a primitive graph with \( n \) vertices and \( L(G) = \{r_1, \ldots, r_\lambda\} \) with \( \lambda \geq 3 \) but \( L(G) \neq \{n, n - 1, (n - 1)/2\} \), then \( \gamma(G) \leq (n^2 - 3n + 6)/2 \).

**Proof.** We divide the proof into the following seven cases:

**Case 1.** If \( s = r_\lambda \leq (n - 5)/2 \), then \( \gamma(G) \leq n + s(n - 2) \leq (n^2 - 5n + 10)/2 \leq (n^2 - 3n + 6)/2 \).

**Case 2.** If \( (n + 1)/2 \leq r_\lambda \leq n - 5 \), then \( \gamma_\lambda + \gamma_{\lambda - 1} > n \) and Lemma 7.1(i) gives us

\[
d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i - 1} - r_i)
\]

\[
\leq n - 1 + (n - 1) - \frac{n + 1}{2} = \frac{3n - 5}{2}.
\]

Also \( r_\lambda \geq (n + 1)/2 \) and \( \lambda \geq 3 \) implies that \( \{r_1, \ldots, r_\lambda\} \) is not a reduced set. Vitek's theorem gives us

\[
\phi(r_1, \ldots, r_\lambda) \leq \left[\frac{1}{2}r_\lambda\right](r_1 - 2) \leq \frac{n - 5}{2}(n - 2) = \frac{n^2 - 7n + 10}{2}.
\]

So

\[
\gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda)
\]

\[
\leq \frac{3n - 5}{2} + \frac{n^2 - 7n + 10}{2} = \frac{n^2 - 4n + 6}{2} \leq \frac{n^2 - 3n + 6}{2}.
\]

**Case 3.** \( r_\lambda = (n - 3)/2 \).

**Subcase 3.1.** If \( \{r_1, \ldots, r_\lambda\} \) is not a reduced set, then

\[
\phi(r_1, \ldots, r_\lambda) \leq \left[\frac{1}{2}r_\lambda\right](r_1 - 2) \leq \frac{n - 3}{4}(n - 2) = \frac{n^2 - 5n + 6}{4}.
\]

Also \( 3r_\lambda = 3(n - 3)/2 > n \), so Lemma 7.1(iii) gives us

\[
d(r_1, \ldots, r_\lambda) \leq 2n + r_1 - 1. \text{ If } r_1 \leq n - r_\lambda, \text{ we then have } d(r_1, \ldots, r_\lambda) \leq 2n + r_1 - 1 \leq 3n - r_\lambda - 1, \text{ and if } r_1 > n - r_\lambda \text{ then Lemma 7.1(ii) gives us } d(r_1, \ldots, r_\lambda) \leq n + r_1 - 1 \leq 3n - r_\lambda - 1. \text{ So in any case we have } d(r_1, \ldots, r_\lambda)
\[ \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq \frac{n^2 - 5n + 6}{4} + \frac{5n + 1}{2} = \frac{n^2 + 5n + 8}{4} \leq \frac{n^2 - 3n + 6}{2}. \]

**Subcase 3.2.** If \( \{ r_1, \ldots, r_\lambda \} \) is a reduced set, then:

(a) If \( r_{\lambda - 1} > (n + 3)/2 \), then \( r_\lambda + r_{\lambda - 1} > n \). By Vitek's theorem \( \phi(r_1, \ldots, r_\lambda) = \phi(r_i, r_\lambda) \) for some \( i \), since \( \{ r_1, \ldots, r_\lambda \} \) is a reduced set. We also know that all \( r_j \)'s are nonnegative integral linear combinations of \( r_i \) and \( r_\lambda \). But \( r_i + r_\lambda \geq r_{\lambda - 1} + r_\lambda > n \geq r_j \), so all \( r_j \)'s except \( r_i \) are multiples of \( r_\lambda \). So we have \( d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i - 1} - r_i) \leq (n - 1)(\lambda - 1) - (n - 3)/2 = (3n - 5)/2 \). Also

\[ \phi(r_1, \ldots, r_\lambda) = \phi(r_i, r_\lambda) = (r_i - 1)(r_\lambda - 1) \leq (n - 1)\left( \frac{n - 3}{2} - 1 \right) = \frac{n^2 - 6n + 5}{2}. \]

So

\[ \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq \frac{3n - 5}{2} + \frac{n^2 - 6n + 5}{2} \leq \frac{n^2 - 3n + 6}{2}. \]

(b) If \( r_{\lambda - 1} \leq (n + 3)/2 \), then \( \phi(r_1, \ldots, r_\lambda) = \phi(r_i, r_\lambda) \) for \( i = \lambda - 1 \), since \( r_{\lambda - 1} \leq (n + 3)/2 \) is not a multiple of \( r_\lambda - (n - 3)/2 \). So

\[ \phi(r_1, \ldots, r_\lambda) = \phi(r_{\lambda - 1}, r_\lambda) = (r_{\lambda - 1} - 1)(r_\lambda - 1) \leq \left( \frac{n + 3}{2} - 1 \right)\left( \frac{n - 3}{2} - 1 \right) = \frac{n^2 - 4n - 5}{4}. \]
Also, that \( \{ r_1, \ldots, r_\lambda \} \) is a reduced set implies \( r_1 \geq 2r_\lambda = n - 3 \), so \( r_1 + r_\lambda > n \). By using Lemma 7.1(ii) we get \( d(r_1, \ldots, r_\lambda) \leq n + r_1 - 1 \leq 2n - 1 \). So \( \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq 2n - 1 + (n^2 - 4n - 3)/4 = (n^2 + 4n - 9)/4 \leq (n^2 - 3n + 6)/2 \).

Case 4. \( r_\lambda = (n - 1)/2 \). Note that if \( r_{\lambda - 1} \geq n - 1 \), then we have either \( \lambda = 2 \) or \( L(G) = \{ n, n - 1, (n - 1)/2 \} \), contradicting the hypothesis. So we must have \( r_{\lambda - 1} \leq n - 2 \).

Subcase 4.1. \( r_{\lambda - 1} = (n + 1)/2 \); then

\[
\phi(r_1, \ldots, r_\lambda) = \phi(r_{\lambda - 1}, r_\lambda) = \left( \frac{n + 1}{2} - 1 \right) \left( \frac{n - 1}{2} - 1 \right)
\]

\[= \frac{n^2 - 4n + 3}{4}.\]

But \( r_1 + r_\lambda > n \), so we have \( d(r_1, \ldots, r_\lambda) \leq n + r_1 - 1 \leq 2n - 1 \) by Lemma 7.1(ii). So

\[
\gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda)
\]

\[\leq 2n - 1 + \frac{n^2 - 4n + 3}{4} = \frac{n^2 + 4n - 1}{4}
\]

\[\leq \frac{n^2 - 3n + 6}{2}.
\]

Subcase 4.2. \((n + 1)/2 < r_{\lambda - 1} \leq n - 2\); then \( r_{\lambda - 1} + r_\lambda > n \), so \( d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i - 1} - r_i) \leq n - 1 + (n - 3)/2 = (3n - 5)/2 \) by Lemma 7.1(i).

(a) If \( \{ r_1, \ldots, r_\lambda \} \) is a reduced set, then the only possibility is \( L(G) = \{ r_1, r_2, r_3 \} = \{ n - 1, r_2, (n - 1)/2 \} \). Hence

\[
\phi(r_1, \ldots, r_\lambda) = \phi(r_2, r_3) = (r_2 - 1) \left( \frac{n - 1}{2} - 1 \right)
\]

\[\leq (n - 3) \frac{n - 3}{2} = \frac{n^2 - 6n + 9}{2}.
\]
So

\[ \gamma(G) \leq d(r_1, r_2, \ldots, r_\lambda) + \phi(r_1, r_2, \ldots, r_\lambda) \]

\[ \leq \frac{3n - 5}{2} + \frac{n^2 - 6n + 9}{2} = \frac{n^2 - 3n + 4}{2} \]

\[ \leq \frac{n^2 - 3n + 6}{2}. \]

(b) If \( \{r_1, \ldots, r_\lambda\} \) is not a reduced set, then

\[ \phi(r_1, \ldots, r_\lambda) \leq \left( \frac{1}{3} r_\lambda \right) (r_1 - 2) \leq \frac{n - 1}{4} (n - 2) \]

\[ = \frac{n^2 - 3n + 2}{4}. \]

So

\[ \gamma(G) \leq d(r_1, r_2, \ldots, r_\lambda) + \phi(r_1, r_2, \ldots, r_\lambda) \leq \frac{(3n - 5) + (n^2 - 3n + 2)}{4} = \frac{(n^2 + 3n - 8)}{4} \]

\[ \leq \frac{(n^2 - 3n + 6)}{2}. \]

**Case 5:** \( r_\lambda = n - 4 \). Then \( r_{\lambda - 1} + r_\lambda > n \), and \( \{r_1, \ldots, r_\lambda\} \) is not a reduced set, because \( n \geq 11 \). So

\[ \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \]

\[ \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i-1} - r_i) + \left( \frac{1}{3} r_\lambda \right) (r_1 - 2) \]

\[ \leq (n - 1) + 3 + \left[ \frac{1}{3} (n - 4) \right] (n - 2) \]

\[ = n + 2 + \frac{1}{3} (n - 5)(n - 2) \]

\[ = \frac{n^2 - 5n + 14}{2} \leq \frac{n^2 - 3n + 6}{2}. \]

**Case 6.** \( r_\lambda = n - 3 \).

Subcase 6.1. \( L(G) = \{n, n - 1, n - 2, n - 3\} \); then

\[ d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i-1} - r_i) = n \]
by Lemma 7.1(i), and

\[ \phi(r_1, \ldots, r_\lambda) = \phi(n, n-1, n-2, n-3) \]

\[ = \left[ \frac{n-2}{3} \right] (n-3) \leq \frac{n^2-5n+6}{3} \]

by Roberts [8], because \( \{r_1, \ldots, r_\lambda\} \) is now an arithmetic progression, so \( \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq n + (n^2-5n+6)/3 = (n^2-2n+6)/3 \leq (n^2-3n+6)/2. \)

**Subcase 6.2.** \( L(G) = \{ n-1, n-2, n-3 \} \), then \( d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_i - r_1) = n \) by Lemma 7.1(i), and

\[ \phi(r_1, \ldots, r_\lambda) = \phi(n-1, n-2, n-3) = \left[ \frac{n-3}{2} \right] (n-3) \]

\[ = \frac{n^2-6n+9}{2} \]

by Roberts [8]. So

\[ \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq n + \frac{n^2-6n+9}{2} \]

\[ = \frac{n^2-4n+9}{2} \leq \frac{n^2-3n+6}{2}. \]

**Subcase 6.3.** \( L(G) = \{ n, n-2, n-3 \} \), then \( d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_i - r_1) = n + 1 \) by Lemma 7.1(i), and

\[ \phi(r_1, \ldots, r_\lambda) \leq (n^2-6n+13)/2 \] by Lemma 7.2. So

\[ \gamma(G) \leq d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq n + 1 + \frac{n^2-6n+13}{2} = \frac{n^2-4n+15}{2} \]

\[ \leq \frac{n^2-3n+6}{2}. \]

**Subcase 6.4.** \( L(G) = \{ n, n-1, n-3 \} \); same proof as Subcase 6.3.
Case 7. \( r_\lambda = n - 2 \). The only possibility is \( L(G) = \{n, n - 1, n - 2\} \). Then by Lemma 7.1(i), we have \( d(r_1, \ldots, r_\lambda) \leq n - 1 + \max_{2 \leq i \leq \lambda} (r_{i-1} - r_i) = n \) and

\[
\phi(r_1, \ldots, r_\lambda) = \phi(n, n - 1, n - 2) = \left[ \frac{n - 2}{2} \right] (n - 2)
\]
\[
= \frac{(n - 3)(n - 2)}{2} = \frac{n^2 - 5n + 6}{2}
\]

by Roberts [8]. So

\[
\gamma(G) d(r_1, \ldots, r_\lambda) + \phi(r_1, \ldots, r_\lambda) \leq n + \frac{n^2 - 5n + 6}{2}
\]
\[
= \frac{n^2 - 3n + 6}{2}.
\]

This completes the proof of Theorem 7.1. ■

Now we consider the exceptional case in Theorem 7.1, namely, the case where \( L(G) = \{n, n - 1, (n - 1)/2\} \).

**Lemma 7.3.** Let \( n \geq 7 \) be an odd number, \( G \) be a primitive graph with \( n \) vertices, and \( L(G) = \{r_1, r_2, r_3\} \) where \( r_1 = n, \ r_2 = n - 1, \ r_3 = (n - 1)/2 \). Then

\[
\phi(r_1, r_2, r_3) + r_1 - 1 \leq \gamma(G) \leq \phi(r_1, r_2, r_3) + n + r_1 - r_2 - 1.
\]

**Proof.** Note that this lemma is in some sense the converse of Theorem 4.1 [in the special case where \( L(G) = \{n, n - 1, (n - 1)/2\} \)]. Also note that \( r_2 = 2r_3 \), so

\[
\phi(r_1, r_2, r_3) = \phi(r_1, r_3) = (n - 1) \left( \frac{n - 1}{2} - 1 \right) = \frac{(n - 1)(n - 3)}{2}.
\]

So the above inequality is actually the inequality \((n^2 - 2n + 1)/2 \leq \gamma(G) \leq (n^2 - 2n + 3)/2\).

Note that we only need the lower bound \( \gamma(G) \geq (n^2 - 2n + 1)/2 \) in the proof of Theorem 7.2 below. The upper bound part is not needed in the rest of this paper.
(a) First we prove \( \gamma(G) \leq \phi(r_1, r_2, r_3) + n + r_1 - r_2 - 1 = \phi(r_1, r_2, r_3) + n \). For this it will suffice to show \( \gamma(x, y) \leq \phi(r_1, r_2, r_3) + n \) for all \( x, y \in V(G) \).

If \( x = y \), we take the closed path along the Hamiltonian circuit of length \( n \) from \( x \) to \( x \). We get \( d_{L(G)}(x, x) \leq n \), so \( \gamma(x, x) \leq \phi(r_1, r_2, r_3) + d_{L(G)}(x, x) \leq \phi(r_1, r_2, r_3) + n \).

If \( x \neq y \), then any circuit of length \( n - 1 \) contains either \( x \) or \( y \). Let \( P(x, y) \) be the shortest path from \( x \) to \( y \) with length \( d(x, y) \). Notice that we have the following two facts:

1. Every integer of the form \( d(x, y) + a_1n + a_2(n - 1) + a_3(n - 1)/2 \) with \( a_1 > 0 \) or \( a_2 > 0 \) is the length of some walk from \( x \) to \( y \).

2. Every integer \( m \geq \phi(n, n - 1, (n - 1)/2) \) can be written as \( m = a_1n + a_2(n - 1) + a_3(n - 1)/2 \) with either \( a_1 > 0 \) or \( a_2 > 0 \).

Fact (1) is true because we can add a Hamiltonian circuit to \( P(x, y) \) or add a circuit of length \( n - 1 \) (at one of the vertices \( x \) and \( y \)) to \( P(x, y) \); then the new walk will meet every other circuit. Fact (2) is true because \( m \geq \phi(n, n - 1, (n - 1)/2) \) means \( m = b_1n + b_2(n - 1) + b_3(n - 1)/2 \) for some nonnegative integers \( b_1, b_2, b_3 \). If \( b_1 > 0 \) or \( b_2 > 0 \), we get what we want. If \( b_1 = b_2 = 0 \), then \( m = b_3(n - 1)/2 \) but

\[
\gamma(G) \leq \phi(r_1, r_2, r_3) + n = \phi(r_1, r_2, r_3) + n + r_1 - r_2 - 1.
\]

(b) Next we prove \( \gamma(G) \geq \phi(r_1, r_2, r_3) + r_1 - 1 = \phi(r_1, r_2, r_3) + n - 1 \). Label the vertices of \( G \) so that \( C = (1, 2, \ldots, n) \) is a Hamiltonian circuit of \( G \). We claim that there is some \( i \in \{1, \ldots, n\} \) such that \( (i, i + 2) \) is an arc of \( G \) (read the integers mod \( n \)), i.e. \( G \) contains a subgraph isomorphic to the graph \( D \) in Figure 6. Suppose not; then \( (i, j) \) can possibly be an arc of \( G \) only when \( j - i = 1 \) or \( j - i = (n + 3)/2 \) (mod \( n \)), since \( (i, j, j + 1, \ldots, i) \) is a circuit of length \( = i - j + 1 \) (mod \( n \)), which is either equal to \( r_3 = (n - 1)/2 \) or equal to \( r_1 = n \). Let \( C' = (t_1, t_2, \ldots, t_n, t_1) \) be an elementary circuit of length \( n - 1 \); then for \( k = 1, 2, \ldots, n - 1 \), we have \( t_{k+1} - t_k = 1 \) or \( (n + 3)/2 \) (mod \( n \)) (if we agree that \( t_n = t_1 \)). Suppose

\[
t_{k+1} - t_k = 1 \quad \text{for} \quad k \in S_1, \quad t_{k+1} - t_k = \frac{n + 3}{2} \quad \text{for} \quad k \in S_2,
\]

so \( m = b_3(n - 1)/2 \) means \( b_3 \geq 2 \). So \( m = (n - 1) + (b_3 - 2)(n - 1)/2 \). This is in the situation where \( a_2 = 1 > 0 \). This proves fact (2).
where $S_1 \cup S_2 = \{1, 2, \ldots, n-1\}$. Let $|S_2| = h$ and $|S_1| = n-1-h$; then

$$(n-1-h) + h \cdot \left( \frac{n+3}{2} \right) \equiv 0 \pmod{n}$$

because $C'$ is a closed path. So

$$2(n-1-h) + h(n+3) \equiv 0 \pmod{n},$$

$$h - 2 \equiv 0 \pmod{n};$$

but $0 \leq h \leq n-1$, so we have $h = 2$.

Now we may assume $t_{a+1} - t_a \equiv (n+3)/2 \pmod{n}$ and $t_{b+1} - t_b \equiv (n+3)/2 \pmod{n}$ for $1 \leq a < b < n-1$:

$$t_1 \to t_2 \to \cdots \to t_a \to t_{a+1} \to \cdots \to t_b \to t_{b+1} \to \cdots \to t_{n-1} \to t_1.$$
d(z, y) = n - 2 or (n - 3)/2 and d(y, x) = n - 2 or (n - 3)/2, since L(G) = \{n, n - 1, (n - 1)/2\}. Hence we get a walk from z to x of length (n - 2)+(n - 2) or n - 2+(n - 3)/2 or (n - 3)/2+(n - 3)/2. Add the arc (x, z) to this walk; we get a closed walk from z to z of length 2n - 3 or (3n - 5)/2 or n - 2, which are respectively 4 \cdot (n - 1)/2 - 2, 3 \cdot (n - 1)/2 - 1, and 2 \cdot (n - 1)/2 - 1. It is easy to see that none of them is of the form a \cdot (n - 1)/2 + bn for nonnegative integers a, b, because for u = 2, 3, 4,
\[ u \left( \frac{n - 1}{2} \right) - 1 + (n - 3 - u) \frac{n - 1}{2} = (n - 3) \frac{n - 1}{2} - 1 = \phi \left( n, \frac{n - 1}{2} \right) - 1, \]
and n ≥ 7 means n - 3 - u ≥ 0. Now φ(n, (n - 1)/2) - 1 is not of the form a(n - 1)/2 + bn; so u(n - 1)/2 - 1 is not of that form. This is a contradiction, because the length of every closed walk must be of the form a(n - 1)/2 + bn. So we have d(z, y) = n - 1 or d(y, x) = n - 1. Say d(z, y) = n - 1; then every elementary path from z to y has length n - 1. It follows that every walk from z to y has length of the form d(z, y) + a(n - 1)/2 + bn = n - 1 + a(n - 1)/2 + bn, because every walk can be obtained by adding elementary circuits to some elementary path. So γ(z, y) = φ(n, (n - 1)/2) + n - 1 = φ(r₁, r₂, r₃) + n - 1. So
\[ γ(G) ≥ γ(z, y) ≥ φ(r₁, r₂, r₃) + n - 1 = φ(r₁, r₂, r₃) + r₁ - 1. \]
This completes the proof of Lemma 7.3.

Now we want to find the necessary and sufficient condition for an odd integer n ≥ 11 to satisfy \([(n² - 4n - 9)/2, \ldots, (n² - 2n + 3)/2] ⊆ Eₙ\). By taking \{r₁, r₂\} = \{n - 2, (n - 1)/2\} and using Theorem 4.1, we have \([(n² - 4n + 3)/2, \ldots, (n² - 3n + 4)/2] ⊆ Eₙ\). By taking \{r₁, r₂, r₃\} = \{n, n - 1, n - 2\} and using Theorem 4.1, we have \([(n² - 3n + 4)/2, \ldots, (n² - 3n + 6)/2] ⊆ Eₙ\). Combine these two results, we have \([(n² - 4n + 9)/2, \ldots, (n² - 3n + 6)/2] ⊆ Eₙ\). By taking \{r₁, r₂, r₃\} = \{n, n - 1, (n - 1)/2\} we have \([(n² - 2n + 1)/2, \ldots, (n² - 2n + 3)/2] ⊆ Eₙ\), so
\[ \left[ \frac{n² - 4n + 9}{2}, \ldots, \frac{n² - 2n + 3}{2} \right] ⊆ Eₙ ⇔ \left[ \frac{n² - 3n + 8}{2}, \ldots, \frac{n² - 2n - 1}{2} \right] \subseteq Eₙ. \]

Theorem 7.2 will give a number theoretical criterion for an integer \(m ∈ \left[ (n² - 3n + 8)/2, \ldots, (n² - 2n - 1)/2 \right] \) to satisfy \(m ∈ Eₙ\).
First we quote the following result by Lewin and Vitek:

**Theorem [7].** Let $G$ be a primitive graph with $n$ vertices. Suppose that $L(G) = \{r_1, r_2\}$ and that $\gamma(G) > (n^2 + 2n + 1)/4$ and $\gamma(G) > 2n - 2$. Then $\phi(r_1, r_2) + r_1 - 1 \leq \gamma(G) \leq \phi(r_1, r_2) + n + r_1 - r_2 - 1$.

**Proof.** See [7, Theorem 4.1, Theorem 4.2, and Corollary 3.2].

Now we can prove the following theorem:

**Theorem 7.2.** Let $n$ be an odd integer $\geq 11$. Suppose $m$ is an integer with $(n^2 - 3n + 8)/2 \leq m \leq (n^2 - 2n - 1)/2$. Then $m \in E_n$ iff there exist positive integers $r_1, r_2$ satisfying $n > r_1 > r_2$ and $\gcd(r_1, r_2) = 1$ such that

$$\phi(r_1, r_2) + r_1 - 1 \leq m \leq \phi(r_1, r_2) + n + r_1 - r_2 - 1.$$ 

**Proof.** The sufficiency follows from Theorem 4.1, so we only need to show the necessity. If $m \in E_n$, then $m = \gamma(G)$ for some primitive graph $G$ with $n$ vertices. Suppose $L(G) = \{r_1, \ldots, r_\lambda\}$; then by Theorem 7.1 and Lemma 7.3, we must have $\lambda = 2$, so $L(G) = \{r_1, r_2\}$ for some positive integers $r_1$ and $r_2$ satisfying $n > r_1 > r_2$ and $\gcd(r_1, r_2) = 1$. Using the previous theorem of Lewin and Vitek and noting that $\gamma(G) = m \geq (n^2 - 3n + 8)/2$ with $n \geq 11$ means $\gamma(G) > (n^2 + 2n + 1)/4$ and $\gamma(G) > 2n - 2$, we get the necessity of Theorem 7.2.

Now we use Theorem 7.2 to give a counterexample to Lewin and Vitek's conjecture:

**Example.** Let $n = 11$; $m = 48$ is below $[\frac{1}{2}w_n] + 1 = (n^2 - 2n + 3)/2 = 51$, but $48 \notin E_{11}$.

**Proof.** $(n^2 - 3n + 8)/2 = 48$, $(n^2 - 2n - 1)/2 = 49$, so $(n^2 - 3n + 8)/2 \leq 48 \leq (n^2 - 2n - 1)/2$. We claim that for any pair of integers $r_1, r_2$ with $n > r_1 > r_2$ and $\gcd(r_1, r_2) = 1$, $48 \notin [\phi(r_1, r_2) + r_1 - 1, \ldots, \phi(r_1, r_2) + n + r_1 - r_2 - 1]$. We can see this by considering the following cases:

(a) $r_2 \leq 4$: $\phi(r_1, r_2) + n + r_1 - r_2 - 1 = (r_1 - 2)r_2 + n \leq 47$.
(b) $r_2 = 5, r_1 \leq 9$: $\phi(r_1, r_2) + n + r_1 - r_2 - 1 = (r_1 - 2)r_2 + n \leq 46$.
(c) $r_2 = 5, r_1 = 11$: $\phi(r_1, r_2) + r_1 - 1 = (r_1 - 1)r_2 \geq 50$.
(d) $r_2 = 6, r_1 \leq 7$: $\phi(r_1, r_2) + n + r_1 - 1 = (r_2 - 2)r_2 + n \leq 41$.
(e) $r_2 = 6, r_1 = 11$: $\phi(r_1, r_2) + r_1 - 1 = (r_1 - 1)r_2 \geq 60$.
(f) $r_2 \geq 7$: then $r_1 \geq r_2 + 1 \geq 8$, so $\phi(r_1, r_2) + r_1 - 1 = (r_1 - 1)r_2 \geq 49$. 

So $48 \not\in E_{11}$ by Theorem 7.2. Thus in the case when $n = 11$, not every integer below $\lceil \frac{1}{2} w_n \rceil + 1$ is an exponent of some $n \times n$ primitive matrix.

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REFERENCES


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