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A NUMERICAL COMPARISON OF HIGH ORDER TRANSFORMATION AND ISOPARAMETRIC TRANSFORMATION METHODS

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Abstract—A numerical comparison is made between the quadratic isoparametric transformation method and a second order example of a high order transformation method for the model problem of Laplace's equation on curved domains. Three curved domains are considered and numerical results for several trial solutions are given. Significantly improved accuracy is attained by the high order transformation method. A finer element discretisation is chosen for one of the domains resulting in more than twice the number of variables. The errors using the high order transformation method on the original mesh remain significantly smaller than those given by the isoparametric method used on the finer mesh.

INTRODUCTION

The isoparametric transformation method gives an ingenious, simple and very useful way of dealing with curved elements in the finite element method. The method has been studied extensively and asymptotic error estimates are well known[1]. It is equally well known, however, that for finite, as opposed to infinitesimal, elements the method reduces to first order[2]. The difference in order between the transformed plane and the plane of problem definition is a function of the distortion of the particular element from its corresponding straight-sided counterpart. New bases have been introduced in an attempt to obtain better geometrical approximations to problem domains. Some of these bases have resulted in significent improvement in accuracy even though being nonconforming[3, 4]. Recently high order bases for curved finite elements have been proposed[5, 6]. These bases are high order in the plane of problem definition and for finite element size. The philosophy behind the development of such bases is the desire to produce more accurate results for a given element size and hence remove the need for element subdivision and the corresponding growth in discretised problem size. In the main, these bases are rational functions and the apparent difficulties in using such functions have deterred their implementation in finite element programs. Techniques, given in [7], facilitate the implementation of these methods and it is hoped that some much needed numerical studies of these methods will soon be forthcoming. Recently an alternative to rational bases has been proposed[8]. This method, called a "High Order Transformation" method, obviates the need for rational bases while retaining the required order. The form of the basis functions can be the same as in the isoparametric case, i.e. polynomial in the transformed coordinates. Thus the numerical integration problem is the same as that in the isoparametric method. The purpose of this note is to report the results of the first, albeit elementary, numerical comparison between a second order example of this basis and the corresponding isoparametric one.

THE BASES

Two-dimensional curved regions were considered. These regions were divided into elements in such a way that only straight-sided triangles and triangles with a single curved side were used. Isoparametric quadratics were used and hence the transformation is quadratic and the curved side a parabola. Illustrating the case where the element has vertices (0,0), (1,0) and (0,1) with the curved side between (1,0) and (0,1) (Fig. 1), the isoparametric transformation is given by

$$\begin{aligned} x &= p(1+2(2X-1)q) \\ y &= q(1+2(2Y-1)p). \end{aligned} \tag{1}$$

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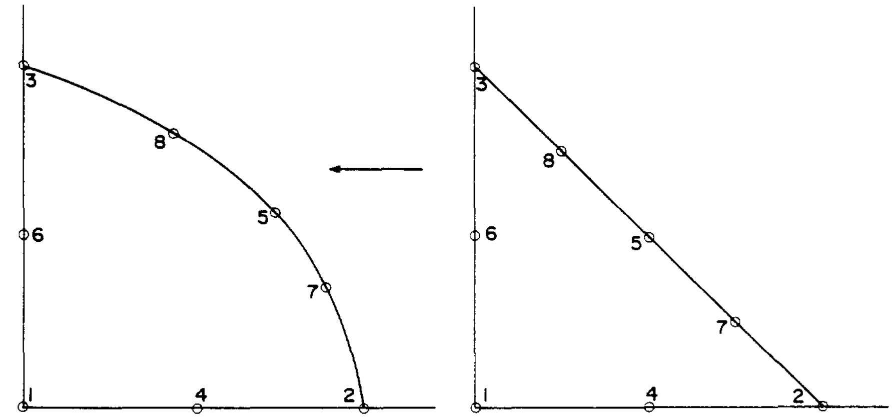

Fig. 1. The positions of the nodes in the X, Y and P, Q planes. Point 5 has coordinates (X, Y) in the X, Y plane and is the image of the point $(\frac{1}{2}, \frac{1}{2})$ under the transformation given by eqn (1). Points 7 and 8 are the images of the points $(\frac{3}{4}, \frac{1}{4})$ and $(\frac{1}{4}, \frac{3}{4})$ respectively, under the same transformation.

The isoparametric basis is given by

$$\begin{aligned} W_1(p,q) &= (1-p-q)(1-2p-2q) \\ W_2(p,q) &= p(2p-1) \\ W_3(p,q) &= q(2q-1) \\ W_4(p,q) &= 4p(1-p-q) \\ W_5(p,q) &= 4pq \\ W_6(p,q) &= 4q(1-p-q). \end{aligned} \tag{2}$$

Though the high order transformation method is not restricted to any particular transformation, for the sake of a controlled comparison the same transformation was used. This ensures us that each method is being used on exactly the same geometry.

Let $C_i(x, y)$, $i = 1, 2, \ldots, 6$, denote the quadratic polynomial which is zero at the points (x_j, y_j), $j \neq i$, and has unit value at (x_i, y_i), i.e.

$$C_i(x_j, y_j) = \delta_{ij}, \qquad i, j = 1, 2, \ldots, 6. \tag{3}$$

Let $(i, j)_k$ denote the linear polynomial which is zero at (x_i, y_i), (x_j, y_j) and has unit value at (x_k, y_k). Then the high order transformation method bases is given by

$$\begin{aligned} W_i(x,y,p,q) &= C_i(x,y) - C_i(x_7,y_7)W_7(x,y,p,q) - C_i(x_8,y_8)W_8(x,y,p,q) \\ W_7(x,y,p,q) &= \frac{16}{3}pq(8;5)_7 \\ W_8(x,y,p,q) &= \frac{16}{3}pq(7;5)_8. \end{aligned} \tag{4}$$

Since (1) gives x and y as polynomials in p and q the basis given by (4) is polynomial in the p, q coordinates. It is in fact a cubic polynomial in the p, q plane. The conformity, order and more detailed discussion of this type of basis is given in [8].

RESULTS

Following tradition we consider Laplace's equation with Dirichlet boundary conditions and a Galerkin formulation. We thus must minimize the functional

$$\int\int_R ((u_x)^2 + (u_y)^2)\,dx\,dy \tag{5}$$

over the problem domain, subject to the boundary conditions

$$u(x, y) = g(x, y) \tag{6}$$

on the domain boundary.

The transformation given by eqn (1) implies that any curved boundaries are approximated by parabolae. This piecewise parabolic boundary was considered to be the problem domain and hence one source of discretization error was removed. The test solutions chosen for the trial were

$$\begin{aligned}
u_1(x, y) &= 1 - x - y\\
u_2(x, y) &= 1 - x - y + x^2 - y^2\\
u_3(x, y) &= 3x^2(x+3) - 3y^2(x+1) + 3x + 1\\
u_4(x, y) &= (x+1)(5y^4 - 10x^2y^2 - 20xy^2 - 10y^2 + x^4 + 4x^3 + 6x^2 + 4x + 1)\\
u_5(x, y) &= 5 \ln(x^2 + y^2) + x^2 - y^2\\
u_6(x, y) &= \exp\left(\frac{x}{2}\right) \sin\left(\frac{y}{2}\right)\\
u_7(x, y) &= \exp(x) \sin(y).
\end{aligned} \tag{7}$$

Exact boundary conditions were used at the boundary nodes. The linear and quadratic trial solutions were chosen so as to provide a reference point in significant figures of accuracy of the numerical solution. The isoparametric method should be exact for the linear trial solution and the high order transformation method exact for both linear and quadratic trial solutions. Since both methods result in integrals of the same form to be calculated, the same method of integration was used in each case. The method involved the use of recurrence relations which have been described elsewhere[8]. The required integrals are of the form

$$\int_0^1 \int_0^{1-q} \sum_{i+j=0}^{m} f_{ij} \frac{p^i q^j}{(1 + \beta p + \alpha q)} \mathrm{d}p \, \mathrm{d}q = \sum_{i+j=0}^{m} f_{ij} M_{ij} \tag{8}$$

where

$$M_{ij} = \int_0^1 \int_0^{1-q} \frac{p^i q^j}{(1 + \beta p + \alpha q)} \mathrm{d}p \quad \mathrm{d}q, \qquad \alpha = 2(2X - 1), \quad \beta = 2(2Y - 1).$$

The M_{ij}'s satisfy the recurrence relation

$$M_{ij} + \beta M_{i+1,j} + \alpha M_{i,j+1} = C_{ij}, \tag{9}$$

where $C_{ij} = [\Gamma(i+1)\Gamma(j+1)]/[\Gamma(i+j+3)]$, Γ represents the gamma function. This relation leads to others which are outlined in [8] and provide a method of calculating the integrals when the element has marked distortion, i.e. $|\alpha|$, $|\beta| > 0.4$ say. For $|\alpha|$, $|\beta| \leq 0.4$, (9) alone provides a technique for calculating the M_{ij} since the procedure defined by

$$M_{ij} = C_{ij} - \alpha M_{i,j+1} - \beta M_{i+1,j} \tag{10}$$

converges quite rapidly. The technique is to assume the M_{ij} are zero at some "layer" given by $i + j = N > m$ and then to use (10) to work towards the required M_{ij}, $i + j < N$. The $M_{ij} \to 0$ as $i + j \to N$ and in practice it is usually sufficient to start at $N = 12$.† The two ways of using (9) yield accurate results and in an attempt to further eliminate a source of error, the integrals were calculated to much higher accuracy than would normally be required. The same effort was applied in the linear equation solver so that any difference in solution could be attributed

†The authors would like to thank Dr. Carl Anderson for his suggestion of applying (9) in this simple and effective way.

entirely to the different bases and not to different geometries, errors in numerical integration or linear equation solution. The computations were carried out on a PRIME 300 minicomputer, which does 32 bit floating point operations. All computations leading to the stiffness matrices were done in double precision, which is done in 48 bit arithmetic. A double precision linear equation solver was used. The application of boundary conditions was done in single precision.

An approximate L^2 norm of the error was calculated in the following way.

$$L^2 = \left[\frac{\sum_{i=1}^{N} (u(x_i, y_i) - \bar{u}_i)^2 \delta A_i}{\sum_{i=1}^{N} u(x_i, y_i)^2 \delta A_i} \right]^{1/2}, \tag{11}$$

where N is the number of elements, the point (x_i, y_i) is the centroid of the element if it is straight-sided or the image of the centroid under the transformation if it is a curved element, δA_i is the area of element i, $u(x_i, y_i)$ is the exact solution at (x_i, y_i) and $\bar{u}_i$ is the approximate solution at the same point.

The numerical results are given in Table 1.

Table 1. The approximate L^2 norm calculated from eqn (8) for each of the seven trial solutions and each of the five trial domains

Figs.	Basis	u_1	u_2	u_3	u_4	u_5	u_6	u_7
2(a)	Isoparametric	0.0000	0.0003	0.0004	0.0032	0.0006	0.0036	0.0200
2(a)	H.O.T.	0.0000	0.0000	0.0000	0.0005	0.0000	0.0008	0.0053
2(b)	Isoparametric	0.0000	0.0006	0.0016	0.0015	0.0013	0.0034	0.0084
2(b)	H.O.T.	0.0000	0.0000	0.0004	0.0010	0.0001	0.0023	0.0059
2(c)	Isoparametric	0.0000	0.0029	0.0036	0.0057	0.0031	0.0077	0.0249
2(c)	H.O.T.	0.0000	0.0000	0.0001	0.0007	0.0000	0.0015	0.0086
2(d)	Isoparametric	0.0000	0.0004	0.0005	0.0013	0.0004	0.0028	0.0188
2(d)	H.O.T.	0.0000	0.0000	0.0000	0.0003	0.0000	0.0008	0.0051
2(e)	Isoparametric	0.0000	0.0001	0.0002	0.0005	0.0002	0.0011	0.0069
2(e)	H.O.T.	0.0000	0.0000	0.0000	0.0002	0.0000	0.0005	0.0036

DISCUSSION

The accuracy of either method is dependent on the geometry and the particular trial solution. The isoparametric method will be second order when the curvature tends to zero, hence for small and mildly curved elements one would expect the isoparametric method to be close to second order. Figure 2(b) illustrates this effect. There are relatively few seriously curved elements. Even in this case, the high order transformation method yields a distinct improvement for several of the trial solutions though only a marginal improvement—as expected—in "difficult" trial solutions $u_6(x, y)$ and $u_7(x, y)$. In the situation of Fig. 2(c), we have a marked improvement even for the most testing of the trial solutions. This region has 12 of the 20 elements having non-trivial curvature and the loss of accuracy of the isoparametric method is quite distinct. Subdivision of the same domain into smaller elements yields further interesting results. With the situation of Fig. 2(d) we have increased the number of unknowns in the linear system from 29 to 60. Both methods yield smaller L^2 errors but the error in the isoparametric case for the 60×60 problem is still significantly larger than the error in the high order transformation case for the 29×29 problem. Subdivision of the same region to give 72 unknowns still resulted in an isoparametric solution with L^2 errors twice those obtained by the high order transformation method with 29 unknowns. Finally subdivision to yield Fig. 2(e), which resulted in 97 unknowns, gave isoparametric L^2 norms slightly better than the high order transformation method norms for the unsubdivided region.

The results in Table 1 give a clear indication of the superiority in accuracy of the high order transformation method. However these tests reported here are far from extensive. Studies should be done on more realistic problems and also on the relative complications of using the two methods. One of the central issues is undoubtedly resulting matrix size. Current indications suggest that a given accuracy could be achieved with a much smaller matrix size if the high order transformation method is used. This benefit however could be offset if the element stiffness matrices took much longer to calculate or if the program size were much larger than in the isoparametric case. These questions are difficult to answer without much more careful investigation. The basis functions themselves are more complicated in the high order transformation method for not only are they higher degree (cubic as opposed to quadratic) polynomials, but the coefficients are functions of the curvature (see eqn 4). These functions are however known and could be coded into the program at, say, the level of the derivative of the basis functions. That is, the coefficients can easily be calculated algebraically for each basis. This would result in a less flexible program for if one desired to use a different order of basis then a new set of coefficients would have to be calculated and a new block of code incorporated into the program. Such a procedure may result in a larger program size but may also result in a fast execution time. Such ideas of course can also be applied to the calculation of stiffness matrices

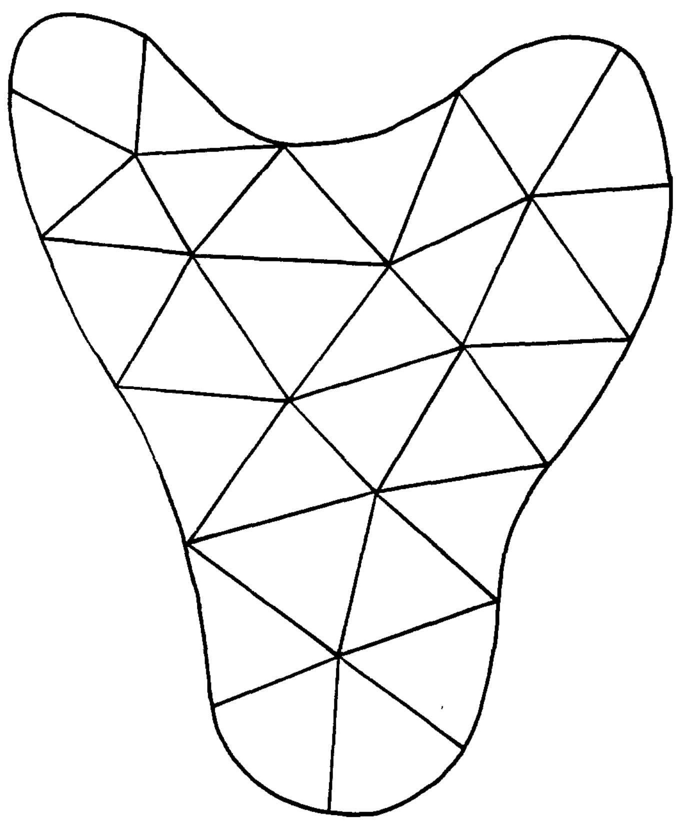

Fig. 2(a). This region comprises 29 elements, 15 of which have one curved side. The resulting linear system is of order 44. The region lies in the square $x, y \in (2, 7)$.

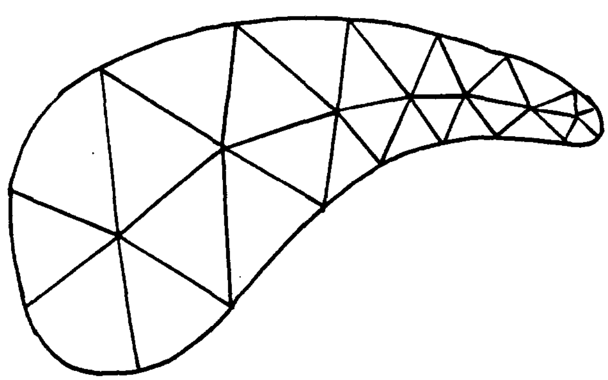

Fig. 2(b). This region comprises 29 elements, 17 of which have one curved side. The resulting linear system is of order 42. The region lies in the square $x, y \in (1, 8)$.

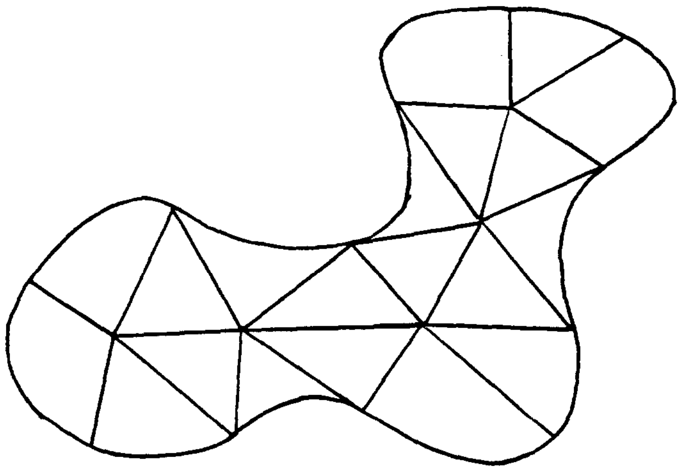

Fig. 2(c). This region comprises 20 elements, 12 of which have one curved side. The resulting linear system is of order 29. The region lies in the square $x, y \in (2, 8)$.

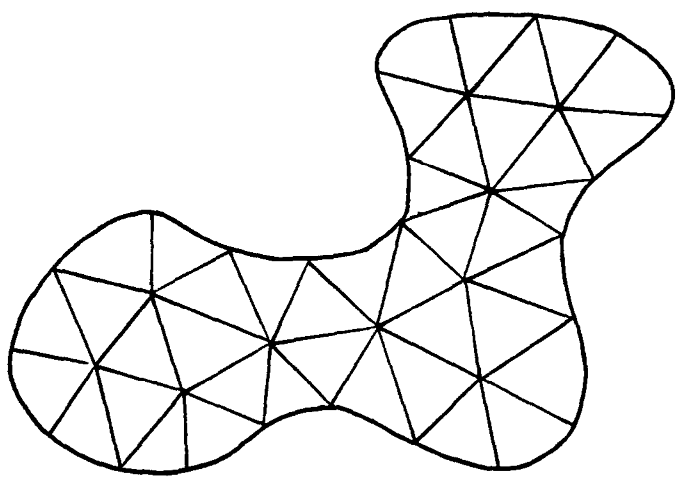

Fig. 2(d). This region comprises 41 elements, 23 of which have one curved side. The resulting linear system is of order 60. The region is a different subdivision of the same shape as that of Fig. 2(c).

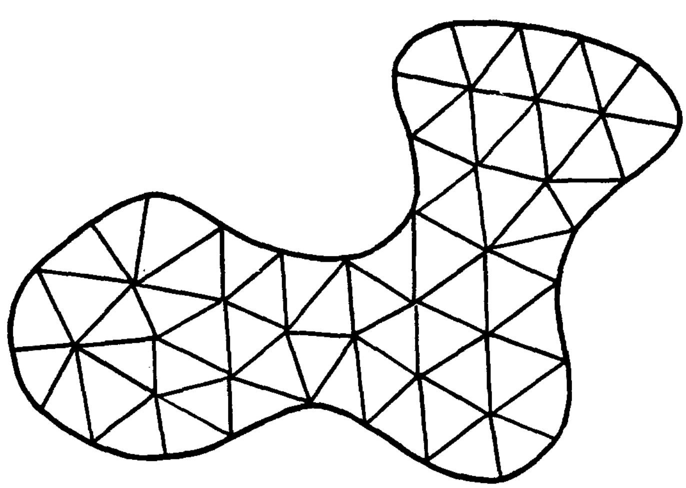

Fig. 2(e). This region comprises 62 elements, 28 of which have one curved side. The resulting linear system is of order 97. The region is a different subdivision of the same shape as that of Fig. 2(c).

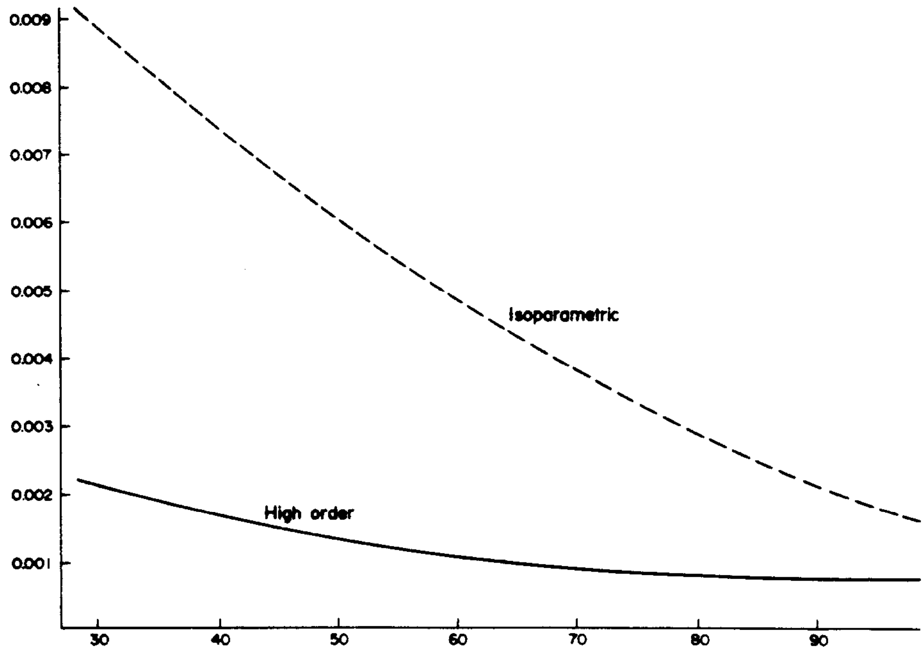

Fig. 3. A schematic representing the average L^2 error norms in trial solution u_3 to u_7 for the isoparametric and high order transformation methods based on Figs. 2(b), 2(c) and 2(d).

for isoparametric bases[9]. It is also possible that further economies could be made by judicious use of the fact that a high order basis satisfies more properties than the isoparametric one. These extra conditions could be used to deduce the number of integrals once a few have been calculated[7]. Also warranting further investigation is the stability of high order bases. Most high order conforming bases for curved elements are singular as the curved sides degenerate to straight sides. This can cause numerical sensitivity in the calculation of the required integrals for elements which are only very slightly curved. In the current tests a check on the element distortion was imposed and the isoparametric bases was used in cases where the use of the high order transformation method would have led to numerical instabilities. This was not a serious problem and occurred in only a few of the curved elements. This is discussed in some depth in the next paper.

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