Newton Polytopes of the Classical Resultant and Discriminant

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This paper is devoted to the following question of a very classical nature. Let \( P(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \) \( Q(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n \) be two polynomials in one variable. Let \( \text{Res}(P, Q) \) be the resultant of \( P \) and \( Q. \) By definition (see any textbook in algebra, e.g., [1]) it is a polynomial in the coefficients \( a_i \) and \( b_j \) uniquely up to sign determined by the conditions that it is irreducible over \( \mathbb{Z} \) and vanishes whenever \( P \) and \( Q \) have a common root. Consider the decomposition of \( \text{Res}(P, Q) \) into the sum of monomials,

\[
\text{Res}(P, Q) = \sum_{p,q} c_{pq} a^p b^q, \tag{1}
\]

where \( p = (p_0, p_1, \ldots, p_m) \in \mathbb{Z}_{\geq 0}^{m+1}, \) \( q = (q_0, q_1, \ldots, q_n) \in \mathbb{Z}_{\geq 0}^{n+1}, \) \( a^p = \prod_i a_i^{p_i}, \) \( b^q = \prod_j b_j^{q_j}. \) We want to find an explicit description of all monomials occurring in (1) and the corresponding coefficients \( c_{pq}. \) In particular, consider the Newton polytope \( N = N_{m,n} \) of \( \text{Res}(P, Q), \) i.e., the convex hull in \( \mathbb{R}^{m+n+2} \) of the set \( \{(p, q): c_{pq} \neq 0\}. \) It turns out to be a very interesting convex polytope, and we describe explicitly its face lattice. Surprisingly enough, although the problem seems to be very natural we were not able to find it in the literature. For example, in [2] there are given several expressions for \( \text{Res}(P, Q) \) but not in terms of monomials. Since the number of monomials in \( \text{Res}(P, Q) \) increases very rapidly with the growth of \( m \) and \( n, \) maybe it required some courage just to begin working on this problem!

Our interest in this kind of problem arose from the study of hypergeometric functions related to toric varieties (see [3–5]). They can be roughly described as follows. Let \( A \) be a finite set of Laurent monomials in several variables. We associate to \( A \) a holonomic system of linear differential equations on the function \( \Phi(v), v = (v_\omega)_{\omega \in A} \in \mathbb{C}^A. \) This system is called \( A \)-hypergeometric system, and its solutions are \( A \)-hypergeometric functions. The study of singularities of the \( A \)-hypergeometric system leads to an
interesting algebraic theory of so-called $A$-discriminants. Roughly speaking the $A$-discriminant is a polynomial function on $\mathbb{C}^A$ defining the variety of singularities of the $A$-hypergeometric system. $A$-discriminants include as special cases ordinary determinants and their multidimensional analogues (so-called hyperdeterminants [6]) as well as classical discriminants and resultants. So the theory of $A$-discriminants is of independent interest; it was developed by the authors in [7–9].Remarkably, the Newton polytope $N_A$ of the $A$-discriminant admits an explicit description in terms of geometry of the set $A$ and its convex hull $Q$. Roughly speaking the vertices of $N_A$ correspond to triangulations of $Q$ with vertices on $A$. Therefore, algebraic properties of $A$-discriminants turn out to be closely connected with combinatorial geometric properties of $Q$ and $A$.

In the process of developing the general theory of $A$-discriminants we discovered that it gives us new interesting results even in the special case of the classical discriminant and resultant for polynomials in one variable. The goal of this paper is to present these results in the most self-contained and elementary manner.

The paper is organized as follows. In Section 1 we formulate all main results on the structure of the convex polytope $N_{m,n}$. It turns out that the polytope $N_{m,n}$ has some resemblance with the hypersimplex $A_{m,n}$ which can be defined as the section of the unit cube in $\mathbb{R}^{m+n}$ by the hyperplane \{$\sum x_i = m$\}. Namely, $N_{m,n}$ and $A_{m,n}$ have the same dimension \{\binom{m+n-1}{m}\} and the same number of vertices (\binom{m+n}{m}). But they are not combinatorially equivalent.

Sections 2 and 3 contain the proofs of the results of Section 1. In Section 2 we give several combinatorial descriptions of the vertices of $N_{m,n}$ deducing them from the general results on $A$-discriminants. Then in Section 3 we describe the face lattice of $N_{m,n}$. This is done in a completely elementary way although it is possible in principle to deduce this also from the general theory.

Section 4 is devoted to the discussion of the coefficients $c_{pq}$ in (1). General theory gives us only the coefficients corresponding to vertices of the Newton polytope; in our case all these coefficients are equal to $\pm 1$. We give an answer for all $c_{pq}$ in terms of the classical theory of symmetric functions. This answer presumably suggests that there is no simple general formula for all the coefficients in the $A$-discriminant.

Finally, in Section 5 we formulate and prove the analogous results for the Newton polytope $N_r$ of the classical discriminant $D(P)$ of a polynomial $P(x)$ of degree $r$ in one variable. All the arguments are essentially the same as for the resultant but are much easier. It turns out that $N_r$ is combinatorially equivalent to a $(r-1)$-cube. The description of all the coefficients in $D(P)$ still remains an interesting open problem.
1. Newton Polytope of the Resultant: Formulation of the Results

Let \( m, \ n \geq 1 \) and \( P(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \ Q(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n \) be two polynomials in one variable. Let \( \alpha_1, \ldots, \alpha_m \) be roots of \( P(x) \), and \( \beta_1, \ldots, \beta_n \) roots of \( Q(x) \) so that \( P(x) = a_0 (x - \alpha_1) \cdots (x - \alpha_m), \ Q(x) = b_0 (x - \beta_1) \cdots (x - \beta_n) \). This immediately implies

\[
a_0^n \prod_{i} Q(\alpha_i) = a_0^m b_0^n \prod_{i,j} (\alpha_i - \beta_j) = (-1)^{mn} b_0^n \prod_{j} P(\beta_j). \tag{2}
\]

The resultant \( \text{Res}(P, Q) \) by definition is equal to each of three expressions in (2). It follows that \( \text{Res}(Q, P) = (-1)^{mn} \text{Res}(Q, P) \).

It is well known that \( \text{Res}(P, Q) \) is a polynomial in \( a_0, \ldots, a_m \), \( b_0, \ldots, b_n \) with integral coefficients. This follows at once from the "fundamental theorem on symmetric polynomials" saying that any symmetric polynomial \( f(\alpha_1, \ldots, \alpha_m) \) is a polynomial in elementary symmetric polynomials \( e_k(\alpha_1, \ldots, \alpha_m) \), and the Vietae theorem which claims that \( e_k(\alpha_1, \ldots, \alpha_m) = (-1)^k a_k/k_0 \).

Our goal is to investigate the decomposition (1) of \( \text{Res}(P, Q) \) into the sum of monomials. Denote by \( S = S_{m,n} \) the set \( \{(p, q) \in \mathbb{Z}^{m+n+2} : c_{pq} \neq 0\} \) and let \( N = N_{m,n} \subset \mathbb{R}^{m+n+2} \) be the convex hull of \( S \). It follows at once from (2) that the resultant satisfies three quasihomogeneity conditions:

**Proposition 1.** For any \((p, q) \in S_{m,n}\) we have

\[
\sum_k p_k = n, \quad \sum_i q_i = m, \quad \sum_{0 \leq k \leq m} (m-k) p_k + \sum_{0 \leq i \leq n} (n-l) q_i = mn. \tag{3}
\]

It follows that \( \dim N \leq (m+n+2) - 3 = m+n-1 \). Denote by \( \text{Ver} N \) the set of vertices of \( N \). Let \( S'_n = \{(p_0, \ldots, p_m) \in \mathbb{Z}^{m+1} : p_i \geq 0, \sum p_i = n\} \). By (3), \( S = S_{m,n} \subset S'_n \times S'_n \).

**Theorem 2.** For any \( p \in S'_n \) there is exactly one \( q \in S'_n \) such that \((p, q) \in \text{Ver} N\), viz. \( q_j = \{i : 0 \leq i < m, \ p_0 + \cdots + p_i = j\} \). Similarly, for any \( q \in S'_n \) there is exactly one \( p \in S'_n \) such that \((p, q) \in \text{Ver} N\), viz. \( p_i = \{j : 0 \leq j < n, \ q_0 + \cdots + q_i = j\} \). In other words, each of the projections \( \text{Ver} N \to S'_n \) and \( \text{Ver} N \to S'_n \) is a bijection.

**Theorem 3.** Let \((p, q)\) and \((p', q')\) be two different vertices of \( N \). Then they are connected by an edge if and only if there are two indices \( 0 \leq i < i' \leq m \) such that \( p_k = p'_k \) for \( k \neq i, i' \) and \( p_k = p'_k = 0 \) for \( i < k < i' \).

**Example.** For \( m = n = 2 \) the convex polytope \( N \) is shown in Fig. 1.
THEOREM 4. $N_{m,n}$ is the set of points $(p, q) \in \mathbb{R}^{m+n+2}$ satisfying linear equations (3) and linear constraints

$$p_i \geq 0 \quad (0 \leq i \leq m),$$

$$q_j \geq 0 \quad (0 \leq j \leq n),$$

$$\sum_{0 \leq k \leq i} (i-k) p_k + \sum_{0 \leq l \leq j} (j-l) q_l \geq ij \quad (0 \leq i \leq m, 0 \leq j \leq n).$$

To describe the face lattice of $N_{m,n}$ we need some terminology and notation. For any two integers $k \leq l$ let $[k, l] = \{ r \in \mathbb{Z} : k \leq r \leq l \}$. Let $i \leq i', j \leq j'$. A subset $L \subset \mathbb{Z}^2$ is called a lattice path from $(i, j)$ to $(i', j')$ if $L = \{ (i_k, j_k) : k = 1, \ldots, l \}$, where $(i_1, j_1) = (i, j)$, $(i_l, j_l) = (i', j')$ and for any $k = 1, \ldots, l-1$ a point $(i_{k+1}, j_{k+1})$ is either $(i_k+1, j_k)$ or $(i_k, j_{k+1})$. We write $L = L[(i, j) \to (i', j')]$, $pr_1 L = [i, i']$, and $pr_2 L = [j, j']$.

Let $L$ be a lattice path, $i \in pr_1 L, j \in pr_2 L$. We define

$$p_i(L) = \left| \{ j : (i, j) \in L \} \right| - 1, \quad q_j(L) = \left| \{ i : (i, j) \in L \} \right| - 1 \quad (7)$$

(see Fig. 2).
For any \((i, j) \in [0, m] \times [0, n]\) let \(h_{ij}\) be the affine-linear function on \(\mathbb{R}^{m+n+2}\) defined by

\[
 h_{ij} = \sum_{0 \leq k \leq i} (i-k) p_k + \sum_{0 \leq l \leq j} (j-l) q_l - ij.
\] (8)

Let \(I \subset [0, m], J \subset [0, n], C \subset [0, m] \times [0, n]\). Denote by \(F(I, J, C)\) the set of points \((p, q) \in \mathbb{R}^{m+n+2}\) satisfying (3) to (6) and the equalities \(p_i = 0\) for \(i \in I\), \(q_j = 0\) for \(j \in J\), and \(h_{ij} = 0\) for \((i, j) \in C\).

We say that a triple \((I, J, C)\) is saturated if it satisfies the following conditions:

- \(C\) is a disjoint union of lattice paths \(L_k[(i_k, j_k) \to (i'_k, j'_k)], \)
- \(k = 1, \ldots, r\), where \((i_1, j_1) = (0, 0)\), \((i'_r, j'_r) = (m, n)\), and \(i'_k < i_{k+1}\), \(j'_k < j_{k+1}\) for \(k = 1, \ldots, r-1\). (9)

Let \(I(C) = \{i_1, i_2, i_3, \ldots, i_r\}\). Then \(p_i \in \text{pr}_1 L_k\) belongs to \(I\) if and only if \(i \notin I(C)\) and \(p_i(L_k) = 0\). Similarly, let \(J(C) = \{j_1, j_2, j_3, \ldots, j_r\}\); then \(q_j \in \text{pr}_2 L_k\) belongs to \(J\) if and only if \(j \notin J(C)\) and \(q_j(L_k) = 0\). (10)

The lattice paths \(L_1, \ldots, L_r\) will be called the components of \(C\), and we write \(r(C) = r\), the number of components.

**Theorem 5.** For any saturated triple \((I, J, C)\) the set \(F(I, J, C)\) is a (non-empty) face of \(N_{m,n}\). Conversely, for any face \(F\) of \(N_{m,n}\) the triple \((I = \{i \in [0, m] : p_i |_{F} = 0\}, J = \{j \in [0, n] : q_j |_{F} = 0\}, C = \{(i, j) \in [0, m] \times [0, n] : h_{ij} |_{F} = 0\})\) is saturated, and \(F = (I, J, C)\). Furthermore, for two saturated triples \((I, J, C)\) and \((I', J', C')\) we have \(F(I, J, C) \subset F(I', J', C')\) if and only if \(I \supseteq I', J \supseteq J', \text{ and } C \supseteq C'\).

**Theorem 6.** \(\dim N_{m,n} = n + m - 1\), and for any saturated triple \((I, J, C)\) we have

\[
\text{codim } F(I, J, C) = |I \cup \text{pr}_1 C| + |J \cup \text{pr}_2 C| - r(C) - 2.
\] (11)

**Theorem 7.** Let \(m, n \geq 2\). Then \(N_{m,n}\) has exactly \((mn + 3)\) different facets (i.e., faces of codimension 1), viz., its intersections with the hyperplanes \(p_i = 0\) \((i \in [0, m])\), \(q_j = 0\) \((j \in [0, n])\), \(h_{ij} = 0\) \(((i, j) \in [1, m-1] \times [1, n-1])\).

Theorems 2 to 7 will be proved in the next two sections.
2. Vertices of $N_{m,n}$: Proof of Theorem 2

We fix natural numbers $m$ and $n$ and keep all the notation of Section 1.

Let $\mathcal{L}$ be the set of all lattice paths from $(0,0)$ to $(m,n)$. Define the mapping $L \mapsto (p(L), q(L))$ from $\mathcal{L}$ to $S^n_m \times S^n_m$ as

$$p(L) = (p_0(L), ... , p_m(L)), \quad q(L) = (q_0(L), ... , q_n(L))$$

(see (7)); in other words, $p_0(L), p_1(L), ...$ are the lengths of successive vertical sections of $L$ scanning from left to right while $q_0(L), q_1(L), ...$ are the lengths of successive horizontal sections of $L$ scanning from bottom to top (see Fig. 2).

Clearly, the mapping $L \mapsto p(L)$ is a bijection between $\mathcal{L}$ and $S^n_m$ while the mapping $L \mapsto q(L)$ is a bijection between $\mathcal{L}$ and $S^n_m$. Therefore, Theorem 2 is equivalent to the following statement:

'\textbf{Theorem 2'.} The mapping $L \mapsto (p(L), q(L))$ is a bijection between $\mathcal{L}$ and $\text{Ver } N$.

We shall deduce Theorem 2' from the general result on $A$-discriminants [9, Theorem 1]. For convenience we reproduce here some of the general definitions and results on $A$-discriminants.

Let $A$ be a finite subset of $\mathbb{Z}^k$. Each point $\omega = (\omega_1, ..., \omega_k) \in A$ determines the Laurent monomial $x^\omega = x_1^{\omega_1} \cdots x_k^{\omega_k}$ in variables $x_1, ..., x_k$. We identify the vector space $C^A = \{(v_\omega), \omega \in A\}$ with the space of Laurent polynomials $f(x) = \sum_{\omega \in A} v_\omega x^\omega$. We say that $f \in C^A$ is degenerate if there is $x(0) \in (\mathbb{C}^*)^k$ such that $f(x(0)) = \partial f/\partial x_1(x(0)) = \cdots = \partial f/\partial x_k(x(0)) = 0$. Let $D \subseteq C^A$ be the Zariski closure of the set of all degenerate $f$. If codim $D = 1$ then by the $A$-discriminant we mean a polynomial $A_A(f) = A_A((v_\omega))$ in the variables $v_\omega$ which is irreducible over $\mathbb{Z}$ and vanishes on $D$ (it is determined up to sign). If codim $D > 1$ we put $A_A(f) = 1$.

The resultant $\text{Res}(P, Q)$ is a special case of an $A$-discriminant corresponding to $A = \{(i, 0), (j, 1): i \in [0, m], j \in [0, n]\}$. Indeed in this case $f(x_1, x_2) \in C^A$ has the form $f(x_1, x_2) = P(x_1) + x_2 Q(x_1)$, and definitions readily imply $A_A(f) = \text{Res}(P, Q)$.

Let $N_A \subseteq \mathbb{R}^4$ be the Newton polytope of $A$, i.e., the convex hull of the set $S_A = \{(p, (p, \omega)) \in \mathbb{Z}^+ :$ the monomial $v^p = \prod_\omega v_\omega^{\omega_\omega}$ occurs in $A_A\}$. Let $Q$ be the convex hull of $A$ (in our case $Q$ is the trapezoid shown in Fig. 3).
Suppose that the toric variety corresponding to $A$ is smooth (it is so in the case of the resultant). Then Theorem 1 from [9] claims that the vertices of $N_A$ are in a natural bijective correspondence with the so-called $D$-equivalence classes of regular triangulations of $Q$ with vertices on $A$. We shall not repeat here the general definitions but just show what they mean when $Q$ is our trapezoid.

A triangulation of $Q$ is simply a decomposition of $Q$ into the union of disjoint triangles with the vertices on $A$ (it is easy to see that any such triangulation is regular in the sense of [4, 5, 8]). The $D$-equivalence relation on the set of triangulations is generated (in our special case) by the following relation: $T \sim T'$ if $T'$ is obtained from $T$ by a subdivision of a triangle into the union of two smaller triangles. We say that a triangulation $T$ is $D$-basic if the union of any two of its triangles is not a triangle. Evidently $D$-basic triangulations form the set of representatives of $D$-equivalence classes. Denote the set of all $D$-basic triangulations of the trapezoid $Q$ by $\mathcal{F}$.

For any $T \in \mathcal{F}$ define the vectors $p(T) \in \mathbb{Z}^{n+1}$, $q(T) \in \mathbb{Z}^{n+1}$ as follows. Let $p_i(T) = |b - c|$ if there is a triangle $\sigma \in T$ with vertices $(i, 0)$, $(b, 1)$, and $(c, 1)$, otherwise $p_i(T) = 0$. Similarly, let $q_i(T) = |b - c|$ if there is a triangle $\sigma \in T$ with vertices $(j, 1)$, $(b, 0)$, and $(c, 0)$, otherwise $q_i(T) = 0$.

Theorem 1 from [9] implies at once

**Theorem 2'.** The mapping $T \mapsto (p(T), q(T))$ is a bijection between $\mathcal{F}$ and $\text{Ver } N$.

To deduce Theorem 2' from Theorem 2" it suffices to construct a bijection $\varphi: \mathcal{F} \to \mathcal{L}$ such that $(p(\varphi(T)), q(\varphi(T))) = (p(T), q(T))$ for any $T \in \mathcal{F}$.

Let $T$ be a $D$-basic triangulation of the trapezoid $Q$. Let $sk_1(T)$ be the union of all edges of all triangles $\sigma \in T$, and $sk^0_1(T) = sk_1(T) - (\text{segments } [(0, 0), (m, 0)] \text{ and } [(0, 1), (n, 1)])$ be the polygonal line formed by the edges of our triangles which connect two horizontal edges of $Q$. Let $w(T)$ be the sequence $(w_1, w_2, \ldots, w_r)$ of vertices of $sk^0_1(T)$ scanning successively from the left end. So $w_i$ is one of the vertices $(0, 0)$ or $(0, 1)$, $w$, is one of the vertices $(m, 0)$ or $(n, 1)$, and for any $k$ the points $w_k$ and $w_{k+1}$ lie on the opposite edges of $Q$. Evidently, $T$ is uniquely determined by $w(T)$.

We assign to $T$ the sequence of points $u_k = (i_k, j_k) \in \mathbb{Z}^2$, $k = 0, 1, \ldots, r$ as follows. Let $u_0 = (0, 0)$, and for $k > 0$ define $u_k = (i, j_{k-1})$ if $w_k = (i, 0)$, and $u_k = (i_{k-1}, j)$ if $w_k = (j, 1)$. It is easy to see that the polygonal line $L = \bigcup_{k \geq 0} [u_{k-1}, u_k]$ is a lattice path from $(0, 0)$ to $(m, n)$, and we define $\varphi(T) = L$. An example is given in Fig. 4.

The proof that $\varphi$ is a bijection between $\mathcal{F}$ and $\mathcal{L}$ such that $(p(\varphi(T)), q(\varphi(T))) = (p(T), q(T))$ for any $T \in \mathcal{F}$ is straightforward. This completes the proof of Theorem 2.
3. Face Lattice of $N_{m,n}$: Proof of Theorems 3 to 7

Theorems 3 to 7 can also be deduced from the corresponding general results on $A$-discriminants. But it is more in spirit of the present paper to deduce them directly from Theorem 2. We shall use it in the equivalent form of Theorem 2', i.e., parametrize the vertices of $N_{m,n}$ by the lattice paths $L \in \mathcal{L}$. We shall denote by $v_L$ the vertex $(p(L), q(L))$ defined by (12).

First we prove that each point $(p, q) \in N_{m,n}$ satisfies the linear constraints from Theorem 4. Inequalities (4) and (5) are obvious.

**Proposition 8.** For any $L \in \mathcal{L}$ the vertex $v_L$ satisfies (6), i.e., $h_{ij}(v_L) \geq 0$ for any $(i, j) \in [0, m] \times [0, n]$. Moreover, $h_{ij}(v_L) = 0$ if and only if $(i, j) \in L$.

**Proof.** Let us treat $L$ as a polygonal line on the plane $\mathbb{R}^2$ connecting the points $(0, 0)$ and $(m, n)$ (see Fig. 2). It immediately follows from definitions that $\sum_{0 \leq k \leq i} (i-k) p_k(L)$ is equal to the area between the path $L$, the $x$-axis, and the line $x = i$; similarly, $\sum_{0 \leq j \leq r} (j-r) q_j(L)$ is equal to the area between the path $L$, the $y$-axis, and the line $y = j$. But it is evident that the sum of these two areas is not less than the area of the rectangle $\{0 \leq x \leq i, 0 \leq y \leq j\}$, i.e., not less than $ij$; moreover, the equality holds if and only if $(i, j) \in L$. This proves our statement.

**Remark.** The above proof shows that the inequality $h_{ij}(v_L) \geq 0$ is a discrete analog of the well-known Young inequality (see [10]).

Denote temporarily by $N'_{m,n}$ the convex polyhedron in $\mathbb{R}^{m+n+2}$ defined by the linear equalities (3) and linear inequalities (4) to (6) (Theorem 4 claims that $N'_{m,n} = N_{m,n}$). Recall the notion of a saturated triple $(I, J, C)$, where $I \subseteq [0, m]$, $J \subseteq [0, n]$, $C \subseteq [0, m] \times [0, n]$, and the notation $F(I, J, C) = \{(p, q) \in N'_{m,n}: p_i = 0$ for $i \in I$; $q_j = 0$ for $j \in J$; $h_{ij} = 0$ for $(i, j) \in C\}$ (see Section 1). Clearly, the faces of $N'_{m,n}$ are exactly the non-empty sets $F(I, J, C)$. The next result follows at once from definitions and Proposition 8.
**Proposition 9.** Let \((I, J, C)\) be a saturated triple, and \(L \in \mathcal{L}\). The vertex \(v_L\) belongs to \(F(I, J, C)\) if and only if \(L \supset C\), \(p_i(L) = 0\) for \(i \in 1 - pr_1 C\), and \(q_j(L) = 0\) for \(j \in J - pr_2 C\).

**Proposition 10.** For any saturated triple \((I, J, C)\) there is a lattice path \(L \in \mathcal{L}\) such that \(v_L \in F(I, J, C)\).

Proof. Let \(L_k[(i_k, j_k) \rightarrow (i_k', j_k')]\) be the components of \(C\) scanning from left to right (see Section 1). Let \(L_k\) be the lattice path from \((i_k, j_k)\) to \((i_{k+1}, j_{k+1})\) which goes horizontally from \((i_k, j_k)\) to \((i_{k+1}, j_k)\) and then vertically to \((i_{k+1}, j_{k+1})\). Let \(L\) be the union \(L_1 \cup \cdots \cup L_r \cup L_1' \cup \cdots \cup L_{r-1}'\). Then \(L \supset C\) and \(p_i \neq 0\) for \(i \notin pr_1 C\), \(q_j \neq 0\) for \(j \notin pr_2 C\), so \(v_L \in F(I, J, C)\) by Proposition 9.

**Proposition 11.** For any face \(F\) of \(N_{m,n}'\) the triple \((I = \{i \in [0, m] : p_i | F = 0\}, J = \{j \in [0, n] : q_j | F = 0\}, C = \{(i, j) \in [0, m] \times [0, n] : h_{ij} | F = 0\})\) is saturated, and \(F = F(I, J, C)\).

Proof. The equality \(F = F(I, J, C)\) is evident, so it remains to prove that \((I, J, C)\) satisfies (9) and (10).

The fact that \(C\) contains \((0, 0)\) and \((m, n)\) is evident from (3). Now we prove that \(C\) cannot contain two points \((i, j)\), \((i', j')\) such that \(i < i'\) and \(j < j'\). Consider the following identity which is an easy consequence of (8):

\[h_{ij} + h_{i'j'} - h_{ij} = (i' - i)(j' - j').\] (13)

Evaluating (13) at \(v \in F\) we see that if \((i, j)\), \((i', j') \in C\) then \(h_{ij}(v) = h_{i'j'}(v) = 0\), and so \(h_{ij}(v) + h_{i'j'}(v) < 0\). But this contradicts (6).

Clearly, the property of \(C\) just proved implies that \(C\) is a disjoint union of lattice paths \(L_k[(i_k, j_k) \rightarrow (i_k', j_k')]\), \(k = 1, \ldots, r\), where \((i_0, j_0) = (0, 0)\), \((i_0', j_0') = (m, n)\), and \(i_k \leq i_{k+1}, j_k \leq j_{k+1}\) for \(k = 1, \ldots, r - 1\). To prove (9) it remains to show that the \(L_k\)'s can be chosen so that the inequalities \(i_k' \leq i_{k+1} - 1, j_k' \leq j_{k+1}\) become strict. Clearly, this follows from the next two statements: (a) if \((i, j)\), \((i', j) \in C\), and \(i < k < i'\) then \((k, j) \in C\); (a') if \((i, j)\), \((i, j') \in C\) and \(j < l < j'\) then \((i, l) \in C\).

To prove (a) we shall use the identity which is again an easy consequence of (8):

\[h_{ij} + h_{i'j} = h_{i+1,j} + h_{i,j'} + \sum_{i < k < i'} p_k.\] (14)

Evaluating (14) at \(v \in F\) we see that if \((i, j)\), \((i', j) \in C\) then \(h_{ij}(v) = h_{i'j}(v) = 0\), and so each summand in the right-hand side of (14) vanishes on \(F\). Therefore, \((i + 1, j)\) and \((i' - 1, j)\) belong to \(C\). Repeating the argument if necessary we see that \((k, j) \in C\) for \(i < k < i'\) which proves (a). The proof of (a') is completely the same. This proves that \(C\) satisfies (9).
As for (10), we shall prove only the statements related to I (the proof of the statements about \( J \) is completely analogous). First we prove the following statement: (b) if \((i, j) \in C, i \in I, \) and \(|i' - i| = 1\) then \((i', j) \in C.\) If \(1 \leq i \leq m - 1\) then (b) follows just as above from the identity

\[
2h_{ij} + p_i = h_{i-1,j} + h_{i+1,j}
\]

which is a special case of (14). For \((i, j) = (0, 0), i' = 1\) the statement (b) is evident since \(h_{10} = p_0,\) and for \((i, j) = (m, n), i' = m - 1\) it follows from the identity

\[
h_{m-1,n} = h_{m,n} + \left( n - \sum_{0 \leq k \leq m} p_k \right) + p_m.
\]

Finally, the identity \(h_{ij} + j = h_{0,j} + p_0\) shows that for \(j \geq 1\) the conditions \((0, j) \in C\) and \(0 \in I\) cannot hold simultaneously; similarly, if \(j \leq n - 1\) then the conditions \((m, j) \in C\) and \(m \in I\) cannot hold simultaneously.

The property (b) just established and the definitions (see (10)) readily imply that if either \(i \in \bar{I}(C),\) or \(i \in \text{pr}_1L_k\) and \(p_i(L_k) \neq 0,\) then \(i \notin I.\) The converse part of (10) follows at once from the next statement: if \((i, j), (i', j) \in C\) and \(i < k < i'\) then \(k \in I.\) But this is an immediate consequence of (14).

Now we are able to prove Theorems 4 and 5. By Theorem 2 and Proposition 8, \( N_{m,n} \subset N'_{m,n}.\) Clearly, \( N'_{m,n} \) is bounded; i.e., it is the convex hull of a finite set. To prove that \( N'_{m,n} = N_{m,n} \) it suffices to show that each vertex of \( N'_{m,n} \) is of the form \( v_L.\) But this follows at once from Propositions 10 and 11. This proves Theorem 4. Moreover, replacing \( N'_{m,n} \) by \( N_{m,n} \) in Proposition 11 we get all statements of Theorem 5 except the last one. The remaining part of Theorem 5 follows at once from the next lemma.

**Lemma 12.** Let \((I, J, C)\) be a saturated triple. Then

(a) The intersection of all lattice paths \( L \in \mathcal{L} \) such that \( v_L \in F(I, J, C) \) is equal to \( C.\)

(b) For any \( i \in [0, m] - (I \cup \text{pr}_1C) \) there is \( L \in \mathcal{L} \) such that \( v_L \in F(I, J, C) \) and \( p_i \neq 0.\)

(b') For any \( j \in [0, n] - (J \cup \text{pr}_2C) \) there is \( L \in \mathcal{L} \) such that \( v_L \in F(I, J, C) \) and \( q_j \neq 0.\)

**Proof.** (a) Let \( L \in \mathcal{L} \) be the lattice path constructed in the proof of Proposition 10. Let \( L' \in \mathcal{L} \) be another lattice path containing \( C \) and such that for any \( k = 1, \ldots, r - 1 \) the part of \( L' \) connecting \((i_k, j_k) \) and \((i_{k+1}, j_{k+1})\) first goes vertically to \((i_k', j_{k+1})\) and then horizontally to \((i_{k+1}', j_{k+1})\). Evidently, \( v_L, \) as well as \( v_{L'} \) belongs to \( F(I, J, C), \) and \( L \cap L' = C,\) which proves our statement.
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(247)

(b) Let $i \in [i_k + 1, i_{k+1} - 1]$. Then the desired $L$ can be constructed as follows: it differs from the one from Proposition 10 only in the part connecting $(i_k', j_k')$ and $(i_{k+1}, j_{k+1})$, and this part goes first horizontally to $(i, j_k')$, then vertically to $(i, j_{k+1})$ and finally horizontally to $(i_{k+1}, j_{k+1})$. The fact that $v_L \in F(I, J, C)$ follows at once from Proposition 9, and we have $p_i(L) = j_{k+1} - j_k' > 0$. The proof of (b') is completely analogous. 

Now we begin to prove Theorem 6. The fact that $\dim N_{m,n} = m + n - 1$ follows by induction on $\min(m, n)$ from the next proposition.

**Proposition 13.** (a) Each of the polytopes $N_{1,n}$ and $N_{n,1}$ is linearly isomorphic to the simplex $\Delta^n$.

(b) If $m, n \geq 2$ and $1 \leq i \leq m - 1, 1 \leq j \leq n - 1$ then the intersection of $N_{m,n}$ with the hyperplane $h_{ij} = 0$ is a facet of $N_{m,n}$ which is affinely isomorphic to $N_{i,j} \times N_{m-i,n-j}$.

**Proof.** (a) Clearly, the projection $(p, q) \to q$ is a linear isomorphism of $N_{1,n}$ onto the standard simplex $\Delta^n$. For $(p', q') \in \mathbb{R}^{i+1} \times \mathbb{R}^{j+1}$, $(p'', q'') \in \mathbb{R}^{m-i+1} \times \mathbb{R}^{n-j+1}$ define $(p, q) = f((p', q'), (p'', q''))$ by $p_k = p_k'$ for $0 \leq k \leq i - 1$, $p_j = p_j'$, $p_k = p_k''$, for $i + 1 \leq k \leq m$; $q_l = q_l'$ for $0 \leq l \leq j - 1$, $q_j = q_j'$, $q_l = q_l''$ for $j + 1 \leq l \leq n$. Conversely, define $((p', q'), (p'', q'')) = g(p, q)$ by the formulas:

- $p_k' = p_k$ for $0 \leq k \leq i - 1$, $p_j' = j - \sum_{0 \leq k \leq i - 1} p_k$, $p_0'' = n - j - \sum_{1 \leq k \leq m} p_k$, $p_k'' = p_{k+1}$ for $1 \leq k \leq m - i$; $q_l' = q_l$ for $0 \leq l \leq j - 1$, $q_j' = i - \sum_{0 \leq i \leq j - 1} q_i$, $q_l'' = m - i - \sum_{1 \leq i \leq n} q_i$, $q_i'' = q_i + j$ for $1 \leq l \leq n - j$. Using the description of $N_{m,n}$ given by Theorem 4 it is straightforward to verify that $f$ and $g$ define mutually inverse isomorphisms between $N_{m,n} \cap \{h_{ij} = 0\}$ and $N_{i,j} \times N_{m-i,n-j}$.

Now we prove the formula (11) from Theorem 6 for codim $F(I, J, C)$. Let $C$ be defined by (9). Using repeatedly Proposition 13(b) we see that $N_{m,n} \cap \{h_{ij} = 0\}$ is naturally isomorphic to the product $\Pi_{1 \leq k \leq r-1} N_{k+1-k-1}$. This allows one to reduce the proof of (11) to the crucial special case when $C = \{(0, 0), (m, n)\}$, $I \subseteq [1, m - 1]$, $J \subseteq [1, n - 1]$. We have to prove that in this case codim $F(I, J, C) = |I| + |J|$. Let $H_{IJ}$ denote the affine subspace of $\mathbb{R}^{m+n+2}$ defined by (3) and the conditions $p_i = 0$ for $i \in I$, $q_j = 0$ for $j \in J$. By definition, $F(I, J, C) = H_{IJ} \cap N_{m,n}$. It is easy to see that $H_{IJ}$ is of codimension $|I| + |J| + 3$ in $\mathbb{R}^{m+n+2}$. Therefore, the codimension of $F(I, J, C)$ in $N_{m,n}$ is not less than $|I| + |J|$. On the other hand, consider all lattice paths $L \in \mathcal{L}$ which contain either a vertical segment $[(i, 0), (i, n)]$ for some $i \notin I$, or a horizontal segment $[(0, j), (m, j)]$ for some $j \notin J$. There are exactly $(m + n - |I| - |J|)$
such lattice paths, and it is easy to see that the corresponding vertices $v_L$ are affinely independent; i.e., their convex hull is a simplex of dimension $(m + n - 1 - |I| - |J|)$. But all these vertices lie in $F(I, J, C)$ so codim $F(I, J, C) = |I| + |J|$. This completes the proof of Theorem 6.

Theorem 3 describing edges of $N_{m,n}$ follows easily from Theorem 6. Indeed, (11) implies at once the following characterization of edges of $N_{m,n}$.

**Proposition 14.** A face $F(I, J, C)$ is an edge of $N_{m,n}$ if and only if $r(C) = 2$, $I \cup pr_1 C = [0, m]$, and $J \cup pr_2 C = [0, n]$.

In other words, Proposition 14 means that two vertices $v_L$ and $v_{L'}$ lie on an edge if and only if there are two points $(i, j), (i', j') \in L \cap L'$ such that $i < i', j < j'$ and $L$ and $L'$ differ only by the part connecting $(i, j)$ and $(i', j')$ and $L \supset [(i, j), (i', j)] \cup [(i', j), (i', j')]$, $L' \supset [(i, j), (i', j')] \cup [(i, j'), (i', j')]$. Theorem 3 is simply a reformulation of this statement.

The remaining Theorem 7 also follows at once from (11).

### 4. Coefficients of the Resultant

In this section we discuss the problem of computation of the coefficients $c_{pq}$ in the decomposition (1) of the resultant $\text{Res}(P, Q)$. There is a general formula for the coefficients of "extreme" monomials in the A-discriminant, i.e., those corresponding to the vertices of the Newton polytope [9, Theorem 1]. For the resultant it is especially simple.

**Proposition 15.** If $(p, q)$ is a vertex of the Newton polytope $N_{m,n}$ then the coefficient $c_{pq}$ is equal to $(-1)^{p_1 + 2p_2 + \cdots + mp_m}$.

Following the general strategy of this paper we shall give an elementary proof of this result. More generally, we shall give a formula for all coefficients in $\text{Res}(P, Q)$. For this we associate to vectors $p$ and $q$ the partitions $\mu = (1^{p_1}, 2^{p_2}, \ldots, m^{p_m})$ and $\nu = (1^{q_1}, 2^{q_2}, \ldots, n^{q_n})$. We shall use the notation and terminology from [11]; in particular the notation above means that $p_i$ parts equal to $i$ for each $i = 1, \ldots, m$.

For any two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ denote by $d^\lambda_\mu$ the number of $(0, 1)$-matrices with row sums $\lambda_1, \lambda_2, \ldots$ and column sums $\mu_1, \mu_2, \ldots$. This is one of the fundamental combinatorial functions (see, e.g., [11]). Evidently, $d^\lambda_\mu = 0$ unless $\lambda$ and $\mu$ are of the same weight (the weight of $\lambda$ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots$). The necessary and sufficient conditions for $d^\lambda_\mu$ to be non-zero are given by the Gale–Ryser theorem (see, e.g., [10]). Recall that the dominance partial order on the set of partitions is defined as $\lambda \geq \mu$ if and only if $|\lambda| = |\mu|$ and $\lambda_i + \cdots + \lambda_i \geq \mu_i + \cdots + \mu_i$ for each $i \geq 1$. The Gale–Ryser theorem claims that $d^\lambda_\mu \neq 0$ if and only if $\lambda \leq \mu'$.
where $\mu'$ is the partition conjugate to $\mu$ (if $\mu = (1^{p_1}, 2^{p_2}, \ldots, m^{p_m})$ then $\mu' = (p_1 + p_2 + \cdots + p_{m'}, \ldots, p_{m-1} + p_m, p_m)$); moreover, $d_{\lambda}^2 = 1$. It follows that the matrix $(d_{\mu}^\lambda)$ with rows and columns indexed by partitions is invertible, and the inverse matrix also has integral entries. Denote the inverse matrix by $(c_{\lambda}^\mu)$.

**Proposition 16.** The coefficient $c_{\mu \nu}$ in (1) is equal to $(-1)^{|\mu|} c_{(n^m)-\nu}^\mu$ where $\mu$ and $\nu$ are the partitions corresponding to $\mu$ and $\nu$, and $(n^m)-\nu$ is the partition $(n^{\ell_0}, (n-1)^{\ell_1}, \ldots, 1^{\ell_{n-1}})$.

**Proof.** For each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ denote by $m_\lambda = m_\lambda(\alpha_1, \ldots, \alpha_m)$ the symmetrized monomial, i.e., the sum of all distinct monomials of the form $\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \cdots$. Denote by $e_\lambda = (e_\lambda(\alpha_1, \ldots, \alpha_m))$ the product of elementary symmetric polynomials $e_{\lambda_1}(\alpha_1, \ldots, \alpha_m) e_{\lambda_2}(\alpha_1, \ldots, \alpha_m) \cdots$. Expanding this product we obtain the well-known formula

$$e_\mu = \sum_\lambda d_{\mu}^\lambda m_\lambda.$$  

(16)

Therefore, we have

$$m_\lambda = \sum_\mu c_{\mu}^\lambda e_\mu.$$  

(17)

Now let $\alpha_1, \ldots, \alpha_m$ be the roots of $P$ so by definition (see (2)) we have $\operatorname{Res}(P, Q) = a_0^n \prod_1 Q(\alpha_i)$. Expanding this product we see that

$$\operatorname{Res}(P, Q) = a_0^n \sum_q b^q m_{(n^m)-\nu}(\alpha_1, \ldots, \alpha_m)$$

$$= \sum_{\nu, \mu} a_0^n b^q c_{(n^m)-\nu}^\mu(e_\mu(\alpha_1, \ldots, \alpha_m)).$$

It remains to use the equality $e_k(\alpha_1, \ldots, \alpha_m) = (-1)^k a_k / a_0$.

This proposition and the Gale–Ryzer Theorem imply

**Proposition 17.** For any $(p, q) \in S_{m,n}$ the partitions $\mu = (1^{p_1}, 2^{p_2}, \ldots, m^{p_m})$ and $\nu = (1^{q_1}, 2^{q_2}, \ldots, n^{q_n})$ satisfy the relation $\mu \geq [(n^m)-\nu]'$. If $\mu = [(n^m)-\nu]'$ then $c_{pq} = (-1)^{|\mu|}$.

Our next result shows that the conditions on $(p, q) \in S_{m,n}$ given by Proposition 17 are equivalent to the conditions (4) to (6) from Theorem 4. More precisely, we have

**Proposition 18.** Let $(p, q) \in S'_{m,n}$, and $\mu, \nu$ be the corresponding partitions. Then $(p, q) \in N_{m,n}$ if and only if $\mu \geq [(n^m)-\nu]'$. Moreover, $(p, q)$ is a vertex of $N_{m,n}$ if and only if $\mu = [(n^m)-\nu]'$. 

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Proof. It is the well-known property of the dominance order that 
\( \mu \succeq [(n^m) - \nu]' \) if and only if \( \mu' \preceq [(n^m) - \nu] \). The definitions readily imply 
that the latter relation is equivalent to

\[
\sum_{0 \leq k \leq i} (i-k) p_k + \sum_{0 \leq l \leq j} (j-l) q_l \geq ij
\]

for all \( i, j \) such that \( 0 \leq j \leq n \) and

\[
qu_0 + \cdots + q_{j-1} \leq i \leq q_0 + \cdots + q_j.
\]  

(6')

By Theorem 4, to prove the first statement of the proposition it remains 
to show that for \( (p, q) \in S^m_n \times S^n_m \), the condition (6') implies (6). So suppose 
that \( (p, q) \in S^m_n \times S^n_m \) satisfies (6') and let us verify (6), i.e., that \( h_{ij} \geq 0 \) for 
all \( 0 \leq i \leq m, 0 \leq j \leq n \) (see (8)). Clearly, for any fixed \( i \) it suffices to con-
sider only those \( j \) such that \( h_{ij} \) is minimal possible. This means in particular 
that \( h_{ij} \leq h_{i,j-1} \) and \( h_{ij} \leq h_{i,j+1} \). But then we have \( q_0 + \cdots + q_{j-1} \leq i \leq q_0 + \cdots + q_j \) so \( h_{ij} \geq 0 \) by (6'), as required. The remaining statement about 
the vertices of \( N_{mn} \) follows directly from definitions and Theorem 2 (or 
Theorem 2').

Proposition 15 follows at once from Propositions 17 and 18.

Remark. It is an intriguing question whether it is possible to simplify 
the expression for the coefficients \( c_{pq} \) given by Proposition 16 and to extend 
it to more general \( A \)-discriminants. At present we do not see such an 
opportunity.

5. NEWTON POLYTOPE OF THE DISCRIMINANT

Let \( P(x) = a_0 x^r + a_1 x^{r-1} + \cdots + a_r \) be a polynomial in one variable. By 
definition the discriminant \( D(P) = D(a_0, a_1, ..., a_r) \) is a polynomial which is 
irreducible over \( \mathbb{Z} \) and vanishes whenever \( P \) has multiple roots. If \( \alpha_1, ..., \alpha_r \) 
are the roots of \( P(x) \) then \( D(P) \) is given by

\[
D(P) = a_0^{2r-2} \prod_{i < j} (\alpha_i - \alpha_j)^2
\]  

(18)

(see, e.g., [1]) so \( \deg D = 2r - 2 \).

Analogously to the resultant we decompose \( D(P) \) into the sum of monomials,

\[
D(P) = \sum_p c_p a^p,
\]  

(19)
where \( p = (p_0, p_1, \ldots, p_r) \in \mathbb{Z}^{r+1}_+ \) and \( a^p = \prod_i a_i^{p_i} \). Let \( S_r = \{ p \in \mathbb{Z}^{r+1}_+ : c_p \neq 0 \} \), and \( N_r \subset \mathbb{R}^{r+1} \) be the convex hull of \( S_r \), i.e., the Newton polytope of \( D(P) \). The following result is completely analogous to Proposition 1.

**Proposition 19.** For any \( p \in S_r \), we have
\[
\sum_k p_k = 2r - 2, \quad \sum_k (r-k)p_k = r(r-1).
\]

We shall show that vertices of \( N_r \) are naturally numerated by subsets \( \Omega \subset [1, r-1] \). To any \( \Omega = \{i_1, \ldots, i_s\} \subset [1, r-1] \) with \( i_1 < \cdots < i_s \) we associate a point \( p = p(\Omega) \in \mathbb{Z}^{r+1}_+ \) as follows: \( p_i = 0 \) for \( i \in [1, r-1] - \Omega \); \( p_{i_k} = i_{k+1} - i_k - 1 \) for \( k = 1, \ldots, s \) (with the convention \( i_0 = 0, \ i_{r+1} = r \)); \( p_0 = i_1 - 1, \ p_r = r - i_s - 1 \).

**Theorem 20.** (a) Each \( p(\Omega) \) is a vertex of \( N_r \), and the correspondence \( \Omega \to p(\Omega) \) is a bijection between the subsets of \([1, r-1]\) and the vertices of \( N_r \).

(b) If \( \Omega = \{i_1, \ldots, i_s\} \) then the corresponding coefficient \( c_{p(\Omega)} \) in (19) is equal to \( \prod_{0 \leq k \leq s} (-1)^{(i_{k+1} - i_k)(i_{k+1} - i_k - 1)/2} (i_{k+1} - i_k)^{i_{k+1} - i_k} \) (with the convention \( i_0 = 0, \ i_{r+1} = r \)).

This follows at once from the theory of \( A \)-discriminants (see [9]). In this case \( A = [0, r] \subset \mathbb{Z} \), and \( Q = \mathbb{R} \) is a segment. Therefore, the subsets of \([1, r-1]\) are in bijective correspondence with triangulations of \((Q, A)\): a subset \( \Omega = \{i_1, \ldots, i_s\} \) corresponds to the triangulation \([0, r] = \bigcup [i_k, i_{k+1}] \). Having this in mind we see that Theorem 20 is a special case of Theorem 1 from [9].

Now let us state the analogues of Theorems 4 to 7. For any \( i \in [0, r] \) let \( g_i \) be the affine-linear function on \( \mathbb{R}^{r+1} \) defined by
\[
g_i = \left( \sum_{0 \leq k \leq i} (i-k)p_k \right) - i(i-1).
\]

**Theorem 21.** \( N \) is the set of points \( p \in \mathbb{R}^{r+1} \) satisfying linear equations (20) and linear constraints
\[
p_i \geq 0, \quad g_i \geq 0 \quad (i \in [1, r-1]).
\]

Now for any \( \Xi, \Omega \subset [1, r-1] \) let \( F(\Xi, \Omega) = \{ p \in N_r : p_i = 0 \text{ for } i \in \Xi, \ g_i(p) = 0 \text{ for } i \in \Omega \} \). By Theorem 21 the faces of \( N_r \) are exactly the non-empty subsets \( F(\Xi, \Omega) \).
Theorem 22. (a) $F(\Xi, \Omega) \neq \emptyset$ if and only if $\Xi \cap \Omega = \emptyset$, and in this case we have $\dim F(\Xi, \Omega) = r - 1 - |\Xi| - |\Omega|$. In particular, $\dim N_r = r - 1$.

(b) A vertex $p(\Omega')$ belongs to a face $F(\Xi, \Omega)$ if and only if $\Omega \subset \Omega' \subset [1, r - 1] - \Xi$.

Corollary 23. $N_r$ is combinatorially equivalent to a $(r - 1)$-cube.

Proof of Theorems 21 and 22. Theorem 21 is an immediate consequence of the general results on $A$-discriminants. But we can deduce it from Theorem 20 essentially the same way as Theorem 4 was deduced from Theorem 2 but much easier.

Denote temporarily the polyhedron defined by (20) and (22) by $N'_r$. To prove that $N_r \subset N'_r$ it suffices to verify that each vertex $p(B)$ satisfies (22). This follows from the next lemma which is proved by direct calculation.

Lemma 24. If $\Omega = \{i_1, \ldots, i_s\}$ and $i \in [i_k, i_{k+1}]$ then $g_i(p(\Omega)) = (i_{k+1} - i)(i - i_k)$.

To prove the inverse inclusion we first observe that $p_0 = g_1$, and $p_r = g_{r-1}$ on $N'_r$. It follows that $N'_r$ is bounded, i.e., is a convex polytope. Now for any $\Xi, \Omega \subset [1, r - 1]$ let $F'(\Xi, \Omega) = \{p \in N'_r: p_i = 0 \text{ for } i \in \Xi, g_i(p) = 0 \text{ for } i \in \Omega\}$. Clearly, the faces of $N'_r$ are exactly the non-empty subsets $F'(\Xi, \Omega)$.

Lemma 25. If $\Xi \cap \Omega \neq \emptyset$ then $F'(\Xi, \Omega) = \emptyset$.

This follows at once from the obvious identity

$$2 - p_i = -g_{i-1} + 2g_i - g_{i+1} \quad (i \in [1, r - 1]). \quad (23)$$

Furthermore, Lemma 24 implies at once.

Lemma 26. For $i \in [1, r - 1]$ we have $g_i(p(\Omega)) = 0$ if and only if $i \in \Omega$.

By Lemmas 25 and 26 if $F'(\Xi, \Omega) \neq \emptyset$ then $F'(\Xi, \Omega) \supset F'([1, r - 1] - \Xi, \Omega) \supset p(\Omega)$. So each vertex of $N'_r$ is some $p(\Omega)$ hence $N'_r = N_r$. This proves Theorem 21. Moreover, all other statements in Theorem 22 except the formula for $\dim F(\Xi, \Omega)$ also follow from our lemmas.

Let $\Xi \cap \Omega = \emptyset$ so $\Omega \subset [1, r - 1] - \Xi$. By Theorem 22(b), $F(\Xi, \Omega)$ contains $p(\Omega)$ and all $p(\Omega \cup \{i\})$, $i \in ([1, r - 1] - \Xi) - \Omega$. But it is clear that these points are affinely independent hence $\dim F(\Xi, \Omega) \geq r - 1 - |\Xi| - |\Omega|$. The opposite inequality follows from the fact that linear parts of all affine-linear functions on $R^{r+1}$ defining $F(\Xi, \Omega)$ are linearly independent which is easy to prove. This completes the proof of Theorems 21 and 22. $\blacksquare$
Looking at (23) we recognize in its right-hand part of the Cartan matrix of the root system of type $A_{r-1}$. This suggests the following interpretation of $N_r$ in terms of a root system. Let $R$ be the root system of type $A_{r-1}$ in a real vector space $V$ with a standard choice of simple roots $\alpha_1, ..., \alpha_{r-1}$ and the corresponding fundamental weights $\omega_1, ..., \omega_{r-1}$ (see [12]). Let as usual $\rho = \sum \omega_i$ be the half-sum of all positive roots. Consider the convex polytope $P(2\rho) \subset V$ consisting of the points $\chi \in V$ such that all the coefficients $p_i, g_i$ in the decompositions $\chi = \sum p_i \omega_i$ and $(2\rho - \chi) = \sum g_i \alpha_i$ are nonnegative.

Now let $U \subset \mathbb{R}^{r+1}$ be the affine subspace of codimension 2 defined by (20). Define the affine mapping $\phi: U \to V$ by $\phi(p) = \sum_{1 \leq i \leq r-1} p_i \omega_i$.

**Proposition 27.**

(a) $\phi$ is an isomorphism of affine spaces.

(b) For any $p \in U$ we have $2\rho - \phi(p) = \sum_{1 \leq i \leq r-1} g_i(p) \alpha_i$.

(c) We have $\phi(N_r) = P(2\rho)$. Moreover, $\chi \in P(2\rho)$ lies in $\phi(N_r \cap \mathbb{Z}^{r+1})$ if and only if all the coefficients $p_i, g_i$ in the decompositions $\chi = \sum p_i \omega_i$ and $(2\rho - \chi) = \sum g_i \alpha_i$ are nonnegative integers.

**Proof.** (a) is obvious, (b) follows at once from (23), and (c) from Theorem 21. □

**Remarks.**

(a) It is well known that the set $\phi(N_r \cap \mathbb{Z}^{r+1})$ just described has the following representation-theoretic interpretation: it consists of the dominant integral weights occurring with non-zero multiplicity in an irreducible $\mathfrak{sl}_r$-module $L_{2\rho}$ with highest weight $2\rho$. It has also another interpretation: $\chi \in \phi(N_r \cap \mathbb{Z}^{r+1})$ if and only if $L_\chi$ is an irreducible constituent of $L_{2\rho} \otimes L_{2\rho}$; the fact that these two interpretations give the same answer was conjectured by B. Kostant and proved quite recently by one of the authors (A.Z.) together with A.D. Berenstein. It would be interesting to give a representation-theoretic interpretation of the coefficients $c_p$ in the decomposition (19) of the discriminant.

(b) The construction of the convex polytope $P(2\rho)$ can be extended in an obvious way to an arbitrary root system $R$ and arbitrary regular dominant weight $\lambda$ instead of $2\rho$. It turns out that this polytope is always combinatorially equivalent to a cube of dimension $rk R$.

In conclusion let us briefly discuss the relationships between the resultant and discriminant. First, it is well known (see [1]) that $D(P) = \pm q^{-1} \text{Res}(P, P_1)$, where $P_1 = dP/dx$. It is easy also to express $D(P)$ as the resultant of two polynomials of degree $r - 1$, viz., $D(P) = \text{const. Res}(P_1, P_2)$, where $P_2(x) = (\partial/\partial y)(y'P(x/y)) \mid_{y = 1}$. Translating this statement into the language of Newton polytopes we obtain
PROPOSITION 28. Let $\pi: \mathbb{R}^r \times \mathbb{R}' \rightarrow \mathbb{R}^{r+1}$ be defined by

$$\pi(p_0, \ldots, p_{r-1}, q_0, \ldots, q_{r-1}) = (p_0, p_1 + q_0, \ldots, p_{r-1} + q_{r-2}, q_{r-1}).$$

Then $\pi(N_{r-1,r-1}) = N_r$, and $\pi(S_{r-1,r-1}) = S_r$.

On the other hand, it follows at once from (2) and (18) that $(\text{Res}(P, Q))^2 = D(PQ)/D(P) D(Q)$. It would be interesting to place this identity into the context of general $A$-discriminants.

REFERENCES