Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematical Modelling 32 (2008) 112–121
APPLIED MATHEMATICAL MODELLING
www.elsevier.com/locate/apm

Pattern memory analysis based on stability theory of cellular neural networks

Zhigang Zeng [a,*], De-Shuang Huang [b], Zengfu Wang [c]

[a] School of Automation, Wuhan University of Technology, Wuhan, Hubei 430070, China
[b] Institute of Intelligent Machines, Chinese Academy of Sciences, Hefei, Anhui 230031, China
[c] Department of Automation, University of Science and Technology of China, Hefei, Anhui 230026, China

Received 1 March 2004; received in revised form 1 October 2006; accepted 14 November 2006
Available online 11 January 2007

Abstract

In this paper, several sufficient conditions are obtained to guarantee that the n-dimensional cellular neural network can have even ($\leqslant 2^n$) memory patterns. In addition, the estimations of attractive domain of such stable memory patterns are obtained. These conditions, which can be directly derived from the parameters of the neural networks, are easily verified. A new design procedure for cellular neural networks is developed based on stability theory (rather than the well-known perceptron training algorithm), and the convergence in the new design procedure is guaranteed by the obtained local stability theorems. Finally, the validity and performance of the obtained results are illustrated by two examples.

Keywords: Stability; Pattern memory; Cellular neural networks; Isolated equilibrium point

1. Introduction

Cellular neural networks, first introduced in 1988 [1], are of great interest due to the fact that they are among the easiest to implement in hardware. Cellular neural networks include the class of feedback neural networks with local interconnections and they are also suitable for very large-scale integration (VLSI) implementations of associative memories. The goal of associative memories is to store a set of desired patterns as stable memories such that a stored pattern can be retrieved when the initial pattern contains sufficient information about that pattern. In practice the desired memory patterns are usually represented by bipolar vectors (or binary vectors). Several salient studies of associative memories based on cellular neural networks can be found in [2–8].

In [5], the synthesis of cellular neural networks with space-invariant cloning-templates is considered; a designed algorithm based on the eigenstructure method [7] is developed. The realization of associative memories is considered in [2] via the class of (zero-input) cellular neural networks introduced in [1]. In addition, a synthesis procedure (designed algorithm) for cellular neural networks with space-invariant

* Corresponding author.
E-mail addresses: zhigangzeng@163.com, zgzeng@gmail.com (Z. Zeng).

0307-904X/$ - see front matter
doi:10.1016/j.apm.2006.11.010

cloning-templates is developed based on the well-known perceptron training algorithm in [2]. The global convergence of some neural networks is considered in [9–15]. This paper provides a new design procedure for cellular neural networks based on local stability theory (rather than the well-known perceptron training algorithm, or the global convergence of equilibrium point). The convergence of the neural networks resulting from the design procedure can be guaranteed by the obtained local stability theorems.

Consider a cellular neural network with multiple time-varying delays

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij}(t))) + u_i, \tag{1}$$

where $i = 1, 2, \ldots, n$, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \Re^n$, is a state vector, $A = (a_{ij}) \in \Re^{n\times n}$, $B = (b_{ij}) \in \Re^{n\times n}$ are connection weight matrices, delay time $\tau_{ij}(t) \leqslant \tau$ (constant), $\forall r \in \Re$,

$$f(r) = \frac{1}{2}(|r+1| - |r-1|), \tag{2}$$

$u = (u_1, \ldots, u_n)^{\mathrm{T}} \in \Re^n$ is an external input vector.

Denote $(-\infty,-1) = (-\infty,-1)^1 \times [-1,1]^0 \times (1,+\infty)^0$; $[-1,1] = (-\infty,-1)^0 \times [-1,1]^1 \times (1,+\infty)^0$; $(1,+\infty) = (-\infty,-1)^0 \times [-1,1]^0 \times (1,+\infty)^1$, $\Re = (-\infty,+\infty) = (-\infty,-1) \bigcup [-1,1] \bigcup (1,+\infty)$. Hence, $(-\infty,+\infty)^n$ can be disassembled into 3^n subspaces:

$$\Omega = \left\{ \prod_{i=1}^{n} (-\infty,-1)^{\delta_1^{(i)}} \times [-1,1]^{\delta_2^{(i)}} \times (1,+\infty)^{\delta_3^{(i)}}, (\delta_1^{(j)}, \delta_2^{(j)}, \delta_3^{(j)}) = (1,0,0) \text{ or } (0,1,0) \text{ or } (0,0,1),\ j = 1,2,\ldots,n \right\}. \tag{3}$$

In the following, denote $-(-\infty,-1) = (1,\infty), -(1,\infty) = (-\infty,-1)$,

$$(-\infty,-1)^{\delta^{(ik)}} = \begin{cases} (-\infty,-1), & i \neq k, \\ (1,\infty), & i = k, \end{cases}$$

$$(-\infty,-1)^{\delta^{(ikl)}} = \begin{cases} (-\infty,-1), & i \neq k \text{ or } i \neq l, \\ (1,\infty), & i = k \text{ or } i = l. \end{cases}$$

Denote a saturation region as

$$\Omega^{(\mathrm{s})} = \left\{ \prod_{i=1}^{n} (-\infty,-1)^{\delta^{(i)}} \times (1,+\infty)^{1-\delta^{(i)}},\ \delta^{(j)} = 1 \text{ or } 0,\ j = 1,2,\ldots,n \right\}.$$

Hence, Ω is made up of 3^n elements, and $\Omega^{(\mathrm{s})}$ is made up of 2^n elements. For example, when $n = 2$, all of the elements of Ω are depicted in Fig. 1.

A vector α is called a (stable) memory vector (or simply, a memory) of the cellular neural network (1) if $\alpha = (f(\beta_1), f(\beta_2), \ldots, f(\beta_n))^{\mathrm{T}}$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^{\mathrm{T}}$ is an asymptotically stable equilibrium point of the cellular neural network (1).

Denote $\mathscr{B}^n$ as the set of n-dimensional bipolar vectors, i.e.,

$$\mathscr{B}^n = \left\{ x \in \Re^n,\ x = (x_1, \ldots, x_n)^{\mathrm{T}},\ x_i = 1, \text{ or } -1,\ i = 1,2,\ldots,n \right\}.$$

Hence, $\mathscr{B}^n$ is made up of 2^n elements. For any $(s_1, s_2, \ldots, s_n)^{\mathrm{T}} \in \mathscr{B}^n$, let

$$\mathscr{L}(s_i) = \begin{cases} (1,\infty), & s_i = 1, \\ (-\infty,-1), & s_i = -1. \end{cases}$$

Consequently, $(s_1, s_2, \ldots, s_n)^{\mathrm{T}}$ and $\prod_{i=1}^{n} \mathscr{L}(s_i) = \mathscr{L}(s_1) \times \mathscr{L}(s_2) \times \cdots \times \mathscr{L}(s_n)$ represent a one-to-one correspondence.

The key problem to be addressed in this paper may be expressed as follows:

Design problem: Given $m (m \leqslant 2^n)$ vectors $\alpha^1, \alpha^2, \ldots, \alpha^m \in \mathscr{B}^n$, determine the connection weight matrices $A = (a_{ij})$, $B = (b_{ij})$ and the external input vector u such that $\alpha^1, \alpha^2, \ldots, \alpha^m$ are stable memory vectors of the cellular neural networks (1).

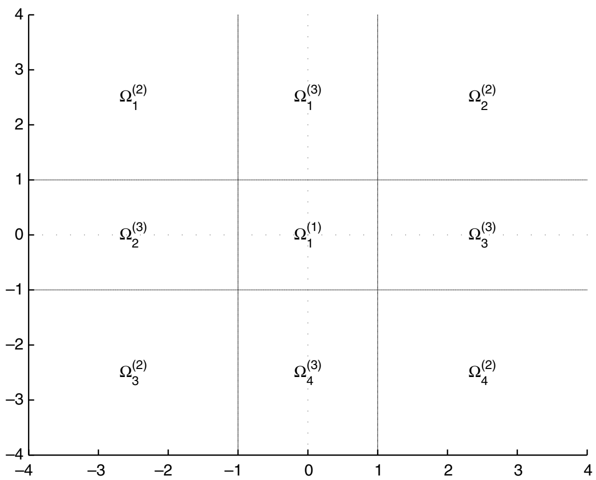

Fig. 1. Disassemble 2-dimensional space.

Definition 1. The set $\mathscr{D} \subset \Re^n$ is said to be a globally exponentially attractive set of (1), if for solution $x(t; t_0, \phi)$ of (1) with any initial condition $\phi(\vartheta) \in C([t_0 - \tau, t_0], \Re^n)$, there exist constants $\alpha > 0$ and $\beta(\phi)$ (depended on ϕ) such that

$$\inf_{\hat{x} \in \mathscr{D}} \|x(t; t_0, \phi) - \hat{x}\| \leqslant \beta(\phi) \exp\{-\alpha(t - t_0)\}.$$

Definition 2. The point x^* is said to be an isolated equilibrium point of (1) if there exists $\delta > 0$ such that x^* is the only equilibrium point of (1) in $\{x | \|x - x^*\| < \delta, x \in \Re^n\}$.

The remaining part of the paper consists of three sections. In Section 2, the main results are derived. In Section 3, two illustrative examples are provided. And in Section 4, concluding remarks are given.

2. The main results

In the following, we always assume $u_i = 0, i \in \{1, 2, \ldots, n\}$.

Theorem 1. *If there exists* $(s_1, s_2, \ldots, s_n)^{\mathrm{T}} \in \mathscr{B}^n$ *such that* $\forall i \in \{1, 2, \ldots, n\}$,

$$\left(\sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j\right) s_i > 1, \tag{4}$$

then (1) *has neither more nor less than* 2 *isolated equilibrium points located in* $\Omega_+ = \prod_{i=1}^n \mathscr{L}(s_i)$ *and* $\Omega_- = \prod_{i=1}^n (-\mathscr{L}(s_i))$, *respectively.*

Proof. Choose $x^* = (x_1^*, \ldots, x_n^*)^{\mathrm{T}}$, where $x_i^* = \sum_{j=1}^n (a_{ij} + b_{ij}) s_j$. From (4), $x^* \in \Omega_+$. In addition, $-x_i^* + \sum_{j=1}^n a_{ij} f(x_j^*) + \sum_{j=1}^n b_{ij} f(x_j^*) = 0$; i.e., x^* is an equilibrium point located in Ω_+. Similarly, $-x^* \in \Omega_-$, and $x_i^* + \sum_{j=1}^n a_{ij} f(-x_j^*) + \sum_{j=1}^n b_{ij} f(-x_j^*) = 0$. Thus, $-x^*$ is an equilibrium point located in Ω_-. In the saturation region $\Omega^{(s)}$, if (1) has an equilibrium point, then this is always an isolated equilibrium point. Hence, (1) has 2 isolated equilibrium points located in $\Omega_+ \in \Omega^{(s)}$ and $\Omega_- \in \Omega^{(s)}$, respectively. □

Theorem 2. $\forall (s_1, s_2, \ldots, s_n)^{\mathrm{T}} \in \mathscr{B}^n$, *denote* $\bar{\Omega} = \prod_{i=1}^n \mathscr{L}(s_i)$. *If* $x^* \in \bar{\Omega}$ *is an equilibrium point of* (1), *then it is a locally exponentially stable equilibrium point, and* $\bar{\Omega}$ *is its locally exponentially attractive region.*

Proof. If $x(t),\ x(t - \tau(t)) \in \bar{\Omega}$, then from (1) and (2),

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + \sum_{j=1}^{n} a_{ij}s_j + \sum_{j=1}^{n} b_{ij}s_j. \quad (5)$$

Obviously, if (5) has an equilibrium point, then this equilibrium point must be globally exponentially stable. Hence x^* is locally exponentially stable, and $\bar{\Omega}$ is its locally exponentially attractive region. □

2.1. Numeration of equilibrium points

Without loss of generality, we assume $a_{ij}, b_{ij} \geqslant 0, i, j \in \{1, 2, \ldots, n\}$ in proving theorems in this section. The proof for the other case can be reasoned in the same manner.

Theorem 3. *If* $\forall i \in \{1, 2, \ldots, n\}$,

$$\sum_{j=1}^{n}(a_{ij} + b_{ij}) > 1, \quad (6)$$

and $\forall k \in \{2, 3, \ldots, n\}$,

$$\sum_{j=1, j\neq k}^{n}(a_{1j} + b_{1j}) - a_{1k} - b_{1k} < 1, \quad (7)$$

then (1) *has neither more nor less than* 2 *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(s)}$.

Proof. Choose $x^* = (x_1^*, \ldots, x_n^*)^{\mathrm{T}}$ or $x^* = -(x_1^*, \ldots, x_n^*)^{\mathrm{T}}$, where $x_i^* = \sum_{j=1}^{n}(a_{ij} + b_{ij})$. From (6), $x_i^* > 1$. Hence, $-x_i^* + \sum_{j=1}^{n} a_{ij} f(x_j^*) + \sum_{j=1}^{n} b_{ij} f(x_j^*) = 0$; i.e., x^* is an equilibrium point located in $(-\infty, -1)^n$ or $(1, \infty)^n$. According to Theorem 2, x^* is locally exponentially stable.

Assume that there exists another equilibrium point $\bar{x}^* = (\bar{x}_1^*, \ldots, \bar{x}_n^*)^{\mathrm{T}}$ located in the saturation region $\Omega^{(s)}$, and moreover, without loss of generality, assume $(1, s_2, \ldots, s_n)^{\mathrm{T}} \in \mathscr{B}^n$, $\bar{x}^* \in \Omega_1 = \mathscr{L}(1) \times \mathscr{L}(s_2) \times \mathscr{L}(s_n) \in \Omega^{(s)}$. Then, there exists $k \in \{2, 3, \ldots, n\}$, such that $s_k = -1$, and $-\bar{x}_i^* + \sum_{j=1}^{n} a_{ij} f(\bar{x}_j^*) + \sum_{j=1}^{n} b_{ij} f(\bar{x}_j^*) = 0$; i.e., $-\bar{x}_i^* + \sum_{j=1}^{n} a_{ij}s_j + \sum_{j=1}^{n} b_{ij}s_j = 0$. Thus, from (7),

$$\bar{x}_1^* = \sum_{j=1}^{n} a_{1j}s_j + \sum_{j=1}^{n} b_{1j}s_j \leqslant \sum_{j=1, j\neq k}^{n}(a_{1j} + b_{1j}) - a_{1k} - b_{1k} < 1.$$

This contradicts that $\bar{x}^*$ is an equilibrium point located in Ω_1. Hence, Theorem 3 holds. □

Theorem 4. *If there exists* $k \in \{2, 3, \ldots, n\}$ *such that* $\forall i \in \{1, 2, \ldots, n\}$,

$$\sum_{j=1, j\neq k}^{n}(a_{ij} + b_{ij}) - a_{ik} - b_{ik} > 1, \quad i \neq k; \quad (8)$$

$$\sum_{j=1, j\neq k}^{n}(a_{kj} + b_{kj}) - a_{kk} - b_{kk} < -1, \quad (9)$$

and $\forall m \in \{2, 3, \ldots, n\}, m \neq k$,

$$\sum_{j=1, j\neq m}^{n}(a_{1j} + b_{1j}) - a_{1m} - b_{1m} < 1, \quad (10)$$

then (1) *has neither more nor less than* $2 + 2 \times 1$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(s)}$.

Proof. $a_{ij}, b_{ij} \geqslant 0$, (8) and (9) imply that (6) holds. It is similar to the proof of Theorem 3 that (1) has 2 isolated and locally exponentially stable equilibrium points located in $(-\infty, -1)^n$ and $(1, \infty)^n$, respectively.

Choose $x^* = (x_1^*, \ldots, x_n^*)^T$, where $x_i^* = \sum_{j=1, j\neq k}^n (a_{ij} + b_{ij}) - a_{ik} - b_{ik}$. (8) and (9) imply that $x_i^* \begin{cases} > 1, & i \neq k \\ < -1 & i = k. \end{cases}$ Hence, $x^* \in \prod_{i=1}^n (-(-\infty, -1)^{\delta^{(ik)}}) \in \Omega^{(s)}$, and $-x_i^* + \sum_{j=1}^n a_{ij} f(x_j^*) + \sum_{j=1}^n b_{ij} f(x_j^*) = 0$; i.e., x^* is an equilibrium point located in $\prod_{i=1}^n (-(-\infty, -1)^{\delta^{(ik)}})$. It is similar to prove that $-x^*$ is an equilibrium point located in $\prod_{i=1}^n (-\infty, -1)^{\delta^{(ik)}} \in \Omega^{(s)}$.

Assume that there exists another equilibrium point $\bar{x}^* = (\bar{x}_1^*, \ldots, \bar{x}_n^*)^T$ located in the saturation region $\Omega^{(s)}$, and moreover, without loss of generality, assume that $(1, s_2, \ldots, s_n)^T \in \mathscr{B}^n$, $\bar{x}^* \in \Omega_2 = \mathscr{L}(1) \times \mathscr{L}(s_2) \times \mathscr{L}(s_n) \in \Omega^{(s)}$. Then, there exists $m \in \{2, 3, \ldots, n\}, m \neq k$ such that $s_m = -1$, and $-\bar{x}_i^* + \sum_{j=1}^n a_{ij} f(\bar{x}_j^*) + \sum_{j=1}^n b_{ij} f(\bar{x}_j^*) = 0$; i.e., $-\bar{x}_i^* + \sum_{j=1}^n a_{ij} s_j + \sum_{j=1}^n b_{ij} s_j = 0$. Thus, from (10),

$$\bar{x}_1^* = \sum_{j=1}^n a_{1j} s_j + \sum_{j=1}^n b_{1j} s_j \leqslant \sum_{j=1, j\neq m}^n (a_{1j} + b_{1j}) - a_{1m} - b_{1m} < 1.$$

This contradicts that $\bar{x}^*$ is an equilibrium point located in Ω_2. Hence, Theorem 4 holds. □

Theorem 5. *If* $\forall i \in \{1, 2, \ldots, n\}, k \in N_1 = \{k_1, k_2, \ldots, k_p\} \subset \{2, 3, \ldots, n\}$ *where* $p \leqslant n - 1$,

$$\sum_{j=1, j\neq k}^n (a_{ij} + b_{ij}) - a_{ik} - b_{ik} > 1, \quad i \neq k; \tag{11}$$

$$\sum_{j=1, j\neq k}^n (a_{kj} + b_{kj}) - a_{kk} - b_{kk} < -1, \tag{12}$$

$\forall m \in \{2, 3, \ldots, n\} - N_1$,

$$\sum_{j=1, j\neq m}^n (a_{1j} + b_{1j}) - a_{1m} - b_{1m} < 1; \tag{13}$$

in addition $\forall l, q \in N_1$,

$$\sum_{j=1, j\neq l,q}^n (a_{1j} + b_{1j}) - a_{1l} - a_{1q} - b_{1l} - b_{1q} < 1, \tag{14}$$

then (1) *has neither more nor less than* $2 + 2 \times p$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(s)}$.

Proof. $a_{ij}, b_{ij} \geqslant 0$, (11) and (12) imply that (6) holds. It is similar to the proof of Theorem 3 that the cellular neural network (1) has 2 isolated and locally stable equilibrium points located in $(-\infty, -1)^n$ and $(1, \infty)^n$, respectively.

$\forall k \in N_1$, it is similar to the proof of Theorem 4 that (1) has 2 isolated equilibrium points located in $\prod_{i=1}^n (-\infty, -1)^{\delta^{(ik)}}$ and $\prod_{i=1}^n (-(-\infty, -1)^{\delta^{(ik)}})$, respectively, which are locally exponentially stable. Since N_1 is made up of p elements, (1) has $2 \times p$ isolated and locally exponentially stable equilibrium points located in $\bigcup_{k\in N_1} (\prod_{i=1}^n (-\infty, -1)^{\delta^{(ik)}} \bigcup \prod_{i=1}^n (-(-\infty, -1)^{\delta^{(ik)}}))$.

Assume that there exists another equilibrium point $\bar{x}^* = (\bar{x}_1^*, \ldots, \bar{x}_n^*)^T$ located in the saturation region $\Omega^{(s)}$, and moreover, without loss of generality, assume that $(1, s_2, \ldots, s_n)^T \in \mathscr{B}^n$, $\bar{x}^* \in \Omega_3 = \mathscr{L}(1) \times \mathscr{L}(s_2) \times \mathscr{L}(s_n) \in \Omega^{(s)}$. Then, there exists $m \in \{2, 3, \ldots, n\}, m \notin N_1$ such that $s_m = -1$, or there exist $l, q \in N_1$ such that $s_l = -1, s_q = -1$, and $-\bar{x}_i^* + \sum_{j=1}^n a_{ij} f(\bar{x}_j^*) + \sum_{j=1}^n b_{ij} f(\bar{x}_j^*) = 0$; i.e., $-\bar{x}_i^* + \sum_{j=1}^n a_{ij} s_j + \sum_{j=1}^n b_{ij} s_j = 0$. Thus, from (13) or (14),

$$\bar{x}_1^* = \sum_{j=1}^n a_{1j} s_j + \sum_{j=1}^n b_{1j} s_j \leqslant \sum_{j=1, j\neq m}^n (a_{1j} + b_{1j}) - a_{1m} - b_{1m} < 1;$$

or

$$\bar{x}_1^* = \sum_{j=1}^{n} a_{1j}s_j + \sum_{j=1}^{n} b_{1j}s_j \leqslant \sum_{j=1,j\neq l,q}^{n} (a_{1j}+b_{1j}) - a_{1l} - b_{1l} - a_{1q} - b_{1q} < 1.$$

This contradicts that $\bar{x}^*$ is an equilibrium point located in Ω_3. Hence, Theorem 5 holds. □

Theorem 6. *If* $\forall i \in \{1,2,\ldots,n\}$ *and* $\forall k,l \in \{2,3,\ldots,n\}$,

$$\sum_{j=1,j\neq k,l}^{n} (a_{ij}+b_{ij}) - a_{ik} - a_{il} - b_{ik} - b_{il} > 1, \quad i \neq k, \text{ and } i \neq l, \tag{15}$$

$$\sum_{j=1,j\neq k}^{n} (a_{ij}+b_{ij}) - a_{kk} - b_{kk} < -1, \tag{16}$$

$$\sum_{j=1,j\neq l}^{n} (a_{ij}+b_{ij}) - a_{ll} - b_{ll} < -1, \tag{17}$$

and $\forall m,p,q \in \{2,3,\ldots,n\}$,

$$\sum_{j=1,j\neq m,p,q}^{n} (a_{1j}+b_{1j}) - \sum_{j=m,p,q} (a_{1j}+b_{1j}) < 1, \tag{18}$$

then the cellular neural network (1) *has neither more nor less than* $2 + 2\times(n-1) + 2\times \mathscr{C}_{n-1}^2$ *isolated equilibrium points located in the saturation region* $\Omega^{(s)}$, *which are locally exponentially stable, where* $\mathscr{C}_{n-1}^2 = \frac{(n-1)\times(n-2)}{2}$.

Proof. $a_{ij}, b_{ij} \geqslant 0$, (15), (16) and (17) imply that (11) and (12) hold. It is similar to the proof of Theorem 5 that (1) has $2+2\times(n-1)$ isolated and locally exponentially stable equilibrium points located in

$$(-\infty,-1)^n \bigcup (1,\infty)^n \bigcup_{k=2}^{n} \left(\prod_{i=1}^{n} (-\infty,-1)^{\delta^{(ik)}} \bigcup \prod_{i=1}^{n} (-(-\infty,-1)^{\delta^{(ik)}}) \right).$$

$\forall k,l \in \{2,3,\ldots,n\}$, choose $x^* = (x_1^*,\ldots,x_n^*)^{\mathrm{T}}$, where $x_i^* = \sum_{j=1,j\neq k,l}^{n}(a_{ij}+b_{ij}) - \sum_{j=k,l}(a_{ij}+b_{ij})$. (15), (16) and (17) imply that $x_i^* \begin{cases} >1, & i\neq k,l \\ <-1 & i=k,l. \end{cases}$ Hence, $x^* \in \prod_{i=1}^{n}(-(-\infty,-1)^{\delta^{(ikl)}}) \in \Omega^{(s)}$, and $-x_i^* + \sum_{j=1}^{n} a_{ij} f(x_j^*) + \sum_{j=1}^{n} b_{ij} f(x_j^*) = 0$; i.e., x^* is an equilibrium point located in $\prod_{i=1}^{n}(-(-\infty,-1)^{\delta^{(ikl)}})$. It is similar to prove that $-x^*$ is an equilibrium point located in $\prod_{i=1}^{n}(-\infty,-1)^{\delta^{(ikl)}} \in \Omega^{(s)}$, which is locally exponentially stable.

Since the set $\{(k,l), k,l \in \{2,3,\ldots,n\}, k\neq l\}$ is made up of $(n-1)\times(n-2)$ elements, (1) has $2\times\mathscr{C}_{n-1}^2$ isolated and locally exponentially stable equilibrium points located in $\bigcup_{k=2}^{n}\left(\bigcup_{l=k+1}^{n}\left(\prod_{i=1}^{n}(-\infty,-1)^{\delta^{(ikl)}} \bigcup \prod_{i=1}^{n}(-(-\infty,-1)^{\delta^{(ikl)}})\right)\right)$.

Assume that there exists another equilibrium point $\bar{x}^* = (\bar{x}_1^*,\ldots,\bar{x}_n^*)^{\mathrm{T}}$ located in the saturation region $\Omega^{(s)}$, and moreover, without loss of generality, assume that $(1,s_2,\ldots,s_n)^{\mathrm{T}} \in \mathscr{B}^n$, $\bar{x}^* \in \Omega_4 = \mathscr{L}(1)\times\mathscr{L}(s_2)\times \mathscr{L}(s_n) \in \Omega^{(s)}$. Then, there exist $m,p,q \in \{2,3,\ldots,n\}$ such that $s_m=-1, s_p=-1, s_q=-1$, and $-\bar{x}_i^* + \sum_{j=1}^{n} a_{ij} f(\bar{x}_j^*) + \sum_{j=1}^{n} b_{ij} f(\bar{x}_j^*) = 0$; i.e., $-\bar{x}_i^* + \sum_{j=1}^{n} a_{ij}s_j + \sum_{j=1}^{n} b_{ij}s_j = 0$. Thus, from (18),

$$\bar{x}_1^* = \sum_{j=1}^{n} a_{1j}s_j + \sum_{j=1}^{n} b_{1j}s_j \leqslant \sum_{j=1,j\neq m,p,q}^{n} (a_{1j}+b_{1j}) - \sum_{j=m,p,q} (a_{1j}+b_{1j}) < 1.$$

This contradicts that $\bar{x}^*$ is an equilibrium point located in Ω_4. Hence, Theorem 6 holds. □

Theorem 7. *If* $\forall i \in \{1,2,\ldots,n\}$,

$$a_{ii} + b_{ii} - \sum_{j=2}^{n} (a_{ij}+b_{ij}) > 1, \tag{19}$$

then the cellular neural network (1) *has neither more nor less than* $2+2\times(n-1)+2\times\sum_{j=2}^{n-1}\mathscr{C}_{n-1}^{j}$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(s)}$, *where* $\mathscr{C}_{n-1}^{j} = \frac{(n-1)\times(n-2)\times\cdots\times(n-j)}{1\times 2\times\cdots\times j}$.

Proof. $\forall(s_1,s_2,\ldots,s_n)^{\mathrm{T}}\in\mathscr{B}^n$, choose $x^*=(x_1^*,\ldots,x_n^*)^{\mathrm{T}}$, where $x_i^*=\sum_{j=1}^n(a_{ij}+b_{ij})s_j$. From (19), $x^*\in\prod_{i=1}^n\mathscr{L}(s_i)\in\Omega^{(\mathrm{s})}$. In addition, $-x_i^*+\sum_{j=1}^n a_{ij}f(x_j^*)+\sum_{j=1}^n b_{ij}f(x_j^*)=0$; i.e., x^* is an equilibrium point located in $\prod_{i=1}^n\mathscr{L}(s_i)$.

Since the set $\mathscr{B}^n$ is made up of 2^n elements, the cellular neural network (1) has 2^n isolated equilibrium points located in the saturation region $\Omega^{(\mathrm{s})}$. According to Theorem 2, these equilibrium points are locally exponentially stable. □

Remark. $2+2\times(n-1)+2\times\sum_{i=2}^{n-1}\mathscr{C}_{n-1}^i=2^n$.

2.2. Design of the connection weights

For convenience, in this subsection, we let $b_{ij}=0,\forall i,j\in\{1,2,\ldots,n\}$.

Corollary 1. *If* $\forall i,j\in\{1,2,\ldots,n\}$,

$$a_{ij}=\begin{cases}2/n, & i=j,\\ 1/n, & i\neq j,\end{cases}$$

then (1) *has neither more nor less than* 2 *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(\mathrm{s})}$.

Proof. According to Theorem 3, Corollary 1 holds. □

Corollary 2. *If there exists a constant* $k\in\{2,3,\ldots,n\}$, *such that* $\forall i,j\in\{1,2,\ldots,n\}$,

$$a_{ij}=\begin{cases}3/n, & i=j\neq k,\\ 1/n, & i\neq j,\ j\neq k,\\ 2, & i=j=k,\\ 1/(2n), & i\neq j,\ j=k,\end{cases}$$

then (1) *has neither more nor less than* $2+2\times1$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(\mathrm{s})}$.

Proof. According to Theorem 4, Corollary 2 holds. □

Corollary 3. *If there exists a set* $N_1=\{k_1,k_2,\ldots,k_p\}\subset\{2,3,\ldots,n\}$ *where* $p\leqslant n-1$, *such that for all* $k\in N_1$, $\forall i,j\in\{1,2,\ldots,n\}$,

$$a_{ij}=\begin{cases}(3+(p-1)/2)/n, & i=j\neq k,\\ 1/n, & i\neq j,\ j\neq k,\\ 2, & i=j=k,\\ 1/(2n), & i\neq j,\ j=k,\end{cases}$$

then (1) *has neither more nor less than* $2+2\times p$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(\mathrm{s})}$.

Proof. According to Theorem 5, Corollary 3 holds. □

Corollary 4. *If* $n\geqslant4$ *and* $\forall i,j\in\{1,2,\ldots,n\}$,

$$a_{ij}=\begin{cases}6/n, & i=j=1,\\ 1/n, & i\neq j,\\ 2, & i=j\neq1,\end{cases}$$

then (1) *has neither more nor less than* $2 + 2 \times (n - 1) + 2 \times \mathscr{C}_{n-1}^{2}$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(s)}$.

Proof. According to Theorem 6, Corollary 4 holds □

Corollary 5. *If* $\forall i, j \in \{1, 2, \ldots, n\}$,

$$a_{ij} = \begin{cases} 2, & i = j, \\ 1/n, & i \neq j, \end{cases}$$

then (1) *has neither more nor less than* $2 + 2 \times (n - 1) + 2 \times \sum_{i=2}^{n-1} \mathscr{C}_{n-1}^{i}$ *isolated and locally exponentially stable equilibrium points located in the saturation region* $\Omega^{(s)}$.

Proof. According to Theorem 7, Corollary 5 holds. □

2.3. A new design procedure

Step 1. Denote m desired patterns by m vectors in $\mathscr{B}^n$; i.e., we obtained m n-dimensional vectors $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(m)} \in \mathscr{B}^n$. n is number of neurons in the designed cellular neural networks.

Step 2. Design connection weight matrix A according to Theorems 3–7 and Corollarys 1–5 such that $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(m)}$ and their allelomorph vectors are stable memory vectors of the following cellular neural network

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + Af(x(t)), \tag{20}$$

while the other vectors are not. The allelomorph vector of β is defined as $-\beta$.

3. Examples

In this section, we provide two examples to illustrate the results obtained in the preceding section.

Example 1. Design a cellular neural network with 4 neurons ($n = 4$) to store three patterns shown in Fig. 2 as stable memories (black $= -1$ and white $= 1$).

Step 1. $\beta^{(1)} = (1, 1, 1, 1)^{\mathrm{T}}$; $\beta^{(2)} = (1, -1, 1, 1)^{\mathrm{T}}$; and $\beta^{(3)} = (1, 1, -1, 1)^{\mathrm{T}}$ are three patterns which are desired to be stable memory vectors of the potential cellular neural network.

Step 2. Choose

$$A_1 = \begin{pmatrix} 7/8, & 1/8, & 1/8, & 1/4 \\ 1/4, & 2, & 1/8, & 1/4 \\ 1/4, & 1/8, & 2, & 1/4 \\ 1/4, & 1/8, & 1/8, & 7/8 \end{pmatrix}.$$

According to Corollary 3, the following cellular neural network

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + A_1 f(x(t)) \tag{21}$$

Fig. 2. Three desired memory patterns for Example 1.

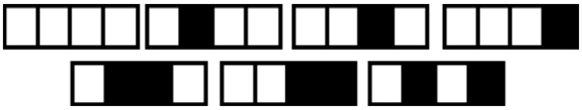

Fig. 3. Seven desired memory patterns for Example 2.

has 6 stable memory patterns; i.e., $\beta^{(1)}$; $\beta^{(2)}$; $\beta^{(3)}$ and their allelomorph vectors. In addition, any other vector is not a stable memory vector of the cellular neural network (21).

Example 2. Design a cellular neural network with 4 neurons ($n = 4$). The objective is to store seven patterns shown in Fig. 3 as stable memories (black $= -1$ and white $= 1$).

Step 1. $\beta^{(1)} = (1,1,1,1)^{\mathrm{T}}$; $\beta^{(2)} = (1,-1,1,1)^{\mathrm{T}}$; $\beta^{(3)} = (1,1,-1,1)^{\mathrm{T}}$; $\beta^{(4)} = (1,1,1,-1)^{\mathrm{T}}$; $\beta^{(5)} = (1,-1,-1,1)^{\mathrm{T}}$; $\beta^{(6)} = (1,1,-1,-1)^{\mathrm{T}}$ and $\beta^{(7)} = (1,-1,1,-1)^{\mathrm{T}}$ are seven patterns which are desired to be stable memory vectors of the potential cellular neural network.

Step 2. Choose

$$A_2 = \begin{pmatrix} 6/4, & 1/4, & 1/4, & 1/4 \\ 1/4, & 2, & 1/4, & 1/4 \\ 1/4, & 1/4, & 2, & 1/4 \\ 1/4, & 1/4, & 1/4, & 2 \end{pmatrix}.$$

According to Corollary 4, the following cellular neural network

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + A_2 f(x(t)) \tag{22}$$

has 14 stable memory patterns; i.e., $\beta^{(1)}$; $\beta^{(2)}$; ...; $\beta^{(7)}$ and their allelomorph vectors. In addition, any other vector is not a stable memory vector of (22).

4. Concluding remarks

In the paper, a new design procedure for cellular neural networks is developed based on the stability theory instead of base on the well-known perceptron training algorithm. The convergence of the designed cellular neural network can be guaranteed by some conditions derived from the local stability analysis of cellular neural networks. With these conditions, an n-dimensional cellular neural network can always have m ($m \leqslant 2^n$) stable memory patterns, where m is an even number. In addition, the attractive domains of such stable memory patterns can be easily estimated. Moreover, the conditions depend only on the parameters of the neural networks, and consequently, are easy to check. So, the new design procedure provides a novel and convenient method to design cellular neural networks for associative memories. Finally, two examples are discussed to illustrate the validity and performance of the results.

Acknowledgement

This work was supported by the Natural Science Foundation of China under Grant 60405002.

References

[1] L.O. Chua, L. Yang, Cellular neural networks: theory, IEEE Trans. Circ. Syst. 35 (1988) 1257–1272.

[2] Z. Lu, D. Liu, A new synthesis procedure for a class of cellular neural networks with space-invariant cloning template, IEEE Trans. Circ. Syst.-II 45 (1998) 1601–1605.

[3] M. Brucoli, L. Carnimeo, G. Grassi, Discrete-time cellular neural networks for associative memories with learning and forgetting capabilities, IEEE Trans. Circ. Syst.-I 42 (1995) 396–399.
[4] L.O. Chua, T. Roska, The CNN paradigm, IEEE Trans. Circ. Syst.-I 40 (1993) 147–156.
[5] D. Liu, Cloning template design of cellular neural networks for associative memories, IEEE Trans. Circ. Syst.-I 44 (1997) 646–650.
[6] D. Liu, Z. Lu, A new synthesis approach for feedback neural networks based on the perceptron training algorithm, IEEE Trans. Neural Networks 8 (1997) 1468–1482.
[7] A.N. Michel, J.A. Farrell, Associative memories via artificial neural networks, IEEE Contr. Syst. Mag. 10 (1990) 6–17.
[8] G. Seiler, A.J. Schuler, J.A. Nossek, Design of robust cellular neural networks, IEEE Trans. Circ. Syst.-I 40 (1993) 358–364.
[9] Z.G. Zeng, J. Wang, Complete stability of cellular neural networks with time-varying delays, IEEE Trans. Circ. Syst.-I: Regular Papers 53 (2006) 944–955.
[10] Z.G. Zeng, J. Wang, Improved conditions for global exponential stability of recurrent neural networks with time-varying delays, IEEE Trans. Neural Networks 17 (2006) 623–635.
[11] Z.G. Zeng, J. Wang, X.X. Liao, Stability analysis of delayed cellular neural networks described using cloning templates, IEEE Trans. Circ. Syst.-I: Regular Papers 51 (2004) 2313–2324.
[12] S. Mohamad, Global exponential stability in continuous-time and discrete-time delayed bidirectional neural networks, Physica D 159 (2001) 233–251.
[13] T. Chen, S. Amari, New theorems on global convergence of some dynamical systems, Neural Networks 14 (2001) 251–255.
[14] J. Cao, Q. Li, On the exponential stability and periodic solutions of delayed cellular neural networks, J. Math. Anal. Appl. 252 (2000) 50–64.
[15] P.V.D. Driessche, X. Zou, Global attractivity in delayed Hopfield neural network models, SIAM J. Appl. Math. 58 (6) (1998) 1878–1890.