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# Building counterexamples

Irena Rusu *

*L.R.I., U.R.A. 410 du C.N.R.S., bât. 490, 91405 Orsay-cedex, France*

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**Abstract**

A conjecture concerning perfect graphs asserts that if for a Berge graph $G$ the following three conditions hold: (1) neither $G$, nor $\bar{G}$ has an even pair; (2) neither $G$, nor $\bar{G}$ has a stable cutset; (3) neither $G$, nor $\bar{G}$ has a star-cutset, then $G$ or $\bar{G}$ is diamond-free. We show that this conjecture is not valid and that, in a way, every weaker version is false too. To this end, we construct a class of perfect graphs satisfying the hypothesis above and indicate counterexamples within this class for the instances of the conjecture obtained by replacing the diamond with any graph $H$ which is the join of a clique and a stable set.

## 1. Introduction

For a graph $G = (V, E)$, let us call a *clique* any set of pairwise adjacent vertices in $G$. The *clique number* $\omega(G)$ of $G$ represents the cardinality of a largest clique in $G$, while the *chromatic number* $\chi(G)$ of $G$ is the minimum number of colours necessary to colour the vertices of $G$ in such a way that any two adjacent vertices have different colours. Using these two parameters, Berge defined a graph $G = (V, E)$ to be *perfect* if for each of its subgraphs the clique number equals the chromatic number. It was proved by Lovász [5] that a graph $G$ is perfect if and only if its complement graph $\bar{G}$ is perfect too. No characterization with minimal forbidden subgraphs is known for perfect graphs, although a conjecture of Berge (called the strong perfect graph conjecture, abbreviated SPGC) has been formulated thirty years ago and it is neither proved, nor invalidated.

**Conjecture 1** (*SPGC*). A graph is perfect if and only if it contains no odd hole and no odd antihole.

A *hole* is a chordless cycle with at least five vertices, while an *antihole* is the complement graph of a hole. A hole is *odd* if it has odd number of edges. Graphs without odd holes and odd antiholes are usually called *Berge graphs*. An equivalent

* E-mail: rusu@chambord.univ-orleans.fr.


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version of the SPGC can be formulated using the notion of *minimal imperfect graph*, which designates an imperfect graph such that every proper induced subgraph is perfect.

**Conjecture 2** (*SPGC′*). Every minimal imperfect graph is an odd hole or an odd antihole.

As a first step toward proving (SPGC′), some properties of minimal imperfect graphs have been investigated.

A subset $C$ of $V$ is called a *cutset* of $G$ if the subgraph of $G$ induced by $V - C$ is not connected. As it was shown by Tucker [8], a minimal imperfect Berge graph cannot contain a *stable cutset*, i.e., a cutset $C$ such that the subgraph $[C]_G$ induced by $C$ in $G$ is edgeless. Moreover, in [1], Chvátal proved that no minimal imperfect graph has a *star-cutset*, that is, a cutset $C$ containing a vertex adjacent to all the other vertices in $C$.

In a graph $G$ two nonadjacent vertices $x, y$ form an *even pair* if there is no odd chordless path joining $x$ and $y$ in $G$. It was proved by Meyniel [6] that no minimal imperfect graph has an even pair.

Unfortunately, the exact importance of these three properties of minimal imperfect graphs is not known. The following conjecture, due to Bruce Reed [7] and involving a well-known class of perfect graphs, namely the line-graphs of bipartite graphs, has been disproved by Hougardy [4]:

**Conjecture 3** (*Reed*). Let $G$ be a Berge graph satisfying the conditions below:
1. neither $G$, nor $\bar{G}$ has an even pair;
2. neither $G$, nor $\bar{G}$ has a star cutset.

Then $G$ or $\bar{G}$ is the line-graph of a bipartite graph.

Several authors proposed a weaker conjecture by substituting the conclusion "$G$ or $\bar{G}$ is the line-graph of a bipartite graph" with "$G$ or $\bar{G}$ is diamond-free", where a diamond is the graph obtained by deleting an edge of a clique on four vertices. Still weaker than the conjecture above is the one we shall examine along this paper.

**Conjecture 4.** Let $G$ be a Berge graph such that
1. neither $G$, nor $\bar{G}$ has an even pair;
2. neither $G$, nor $\bar{G}$ has a star cutset;
3. neither $G$, nor $\bar{G}$ has a stable cutset.

Then $G$ or $\bar{G}$ is diamond-free.

Obviously, since the diamond-free Berge graphs are perfect [9], a proof of this conjecture would also be a proof of the SPGC. Our purpose here is to give a counterexample not only for this conjecture, but also for every weaker version of it obtained by replacing the diamond with any graph $H$ which is the join of a clique and a stable set (that is, all possible edges between the two sets are present in $H$). The counterexamples will be iteratively built using a composition of two graphs $G_1$ and $G_2$ into another graph $G$ with the same properties as the initial graphs.

## 2. The 2-join

Let $G = (V, E)$ be a graph and consider a partition of its vertex set $V$ in $p$ subsets $V_1, V_2, \ldots, V_p$. This is a natural way to indicate the classes of vertices in $G$ having the same behaviour relative to a certain property, whatever is this property. In our case, the vertices in a class will have the same neighbours in the graph joined to $G$ by the operation defined below. Some reasons of simplicity decided us to limit the approach to the case $p = 3$. The partition sets will be denoted by $R$, $W$, $B$, each of them representing the set of vertices in $G$ coloured respectively in red, white and blue. We shall also use the term of *Q-vertex* for a vertex coloured in $Q$, where $Q \in \{R, W, B\}$. A *QP-edge* $xy$ is an edge whose extremities are coloured in $Q$, respectively in $P$, for any $Q$, $P \in \{R, W, B\}$.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ coloured in $R$, $W$ and $B$, denote by $G = G_1 \phi G_2$ the graph $G = (V, E)$ defined as follows:

$$V = V_1 \cup V_2$$

$$E = E_1 \cup E_2 \cup \{xy \mid (x, y) \in R_1 \times R_2\} \cup \{zt \mid (z, t) \in W_1 \times W_2\},$$

where $R_i$ (resp. $W_i$) is the set of red (resp. white) vertices in $G_i$, for $i$=1, 2. The blue vertices in $G$ have precisely the same neighbours as they had in the two initial graphs.

Notice that the new graph $G$ may be also obtained from the graphs

$$G_1' = (V(G_1) \cup \{x_1, y_1\}, E(G_1) \cup \{x_1 v, v \in R_1\} \cup \{y_1 w, w \in W_1\} \cup \{x_1 y_1\})$$

and

$$G_2' = (V(G_2) \cup \{x_2, y_2\}, E(G_2) \cup \{x_2 v, v \in R_2\} \cup \{y_2 w, w \in W_2\} \cup \{x_2 y_2\})$$

using the 2-join operation defined by Cornuéjols and Cunningham [3]. According to that definition, the 2-join of $G_1'$ and $G_2'$ is the graph $H$ resulted by eliminating $x_i, y_i$ ($i = 1, 2$) and joining every neighbour of $x_1$ (resp. of $y_1$) in $G_1$ to every neighbour of $x_2$ (resp. of $y_2$) in $G_2$. A brief verification shows that $G$ and $H$ are in fact the same graph. That is why, for convenience, we shall say along this paper that $G = G_1 \phi G_2$ is the *2-join* of $G_1$ and $G_2$.

For every $n \geqslant 2$, consider now the graph $F_n$ (already coloured) with the vertex set

$$V(F_n) = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{z_1, z_2, \ldots, z_n\},$$

where $R = \{x_1, x_2, \ldots, x_n\}$, $W = \{y_1, y_2, \ldots, y_n\}$, $B = \{z_1, z_2, \ldots, z_n\}$ are the sets of vertices coloured, respectively, in red, white and blue.

The edge set $E(F_n)$ is defined such that $[R]_{F_n}$, $[W]_{F_n}$, $[B]_{F_n}$ are $n$-cliques and for all $i \in \{1, 2, \ldots, n\}$, $[x_i, y_i, z_i]_{F_n}$ is a 3-clique (see Fig. 1).

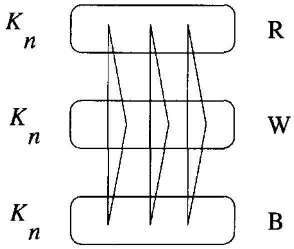



Fig. 1. The graph $F_n$.

It is easy to verify that, for $n \geqslant 3$, $F_n$ satisfies the hypothesis of the conjecture (the case $n = 2$ is particular). In order to define a class of graphs using $F_n$ $(n \geqslant 2)$ and the 2-join operation, let us see which general conditions are sufficient to preserve the hypothesis.

In the four lemmas below, we shall say that a graph is *coloured* in $R$, $W$, $B$ (or simply *coloured*) if its vertex set is partitioned in $R$, $W$, $B$ such that the following condition holds ($Q$ is an arbitrary colour of the set $\{R, W, B\}$):

**(C1)**: For every $Q$-vertex $x \in V$, the neighbourhood $N_G(x)$ intersects each of $R$, $W$, $B$, but contains none of them, except possibly for $Q$.

As we can easily notice, the 2-join of two coloured graphs is a coloured graph too.

The proofs of the lemmas are symmetrical for $R$, $W$ and for the two graphs $G_1$, $G_2$, therefore we shall analyse only the nonsymmetrical cases.

**Lemma 1.** *Let $G_i$ $(i = 1, 2)$ be two coloured Berge graphs satisfying the following conditions*:

**(C2)**: *If $x \in B_i$, then $N_{G_i}(x) = K_1 \cup K_2$, where $K_1 \subset B_i$ and $K_2 \subset R_i \cup W_i$ are two disjoint cliques with no edge between them*;

**(C3)**: *For every odd chordless path $Ru_1 \ldots u_{2k}R$ (resp. $Wu_1 \ldots u_{2k}W$) in $G_i$, there is some $p \in \{1, 2, \ldots, 2k\}$ such that $u_p \in R_i$ (resp. $W_i$)*;

**(C4)**: *There is no odd chordless path $RWW \ldots WR$ (resp. $WRR \ldots RW$) in $\bar{G}_i$.*

*Then $G = G_1 \phi G_2$ is a coloured Berge graph and satisfies* (C2), (C3) *and* (C4).

**Proof.** We firstly show that $G$ has no odd holes and no odd antiholes. Suppose there is an odd hole $C$ induced in $G$ and let $V(C)$ be its vertex set. Obviously, $C$ is not entirely contained in a graph $G_i$ since these graphs are Berge. To pass from a graph into the other one, an edge $xy$ such that $x$ and $y$ have the same colour is needed. Without loss of generality we may suppose that $x \in R_1$ and $y \in R_2$. Let $z$ be the other neighbour of $x$ along the cycle $C$. If $z \in R_1$ then $yz \in E(G)$ is a chord in $C$, a contradiction. Thus $z \in R_2$ or $z$ is not a red vertex.

Three positions of the cycle are then possible with respect to the initial graphs.

- $C = xyP_2zx$, where $x \in R_1$, $z, y \in R_2$ and $P_2$ is a chordless path in $G_2 - R_2$. In this case, $yP_2z$ is an odd chordless path in $G_2$ in which only the extremities are red vertices. By (C3), such a path is not induced in $G_2$.
- $C = xyP_2tuvP'_2zx$, where $x \in R_1$, $z, y \in R_2$, $u \in W_1$, $t, v \in W_2$ and $P_2$, $P'_2$ are chordless paths in $B_2$. Suppose that $yP_2t$ is even. Then $V(P_2)$ must contain an odd number of $B$-vertices. But this is not possible because of (C2).
- $C = xyP_2tvP_1x$, where $x \in R_1$, $y \in R_2$, $v \in W_1$, $t \in W_2$ and $P_i$ $(i = 1, 2)$ is a chordless path in $B_i$. Again, one of the two paths must contain an odd number of $B$-vertices and that is not possible.

We deduce that $G$ contains no odd holes.

Suppose now that $G$ contains some odd antihole $\bar{C}$ and that $\bar{C}$ has a $B$-vertex. If $\bar{C}$ has more than five vertices, then in $\bar{G}$ the $B$-vertex is nonadjacent to at least one path on four vertices. Therefore, in $G$ the $B$-vertex has such a path in its neighbourhood. But that contradicts (C2). Consequently, $\bar{C}$ induces a 5-cycle in $\bar{G}$, thus its complement is a 5-cycle in $G$. But $G$ has no odd holes, a contradiction.

We may then suppose that the antihole $\bar{C}$ contains no $B$-vertices. By (C4), a reasoning similar to the one we used to deduce that $G$ has no odd holes proves that in fact there is no odd hole in $\bar{G}$. Then $G$ is a Berge graph.

We subsequently prove that $G$ satisfies the conditions (C2), (C3), (C4).

(C2): By definition, the operation $\phi$ does not change the neighbour set of a $B$-vertex.

(C3): Assume the contrary and let $x, y$ be two nonadjacent $R$-vertices in $G$ joined by an odd chordless path $P$ with no $R$-vertices. Then $x$ and $y$ are both in $G_1$ or both in $G_2$ (say they are in $G_1$). By (C2), if there are any $B$-vertices on $P$, their number is even. An even number of $W$-vertices is then required in order to have an odd chordless path (notice that if there are no $W$-vertices, $P$ is entirely contained in $G_1$, a contradiction). Whatever would be the repartition of the $W$-vertices in $G_1$ and $G_2$, there is a chord in $P$.

(C4): Assume the contrary and let $x, y$ be two nonadjacent $R$-vertices in $\bar{G}$ joined by an odd chordless path $P'$ containing only $W$-vertices. Notice that in $\bar{G}$, the red vertices in a graph are joined to the white and to the blue vertices in the other one. Consequently, if $x, y$ are both in $G_1$, then every $W$-vertex of $P'$ in $G_2$ (and there exists at least one) would be adjacent to both $x$ and $y$, a contradiction. The only possible case is $x \in R_1$, $y \in R_2$ such that their neighbours along $P'$ ($u$ and, respectively, $v$) are in $W_2$, respectively in $W_1$. The $W$-vertices $u$ and $v$ are nonadjacent in $\bar{G}$, therefore at least another $W$-vertex occurs on $P'$. But then $P'$ would have chords. □

**Remark 1.** Notice that the condition (C2) could be relaxed, but such a modification would also be relaxing the conclusion of Lemma 1. Later reasonings will involve precisely the indicated form of (C2).

**Lemma 2.** *Let $G_i$ $(i = 1, 2)$ be two coloured graphs such that neither $G_i$, nor $\bar{G}_i$ has an even pair and*

**(C2)**: *If $x \in B_i$, then $N_{G_i}(x) = K_1 \cup K_2$, where $K_1 \subset B_i$ and $K_2 \subset R_i \cup W_i$ are two disjoint cliques with no edge between them*;

**(C5)**: *For every R-vertex (resp. W-vertex) x in $G_i$, there is an even chordless path $xRR \ldots RW$ (resp $xWW \ldots WR$) in $G_i$.*

*Then neither $G = G_1 \phi G_2$, nor $\bar{G}$ has an even pair and G satisfies* (C2) *and* (C5).

**Proof.** For every pair of vertices $u, v$ in the same graph $G_i$, if $u, v$ are nonadjacent (resp. adjacent) then there is an odd chordless path joining them in the same graph (resp. in its complement graph). This path is also an odd chordless path in $G$ (resp. in $\bar{G}$).

Let now $u, v$ be a pair of vertices in $G$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Four nonsymmetrical cases can occur:

- $u \in R_1$, $v \in R_2$; by (C1), there is a $W$-vertex $t$ in $G_1$ adjacent to $u$ and, obviously, nonadjacent to $v$. Also, there is a $B$-vertex $w$ in $G_2$ adjacent to $v$ and nonadjacent to $u$. Moreover, $t$ and $w$ are nonadjacent, so $tuvw$ is a chordless path on four vertices in $G$. The complement of this graph is the odd chordless path on four vertices $uwtv$ joining $u$ and $v$ in $\bar{G}$. We deduce that $(u, v)$ is not an even pair in $\bar{G}$.
- $u \in R_1$, $v \in W_2$; by (C5) there is an even chordless path $uRR...RW$ in $G_1$; then $uRR...Wv$ is an odd chordless path in $G$.
- $u \in R_1$, $v \in B_2$; by (C1) there exists a $B$-neighbour $w$ of $v$ in $G_2$ and by (C2) $w$ has an R-neighbour $t$ that is nonadjacent to $v$. Then $utwv$ is an odd chordless path joining $u$ and $v$ in $G$.
- $u \in B_1$, $v \in B_2$; if $t$ is a $W$-vertex in $G_1$ adjacent to $u$ (according to (C1)) and $w$ a $W$-vertex in $G_2$ adjacent to $v$, then $utwv$ is an odd chordless path joining $u$ and $v$ in $G$.

Thus $G$ and $\bar{G}$ have no even pair. Obviously, the conditions (C2) and (C5) are also valid for $G$. □

**Remark 2.** In fact, the $RW$-, $RB$- and $WB$-edges in $G_i$ may be even pairs in $\bar{G}_i$ and the conclusion is still valid. While 2-joining $G_1$ and $G_2$, by (C1) we can find for a $WR$-edge $xy$ in $G_1$ a $W$-vertex $u$ and an $R$-vertex $v$ in $G_2$ such that $uv \notin E(G)$, so $xvuy$ is an odd chordless path joining $x$ and $y$ in $\bar{G}$. For a $WB$-edge $zt$ in $G_1$, since (C2) is true the $B$-vertex $t$ has a $B$-neighbour $a$ in $G_1$ nonadjacent to $z$. As before, $z$ has a $W$-neighbour $b$ in $G_2$ and $atzb$ is a $P_4$ in $G$ that induces an odd chordless path in $\bar{G}$ joining $z$ and $t$.

**Lemma 3.** *Let $G_i$ ($i = 1, 2$) be two coloured graphs such that neither $G_i$, nor $\bar{G}_i$ has a stable cutset. Then the same property holds for $G = G_1 \phi G_2$.*

**Proof.** Assume the contrary and let $S$ be a stable cutset in $G$. Since neither $G_1$, nor $G_2$ has a stable cutset, $G_1' = G_1 - S$ and $G_2' = G_2 - S$ are connected. Moreover, they can have at most one colour in common and that one is $B$ (otherwise $G - S$ is connected). Consequently, $S$ contains at least one of $R_1, R_2$, so by (C1) it is not a stable.

Let now $S'$ be a stable cutset in $\bar{G}$ and let $\bar{G}_1'$, $\bar{G}_2'$ be the connected components of $\bar{G} - S'$. Neither $\bar{G}_1'$, nor $\bar{G}_2'$ contains a $B$-vertex (otherwise the two subgraphs are connected), therefore all the $B$-vertices are in $S'$. Since none of $B_1$, $B_2$ is empty, $S'$ cannot be a stable set, a contradiction. □

**Remark 3.** We can notice that in the proof we do not really need to use the hypothesis that $\bar{G}_i$ $(i = 1,2)$ has no stable cutset. Indeed, suppose that $\bar{G}_i$ may have stable cutsets, let $S'$ be a stable cutset of $\bar{G}$ and consider $A_1, A_2, \ldots, A_s$ the connected components of $\bar{G} - S'$. The graph $\bar{G} - S'$ contains at least one $B$-vertex $v$. Without loss of generality we may assume that $v$ is a $B$-vertex in $\bar{G}_1$ contained in the connected component $A_1$. Then $V(G_2) \subseteq A_1 \cup S'$, so $A_2, \ldots, A_s \subseteq V(G_1)$. This implies that $B_2$ and one of $R_2$, $W_2$ are included in $S'$. But since $S'$ is a stable set we get a contradiction to (C1).

**Lemma 4.** *Let $G_i$ $(i = 1,2)$ be two coloured graphs such that neither $G_i$, nor $\bar{G}_i$ has a star-cutset. Then the same property holds for $G = G_1 \phi G_2$.*

**Proof.** We shall use a result of Chvátal [1] stating that a graph $G$ has a star-cutset if and only if it has at least one of the following properties:

(i) $G$ has a vertex $w$ such that the set of all the vertices distinct from $w$ and not adjacent to $w$ induces a disconnected subgraph of $G$;

(ii) $G$ has at least two nonadjacent vertices, and it has adjacent vertices $v, w$ such that $w$ dominates $v$.

Assume now that the lemma is not true and let $S$ be a star-cutset of $G$. We denote by $x$ the vertex of $S$ adjacent to all the other vertices in $S$. Two nonsymmetrical cases are possible:

- $x \in B_1$; then $S \subset V(G_1)$ and $G_1 - S$ is connected. The two connected components are precisely $G_1 - S$ and $G_2$, therefore $G_1 - S$ contains no $R$-vertices and no $W$-vertices. The condition (C1) is violated.
- $x \in R_1$; denote $N_G'(x) = N_G(x) \cup \{x\}$ and $G_1' = G_1 - N_G'(x)$. Then $V(G) - N_G'(x) = W_2 \cup B_2 \cup V(G_1')$. By (C1), every $B$-vertex of $G_2$ has a neighbour in $G_2$ coloured $W$. This one is adjacent to all the $W$-vertices in $G_1'$ (at least one such vertex exists, by (C1)). Moreover, $G_1'$ is a connected graph (otherwise $N_G'(x) \cap V(G_1')$ would be a star-cutset for $G_1$, a contradiction). Conclusion: $G - N_G'(x)$ is connected.

To prove that no star-cutset $S$ exists, it is sufficient to show that every neighbour of $x$ in $G$ is adjacent to at least one nonneighbour of $x$. For the neighbours of $x$ in $G_1$, this property is insured by the fact that $G_1$ has no star-cutset. The neighbours of $x$ in $G_2$ are $R$-vertices, which are adjacent to some of the $W$-vertices in $G_2$.

Let us now prove that $\bar{G}$ has no star-cutset. Suppose the contrary.

- $x \in B_1$; as before, denote $N_{\bar{G}}'(x) = \{x\} \cup N_{\bar{G}}(x)$ and let $\bar{G}_1' = \bar{G}_1 - N_{\bar{G}}'(x)$ (connected) be the graph induced in $\bar{G}_1$ by the vertices nonadjacent to $x$. One has $\bar{G}_1' = \bar{G} - N_{\bar{G}}'(x)$ (since all vertices in $\bar{G}_2$ are adjacent to $x$), so $N_{\bar{G}}'(x)$ is not a star-cutset of $\bar{G}$. By (C1), there is at least one $W$-vertex and at least one $R$-vertex in $G_1$ adjacent to $x$. Therefore, in $\bar{G}_1'$ there exist at least one $R$-vertex and at least one $W$-vertex.

Consequently, every neighbour $y \in \bar{G}_2$ of $x$ is adjacent to a nonneighbour of $x$ (the $R$-vertex or the $W$-vertex in $\bar{G}_1$). Since $N'_{\bar{G}}(x) \cap V(\bar{G}_1)$ is not a star-cutset in $\bar{G}_1$, the same holds for the neighbours of $x$ in $\bar{G}_1$.

- $x \in R_1$; the graph $\bar{G}'_1$ is connected and contains at least one $W$-vertex, by (C1). This vertex is adjacent to all the $R$-vertices in $\bar{G}_2$, so $\bar{G} - N'_{\bar{G}}(x)$ is connected (since $\bar{G} - N'_{\bar{G}}(x) = \bar{G}'_1 \cup R_2$). Every neighbour $y \in V(G_2)$ of $x$ is a $W$-vertex or a $B$-vertex and, again by (C1), is adjacent to at least one $R$-vertex $z$ in $\bar{G}_2$ (nonadjacent to $x$). Also, every neighbour $y \in V(G_1)$ of $x$ is adjacent to a nonneighbour of $x$ in $\bar{G}_1$, since $\bar{G}_1$ has no star-cutset. □

The five conditions identified in the lemmas above are now sufficient to define the class of graphs generated by $F_n$ ($n \geqslant 2$) using the 2-join operation. Let us firstly say that a graph $G$ is *finely-colourable* if its vertex set may be partitioned into three sets $R$, $W$, $B$ such that the conditions (C1)–(C5) hold. A colouring of $G$ with these properties is called a *fine-colouring* and a graph $G$ provided with a fine-colouring is said to be *finely-coloured.*

We define the class $\Gamma$ generated by the *basic graphs* $F_n$ ($n \geqslant 2$) as following:

- for every $n \geqslant 2$, $F_n \in \Gamma$;
- if $G_1$ and $G_2$ in $\Gamma$ are finely-coloured in $R$, $W$, $B$, then $G_1 \phi G_2 \in \Gamma$.

The theorem below uses the four previous lemmas to prove that $\Gamma$ is correctly defined and, moreover, that every graph in $\Gamma \setminus \{F_2\}$ satisfies the hypothesis of the conjecture:

**Theorem 1.** *The graphs in $\Gamma \setminus \{F_2\}$ are finely-colourable Berge graphs with the following properties*:

1. *neither $G$, nor $\bar{G}$ has an even pair*;
2. *neither $G$, nor $\bar{G}$ has a star cutset*;
3. *neither $G$, nor $\bar{G}$ has a stable cutset.*

**Proof.** It is a routine matter to verify that, for $n \geqslant 3$, $F_n$ satisfies the hypothesis of Lemmas 1, 2, 3, 4.

For $n = 2$ not all the properties hold. Namely, there are two exceptions, both in $\bar{F_2}$:
– the nonadjacent pairs of vertices coloured in $RW$, $RB$ or $WB$ are even pairs in $\bar{F_2}$;
– the two 3-stables are stable cutsets in $\bar{F_2}$.

According to Remarks 2 and 3, the two exceptions above do not disturb the proofs of Lemmas 2 and 3.

Then we can easily prove by induction that every graph in $\Gamma$ has the indicated properties. □

## 3. A class of perfect graphs

The purpose of this section is to show that the class $\Gamma$ defined in Section 2 is a class of perfect graphs. To this end we prove that the fine-colourings of the graphs in

$\Gamma$ have a common structure and use this property to deduce that no $P_4$ of type *RRRR*, *WWWW* exists in a fine-colouring. The perfection results as a simple application of a theorem due to Chvátal et al. [2].

Since all the graphs in $\Gamma$ are obtained from the basic graphs by repeated 2-joins, we can regard every $G \in \Gamma$ as a set of basic graphs whose vertices are joined accordingly to the indicated rule. Apparently, in a fine-colouring of $G$ it is no need for a basic graph to be coloured as described before. Actually, they are coloured in this way.

**Lemma 5.** *In a fine-colouring of $G \in \Gamma$, every basic graph is finely-coloured.*

**Proof.** A brief verification shows that a fine-colouring of a basic graph $F_n$ is a one-to-one application from the set of $n$-cliques to the set of colours; therefore, for this easy case, the lemma is proved.

Let $G = G_1 \phi G_2$ be a graph of $\Gamma$ obtained by 2-joining the finely-coloured graphs $G_1$ and $G_2$. We denote by $R_i^{\text{old}}$ (resp. $W_i^{\text{old}}, B_i^{\text{old}}$) the sets of $R$-vertices (resp. $W$-, $B$-vertices) in the fine-colouring of $G_i$ used to obtain $G$ and by $R_i^{\text{new}}$ (resp. $W_i^{\text{new}}, B_i^{\text{new}}$) the vertices in $G_i$ $(i = 1,2)$ coloured in $R$ (resp. $W$, $B$) in an arbitrary fine-colouring of $G$. Then every vertex in $R_1^{\text{old}}$ is adjacent to every vertex in $R_2^{\text{old}}$ and the same holds for the $W$-vertices. The vertices in $B_i^{\text{old}}$ $(i = 1,2)$ have neighbours only in $G_i$.

**Claim 1.** *No $R$- or $W$-vertex in $G_i$ (old) becomes a $B$-vertex in $G$ (new).*

**Proof.** Suppose there is such a vertex $x \in B_2^{\text{new}} \cap W_2^{\text{old}}$. Then $x$ has at least one neighbour $y$ in $B_2^{\text{old}}$. We have to consider the following three cases:

- $x$ has a neighbour $y \in B_2^{\text{old}} \cap R_2^{\text{new}}$. Then all the neighbours of $x$ in $G_1$ must be $B$-vertices (new), otherwise the $R$- and $W$-neighbours of $x$ in $G$ do not induce a clique and this contradicts (C2). Consequently, all the vertices in $W_1^{\text{old}}$ are now $B$-vertices and form a clique. Let $z$ be a vertex in this set. By (C1), $z$ has an R-neighbour $t$ in $G$. If $t$ is in $G_1$, then $tzxy$ is a $P_4$ coloured *RBBR* and (C3) is violated. Then all the R-neighbours of $z$ in $G$ are in fact in $W_2^{\text{old}}$. We deduce that no W-neighbour (new) $u$ of $z$ occurs in $G_1$. Otherwise $u$ is in $R_1^{\text{old}}$ or in $B_1^{\text{old}}$ and it is not adjacent to $t$, therefore $z$ contradicts (C2). Now, the neighbourhood of $z$ coloured in $B$ contains $x$ and at least a vertex in $B_1^{\text{old}}$ (according to (C1) in $G_1$), thus it is not a clique, a contradiction.
- $x$ has a neighbour $y \in B_2^{\text{old}} \cap W_2^{\text{new}}$. The reasoning is similar to the preceding one.
- every neighbour of $x$ in $B_2$ (old) is also coloured with $B$ in $G$. Then all the vertices in $W_1^{\text{old}}$ are now coloured in $R$, $W$ and form a clique. Also, every neighbour of $x$ in $R_2^{\text{old}}$ is a $B$-vertex in $G$.

  We shall firstly prove that in $W_1^{\text{old}}$ there exist both new $W$-vertices and new $R$-vertices. Suppose this is not the case and all the vertices in $W_1^{\text{old}}$ are $W$ in the new colouring. We then consider an old $R$-neighbour $y$ of $x$ (which must be a $B$-vertex in the new colouring) and a neighbour $q$ of $y$ in $R_1^{\text{old}}$, which is not adjacent to a fixed vertex $s$ of $W_1^{\text{old}}$ (such a vertex exists, otherwise $s$ would be adjacent to the entire set $R_1^{\text{old}}$, a contradiction). The $P_4$ $qyxs$ implies $q \in R_1^{\text{new}}$. Let $r$ be an $R$-neighbour (new) of $x$, so $r \in W_2^{\text{old}}$ (otherwise the new $W$- and $R$-neighbours of $x$ would not form a clique). We deduce that $qyxr$ is a $P_4$ coloured $RBBR$, except if $yr \in E$; but then $r$ and $y$ are two adjacent neighbours of the $B$-vertex $x$ coloured in $R$ and $B$, a contradiction. The reasoning is similar if we suppose that all the vertices in $W_1^{\text{old}}$ are $R$ in the new colouring.

Consequently, in $W_1^{\text{old}}$ there exist vertices of both colours $R$ (new) and $W$ (new). Let $z$ be a neighbour of $x$ in $B_2^{\text{old}}$. Then $z$ is a $B$-vertex in $G$ too and its $R$- or $W$-neighbours are all in $G_2$. Suppose it has an $R$-neighbour $v$ in $B_2^{\text{old}}$ or in $R_2^{\text{old}}$. For every vertex $w \in W_1^{\text{old}} \cap R_1^{\text{new}}$ we have then the $P_4$ $wxzv$ which is coloured $RBBR$, a contradiction. We deduce that every $R$-neighbour of $z$ was coloured with $W$ in the fine-colouring of $G_2$. In the same way, we deduce that every $W$-neighbour of $z$ was coloured with $W$ in the fine-colouring of $G_2$.

Consider now a neighbour $t$ of $x$ in $R_2^{\text{old}}$. Then $t$ is coloured in $B$ (new) and its neighbours in $G_1$ cannot be $B$-vertices (since $x$ is also a $B$-neighbour of $t$, the $B$-vertices adjacent to $t$ in $G$ would not form a clique). Let $u \in R_1^{\text{new}} \cap R_1^{\text{old}}$ (if there is one) be a neighbour of $t$ and $v \in W_2^{\text{old}}$ an $R$-neighbour of $z$. Since $tz \in E(G)$ (both are $B$-vertices in $N_G(x)$), then $utzv$ is a $P_4$ coloured $RBBR$, again in contradiction with (C3). We have necessarily $tv \in E(G)$ and this contradicts the fact that the $R$- and $W$-vertices adjacent to $t$ form a clique (since $v$ and $u$ are nonadjacent). The same reasoning is valid for the W-neighbours of $t$ in $R_1^{\text{old}}$, so the former $R$-vertices in $G_1$ cannot be coloured in $R$, $W$ or $B$, a contradiction. □

**Claim 2.** *No $B$-vertex in $G_i$ (old) can change its colour in a fine-colouring of $G$.*

**Proof.** In the basic graphs $F_n$, every $R$- or $W$-vertex is adjacent to exactly one $B$-vertex. By Claim 1, no $B$-vertex can be added to the initial class, so by induction we may suppose that in $G_1$ and $G_2$ every $R$- or $W$-vertex is also adjacent to exactly one $B$-vertex. If a $B$-vertex (old) in $G_i$ ($i$=1 or 2) changes its colour, every former $R$-neighbour of it would have no more $B$-neighbours in $G$ and that would contradict (C1). □

**Proof of Lemma 5** (*continued*). We show that in every fine-colouring of $G$, each basic graph has the vertices in an $n$-clique of the same colour. By Claims 1 and 2, the set of $B$-vertices in every basic graph is unchanged while using the 2-join operation. Consequently, one of the three $n$-cliques of each $F_n$ (say $\{z_1, z_2, \ldots, z_n\}$) is always coloured in $B$ and all its neighbours are vertices of the same $F_n$ (see Fig. 1). By (C1) and (C2) for $G$, any $B$-vertex $z_i$ has the property that its neighbours $x_i, y_i$ are coloured in $R$ and $W$. Suppose that for two indices $i$ and $j$, $x_i$ and $x_j$ have not the same colour. If $x_i$ is an $R$-vertex and $x_j$ a $W$-vertex, then $y_j$ is an $R$-vertex and $x_iz_iz_jy_j$ is a $P_4$ coloured $RBBR$, a contradiction. Therefore, all the vertices in an $n$-clique have the same colour and this is a fine-colouring of $F_n$. In fact, every fine-colouring of $G$ may be obtained from another fine-colouring by interchanging the colours $R$ and $W$ in some basic graphs. □

**Remark 4.** Lemma 5 insures that the vertices of an $n$-clique in a basic graph $F_n$ are always of the same colour in a fine-colouring of $G$ and have the same neighbour set in $G-F_n$. Also, the neighbourhood in $G$ of each $B$-vertex is precisely its neighbourhood in the basic graph containing it. Moreover, given two basic subgraphs in a graph $G \in \Gamma$, either the connection between them is $R$–$R$, $W$–$W$, or it is $W$–$R$, $R$–$W$.

**Lemma 6.** *No $P_4$ of type RRRR or WWWW is induced by a fine-coloring of $G \in \Gamma$.*

**Proof.** Suppose the contrary and let $xyzt$ be a $P_4$ of type $RRRR$.

If any two vertices of this path are contained in the same basic graph $F_n$, then they are contained in the same $n$-clique of this graph, so they have precisely the same neighbours in $G - F_n$, according to Remark 4. Consequently, the other two vertices must be in the same basic graph, and this contradicts the fact that in a basic graph the vertices of the same colour form a clique.

Then every two vertices are in different basic graphs. Consider now $F^x$ and $F^z$ the basic graphs containing $x$ and $z$, respectively. The $R$-vertices in $F^z$ are not adjacent to the $R$-vertices in $F^x$, therefore, by Remark 4, they must be adjacent to the $W$-vertices in $F^x$. If $x'$ is a $W$-vertex in $F^x$, then $x'z \in E(G)$. The same holds for $y$ and $t$, so if $t'$ is a $W$-vertex in $F^t$ then $yt' \in E(G)$. Moreover, $x'$ and $t'$ are nonadjacent (otherwise the edges between $F^x$ and $F^t$ would be $WW$-edges or $RR$-edges and $x$ would be adjacent to $t$ since they are both $R$-vertices), and the same holds for $x'$ and $y$, respectively $t'$ and $z$. We deduce that $x'zyt'$ is a $P_4$ coloured in $WRRW$, a contradiction. □

**Theorem 2.** *The graphs in $\Gamma$ are perfect graphs.*

**Proof.** Let $G \in \Gamma$ be a graph provided with a fine-colouring. By Lemma 6, the fine-colouring is a partition of $V(G)$ into three sets $R, W, B$ which satisfies:
– no $P_4$ induced in $G$ is coloured $RRRR$, $WRRW$, $RWWR$ or $WWWW$.
– for each $B$-vertex $x$ in $G$, $N_G(x) = K_1 \cup K_2$, where $K_1$ and $K_2$ are two disjoint cliques with no edge between them.

If $G$ was not perfect, then it would contain a minimal imperfect subgraph $G'$. Two cases occur:

*Case* 1: $G'$ contains no $B$-vertex. If all the vertices of $G'$ have the same colour, then $G'$ is $P_4$-free, thus it is perfect. If both colours $R$ and $W$ are present in $G'$, we have a partition of $V(G')$ into two sets $R$, $W$ such that no induced $P_4$ is coloured $RRRR$, $RWWR$, $WRRW$ or $WWWW$. By a theorem of Chvátal et al. [2], the graph $G'$ is perfect if and only if the subgraphs induced by the vertices coloured in $R$ and, respectively, in $W$ are perfect. In our case, the two subgraphs are $P_4$-free, thus they are perfect. Conclusion: $G'$ is also perfect.

*Case* 2: $G'$ contains at least one $B$-vertex. Since $G'$ is minimal imperfect, the $B$-vertex is contained in exactly $\omega$ $\omega$-cliques, where $\omega = \omega(G')$ is the clique number of $G'$. But every $B$-vertex is contained in exactly two cliques in $G$, so it is contained in at most two cliques in $G'$, therefore $\omega = 1$ or 2. In the first case, $G'$ is perfect; in the second one, it is an odd hole and this contradicts the fact that $G$ is a Berge graph.

In both cases we obtain that every subgraph of $G$ is perfect, so $G$ itself is a perfect graph. □

## 4. Counterexamples

For an arbitrary graph $H$, consider the following conjecture:

**Conjecture** ($C_H$). If $G$ is a Berge graph such that
1. neither $G$, nor $\bar{G}$ has an even pair;
2. neither $G$, nor $\bar{G}$ has a star cutset;
3. neither $G$, nor $\bar{G}$ has a stable cutset

then $G$ or $\bar{G}$ is $H$-free.

For all graphs $H$ such that there is a graph $G_H$ in $\Gamma$ containing both $H$ and $\bar{H}$, the corresponding conjecture ($C_H$) is false. We don't know exactly which are these graphs, but we indicate a constructive method to found counterexamples for the cases when $H$ is a join of a clique and a stable set.

If $G = G_1 \phi G_2$, let $(G_1 \phi G_2)_{(R \leftrightarrow W)}$ or $G_{(R \leftrightarrow W)}$ be the graph $G$ with the colouring obtained from the initial one by interchanging the colours $R$ and $W$ in $G_2$. Consequently, the edges between $G_1$ and $G_2$ are no more $RR$-edges or $WW$-edges, but $RW$- or $WR$-edges. Obviously, the new colouring is a fine-colouring for the graphs $G_1$ and $G_2$. Is it a fine-colouring for the graph $G$? A reasoning similar to the one in Lemma 1 gives the affirmative answer for the conditions (C3), (C4) (notice that in $G_{(R \leftrightarrow W)}$ the adjacency is $R$–$W$, $W$–$R$, as in $\bar{G}$ before switching the colours $R$ and $W$). The other conditions are insured by the internal properties of $G_i$ ($i$=1, 2).

For a fixed $n \geqslant 2$, consider now the sequence of graphs defined by induction as follows:

$$G_n^1 = F_n$$

$$G_n^2 = F_n \phi F_n$$

$$\cdots\cdots$$

$$G_n^k = G_{n,(R \leftrightarrow W)}^{k-1} \phi F_n$$

$$\cdots\cdots$$

The graph $G_n^k$ is obtained from $G_n^{k-1}$ by interchanging the colours $R$ and $W$ in *its* subgraph $F_n$ (i.e. the subgraph used to build $G_n^{k-1}$) and 2-joining with a new graph $F_n$; therefore $G_n^k \in \Gamma$.

The iterative construction above allows us to identify a certain structure of these graphs. More precisely, if we denote by $F_n^i$ the graph $F_n$ used during the step $i$ of the composition (that is, the graph composed with $G_{n,(R\leftrightarrow W)}^{i-1}$ to obtain $G_n^i$) and by $R_i$, $W_i$, $B_i$ the $n$-cliques of $F_n^i$ coloured in $R$, $W$, $B$ at the present moment of the composition, then the graph $G_n^k$ ($k \geqslant 2$) has the structure in Fig. 2 (The $n$-cliques are represented by points, the join of two subgraphs by continuous line and the already known connections in $F_n^i$ by dashed line.)

Indeed, the graph $G_n^2$ has precisely this structure. By induction, suppose that $G_n^k$ has the configuration in Fig. 2. Its successor $G_n^{k+1}$ is obtained from $G_n^k$ by interchanging the colours $R$, $W$ in $F_n^k$ (that is, in Fig. 2, $W_k$ becomes $R_k$ and vice versa) and by composing with $F_n^{k+1}$ according to the new colouring. The structure obtained in this way is the one in Fig. 2.

In $G_n^{k+1}$ every vertex in the clique $W_{k+1}$ is adjacent to every vertex in the cliques $W_1$, $W_2$, …, $W_k$, for all $k \geqslant 1$. We deduce that $G_n^{n+1}$ contains as an induced subgraph the graph $H_n$ which is the join of an $n$-clique and an $n$-stable. Also, $G_n^{n+1}$ contains $\bar{H}_n$, since the $n$-clique $W_{n+1}$ is nonadjacent to any of the cliques $R_1$, $R_2$, …, $R_n$. So neither $G_n^{n+1}$, nor $\bar{G}_n^{n+1}$ is $H_n$-free.

We can then state that the conjecture $(C_H)$ is false for every $H$ which is the join of a clique $K$ and a stable set $S$. A counterexample is $G_p^{p+1}$, where $p = max\{|K|, |S|\}$.

**Remark 5.** Another conjecture says that a minimal imperfect graph cannot contain an *odd pair*, i.e., a pair of nonadjacent vertices joined by no even chordless path. Similarly to Lemma 2 and Remark 2, one can show that if the only odd pairs in $G_i$, $\bar{G}_i$ ($i = 1$,

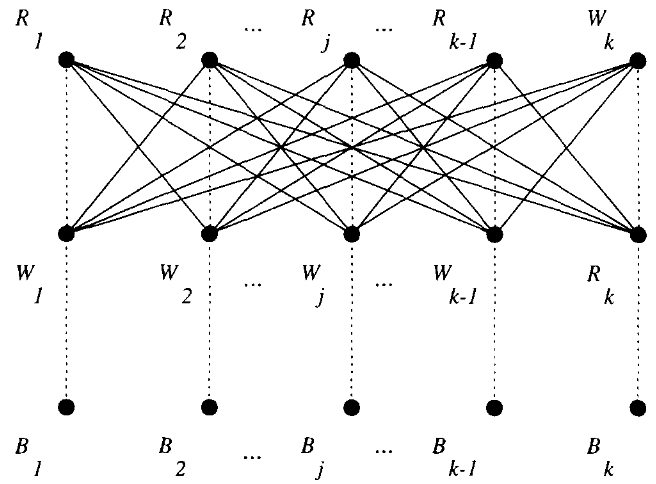
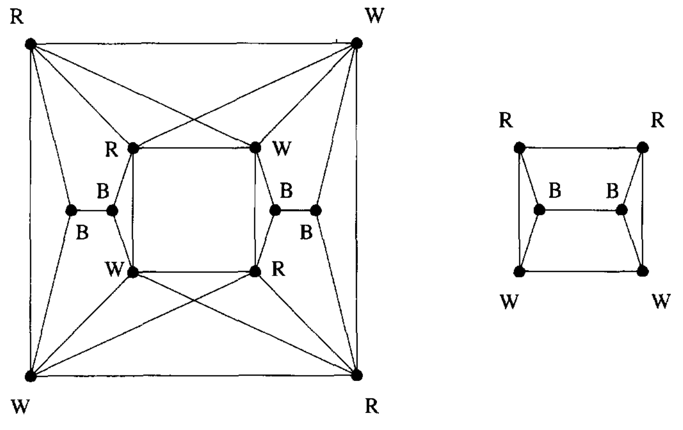



Fig. 2.




$G^{2}_{2,(R \leftrightarrow W)}$ $F_2$

Fig. 3.

2) correspond to the *RR*-, *BB*-, *WW*- edges in $G_i$, then the 2-join $G$ has the property that neither $G$, nor $\bar{G}$ has an odd pair. We deduce that all $G_n^k$ (except for $G_2^1 = F_2$) have this property and that even if we add to $(C_H)$ the following hypothesis:

4. Neither $G$, nor $\bar{G}$ has an odd pair,

the conjecture is false.

For $n = 2$, the graph $H_n = H_2$ is precisely the diamond and the counterexample $G_2^3$ is the 2-join of the two finely-coloured Berge graphs in Fig. 3.

Chvátal noticed that the graph $G_2^3$ can be described in the following easier manner: in the induced cycle on six vertices $w_1, w_2, \ldots, w_6$, substitute the vertex $w_i$ with the 2-clique induced by the vertices $u_i, v_i$; then, for any $i = 1,2,3$, add the vertices $x_i, y_i$ and the edges $x_iy_i, x_iu_i, x_iu_{i+3}, u_iu_{i+3}, y_iv_i, y_iv_{i+3}, v_iv_{i+3}$.

## 5. Conclusions

It is quite hard to believe that a conjecture of type $(C_H)$ may be true, although the counterexamples we have built do not exhaust the subject. The existence of such a large class of "nonconventional" perfect graphs contributes to confirm the idea that we really need some new properties of minimal imperfect graphs before being able to solve the difficult perfect graph problems.

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