Eventual practical stability of impulsive differential equations with time delay in terms of two measurements

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Abstract

In this paper we introduce a new stability—eventual practical stability for impulsive differential equations with time delay. By using Lyapunov functions and comparison principle, we will get some criteria of eventual practical stability, eventual practical quasistability and strong eventual practical stability for impulsive differential equations with time delay in terms of two measurements.
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1. Introduction

In recent years, significant progress has been made in the theory of impulsive differential [1,4–12] and references therein. Since practical stability only needs to stabilize a system into a region of phase space, it has been widely used in application. And the theory of practical stability has developed rather intensively [10,1–3] and references therein. In Ref. [10,1], the authors have gotten some results for practical stability of impulsive differential equations. In Ref. [5], the author has obtained some results for eventual stability of impulsive differential equations. In Ref. [2], the authors have obtained some results for the practical stability of finite delay differential systems in terms of two measures, but there does not exist impulses.

However, in the present paper, we will introduce a new stability for impulsive differential equation with time delay—eventual practical stability. And we will consider this stability for impulsive differential
2. Preliminaries

Consider the following impulsive differential equations with time delay:

\[ \dot{x}(t) = f(t, x(t), x(t - \tau)), \quad t > t_0, t \neq \tau_k, \]
\[ x(t_0 + t) = \phi(t), \quad t \in [-\tau, 0], \]
\[ \Delta x(t) = x(t^+) - x(t) = I_k(x(t)), \quad t = \tau_k, k = 1, 2, \ldots, \]

where \( t \in \mathbb{R}^+, \tau = \text{const} > 0, f \in C[\mathbb{R}^+ \times \Omega \times \Omega, \mathbb{R}^n], \Omega \) is a domain in \( \mathbb{R}^n \) containing the origin, \( \phi \in C([-\tau, 0], \Omega) \), \( f(t, 0, 0) \equiv 0, I_k(0) = 0, 0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots, \tau_k \to \infty \) for \( k \to \infty \), and \( x(t^+) = \lim_{s \to t^+} x(s), x(t^-) = \lim_{s \to t^-} x(s) \). The functions \( I_k : \mathbb{R}^n \to \mathbb{R}^n, k = 1, 2, \ldots, \) are such that if \( \|x\| < H \) and \( I_k(x) \neq 0 \), then \( \|x + I_k(x)\| < H \), where \( H = \text{const} > 0 \).

Denote by \( PC([-\tau, 0], \mathbb{R}^n) \) the set of piecewise left continuous functions \( \phi : [-\tau, 0] \to \mathbb{R}^n \) with the sup-norm \( \|\phi\| = \sup_{-\tau \leq s \leq 0} \|\phi(s)\| \), where \( \|\cdot\| \) is a norm in \( \mathbb{R}^n \), and \( R_t = [-\tau, \infty) \).

Throughout this paper we let the following hypotheses hold:

H1) For each function \( x(s) : [t_0 - \tau, \infty) \to \mathbb{R}^n \), which is continuous everywhere except the points \( \tau_k \) at which \( x(\tau_k^+) \) and \( x(\tau_k^-) \) exist and \( x(\tau_k^+) = x(\tau_k^-) \), \( f(t, x(t), x(t - \tau)) \) is continuous for almost all \( t \in [t_0, \infty) \) and at the discontinuous points \( f \) is left continuous.

H2) \( f(t, \phi) \) is Lipschitzian in \( \phi \) in each compact set in \( PC([-\tau, 0], \mathbb{R}^n) \).

Under the conditions (H1) and (H2), there is a unique solution of Eq. (1) through \( (t_0, \phi) \).

Together with Eq. (1), we consider the following comparison system:

\[ \dot{u}(t) = g(t, u), \quad t > t_0, t \neq \tau_k, \]
\[ u(t_0 + 0) = u_0, \]
\[ \Delta u(\tau_k) = u(\tau_k^+) - u(\tau_k) = J_k(u(\tau_k)) \quad k = 1, 2, \ldots, \]

where \( g : \mathbb{R}^+ \times G \to \mathbb{R}^n, J_k : G \to \mathbb{R}^n, k = 1, 2, \ldots, \) is a domain in \( \mathbb{R}^n \) containing the origin, \( u_0 \in G \). We denote \( J^+(t_0, u_0) \) the maximal interval of the type \( [t_0, w) \) in which the solution of system (2) is defined. Under the upper hypotheses, we know that \( J^+(t_0, u_0) = [t_0, \infty) \) [1].

Let
\[
S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}, \\
K = \{a \in C[\mathbb{R}^+, \mathbb{R}^+] : a(t) \text{ is monotone strictly increasing and } a(0) = 0\}, \\
\Gamma^n = \{h \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+] : \forall t \in \mathbb{R}^+, \inf_x h(t, x) = 0\}, \\
\Gamma^n_\tau = \{h \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+] : \forall t \in \mathbb{R}_+, \inf_x h(t, x) = 0\}, \\
PC(\rho) = \{\phi \in PC([-\tau, 0], \mathbb{R}^n) : |\phi| < \rho\}. 
\]
We have the following definitions:

**Definition 1 (Kou and Zhang [2])**. Let \( h_0 \in I^n_\tau, \phi \in PC([-\tau, 0], \mathbb{R}^m) \), for any \( t \in \mathbb{R}^+ \), \( \tilde{h}_0(t, \phi) \) is defined by
\[
\tilde{h}_0(t, \phi) = \sup_{-\tau \leq \theta \leq 0} h_0(t + \theta, \phi(\theta)).
\]

**Definition 2.** Let \( h_0 \in I^n, h \in I^n \), system (1) is said to be

(A_1) \((\tilde{h}_0, h)\) — eventually practically stable if for given \( (u, v) \) with \( 0 < u < v \), and some \( t_0 \in \mathbb{R}^+ \), there exists a \( \tau(u, v) > 0 \), such that \( \tilde{h}_0(t_0, \phi) < u \) implies that \( h(t(x(t))) < v, t \geq t_0 \geq \tau(u, v) \);

(A_2) \((\tilde{h}_0, h)\) — uniformly eventually practically stable if (A_1) holds for all \( t_0 \in \mathbb{R}^+ \);

(A_3) \((\tilde{h}_0, h)\) — eventually practically quasistable if for given \( (u, v, T) \) with \( u > 0, v > 0, T > 0 \), and some \( t_0 \in \mathbb{R}^+ \), there exists a \( \tau(u, v) > 0 \), such that \( \tilde{h}_0(t_0, \phi) < u \) implies that \( h(t(x(t))) < v, t \geq t_0 + T, t_0 \geq \tau(u, v) \);

(A_4) \((\tilde{h}_0, h)\) — uniformly eventually practically quasistable if (A_3) holds for all \( t_0 \in \mathbb{R}^+ \);

(A_5) \((\tilde{h}_0, h)\) — strongly eventually practically stable if both (A_1) and (A_3) hold;

(A_6) \((\tilde{h}_0, h)\) — strongly uniformly eventually practically stable if both (A_2) and (A_4) hold.

**Definition 3 (Yang [10])**. The function \( V : [t_0, \infty) \times S(\rho) \to \mathbb{R}^+ \) belongs to class \( v_0 \) if

1. the function \( V \) is continuous on each of the sets \([\tau_{k-1}, \tau_k) \times S(\rho)\) and for all \( t \geq t_0 \), \( V(t, 0) \equiv 0 \);
2. \( V(t, x) \) is locally Lipschitzian in \( x \in S(\rho) \);
3. for each \( k = 1, 2, \ldots \), there exist finite limits
\[
\lim_{(t, y) \to (\tau_k, x)} V(t, y) = V(\tau_k, x).
\]

**Definition 4 (Bainov and Stamova [1])**. Let \( V \in v_0, D_- V \) is defined as
\[
D_- V(t, x(t)) = \lim_{\delta \to 0^-} \inf_{\delta} \{V(t + \delta, x(t + \delta) + \delta f(t, x(t), x(t - \tau))) - V(t, x(t))\}.
\]

**Definition 5 (Bainov and Stamova [1])**. The function \( g : (t_0, \infty) \times G \to \mathbb{R}^m (G \subset \mathbb{R}^m) \) is said to be quasimonotone increasing in \((t_0, \infty) \times G \) if for each pair of points \((t, u)\) and \((t, v)\) from \((t_0, \infty) \times G \) and for \( j \in \{1, 2, \ldots, m\} \) the inequality \( g_j(t, u) \geq g_j(t, v) \) holds whenever \( u_j = v_j \) and \( u_i \geq v_i \), for \( i = 1, 2, \ldots, m, i \neq j \), i.e., for any \( t \in (t_0, \infty) \) fixed and any \( j \in \{1, 2, \ldots, m\} \) the function \( g_j(t, u) \) is nondecreasing with respect to \((u_1, \ldots, u_j-1, u_{j+1}, \ldots, u_m)\).

3. Main results

First, we introduce a lemma that has been given in Ref. [1].

**Lemma 1 (Bainov and Stamova [1])**. Let the following conditions hold:

1. The function \( g \) is quasimonotone increasing, continuous in the set \((\tau_k, \tau_{k+1}] \times G, k = 1, 2, \ldots, \) and \( r : J^+(t_0, u_0) \times \mathbb{R}^m \) is the maximal solution of the Eq. (2);
2. The functions $\psi_k : G \to \mathbb{R}^n, \psi_k(u) = u + J_k(u), k = 1, 2, \ldots$ are monotone increasing in $G$;
3. For each $k \in \mathbb{N}$ and $v \in G$ there exists the limit $\lim_{(t,u) \to (t,v)} g(t, u)$;
4. The function $V \in v_0$ is such that $V(t_0, \varphi) \leq u_0$ and let $x(t) = x(t, t_0, \varphi)$ denote the solution of Eq. (1).
5. The inequality
   
   \begin{align*}
   D_- V(t, x(t)) & \leq g(t, V(t, x(t))), \quad t \neq \tau_k, \\
   V(t^+, x(t) + I_k(x(t))) & \leq \psi_k(V(t, x(t))), \quad t = \tau_k
   \end{align*}

   is valid for each $t \geq t_0$ and any function $x \in C([-\tau, 0], \Omega)$ for which
   
   $V(t + s, x(t + s)) \leq V(t, x(t)) \quad s \in [-\tau, 0].$

   Then
   
   $V(t, x(t, t_0, \varphi)) \leq r(t, t_0, u_0) \quad t \in J^+(t_0, u_0).$

Now, when we consider the eventual practical stability of system (1), we have the following results:

**Theorem 1.** Let the following conditions hold:

(i) The conditions of Lemma 1 hold.
(ii) $0 < u < v$ are given.
(iii) $h_0 \in \Gamma^n, h \in \Gamma^n, \text{and } h(t, x) \leq \phi(\tilde{h}_0(t, x)) \text{ with } \phi \in K \text{ whenever } \tilde{h}_0(t, x) < u.$
(iv) $V \in v_0$ and there exist $\alpha, \beta \in K$ such that
   
   $\beta(h(t, x)) \leq V(t, x) \leq \alpha(\tilde{h}_0(t, x)).$

(v) $\phi(u) < v \text{ and } \alpha(u) < \beta(v).$

Then

(a) If system (2) is uniformly eventually practically stable with respect to $(\alpha(u), \beta(v))$, then system (1) is $(\tilde{h}_0, h)$-uniformly eventually practically stable with respect to $(u, v);$  
(b) If system (2) is strongly uniformly eventually practically stable with respect to $(\alpha(u), \beta(v))$, then the system (1) is $(\tilde{h}_0, h)$-strongly uniformly eventually practically stable with respect to $(u, v).$

**Proof.** (a) For given $(u, v)$ with $0 < u < v$, since the comparison system (2) is uniformly eventually practically stable with respect to $(\alpha(u), \beta(v))$, then for any $t_0 \in \mathbb{R}^+$, there exists a $\tau(u, v) > 0$ such that $u_0 < \alpha(u)$ implies $u(t, t_0, u_0) < \beta(v), t \geq t_0 \geq \tau(u, v)$, where $u(t, t_0, u_0)$ is any solution of system (2).

For any $(t_0, \varphi) \in \mathbb{R}^+ \times PC([-\tau, 0], \mathbb{R}^n)$ such that $\tilde{h}_0(t_0, \varphi) < u$, we have from condition (iii) and (v)

\[ h(t_0, \varphi) \leq \phi(\tilde{h}_0(t_0, \varphi)) \leq \phi(u) \leq v. \]

We then have

\[ h(t, x(t)) < v \quad \forall t \geq t_0 \geq \tau(u, v). \quad (3) \]

If this is not the case, then there exists an $s_1 > t_0$ such that

\[ h(s_1, s_1) \geq v \quad \text{and} \quad h(t, x(t)) < vt \in [t_0, s_1). \]
Let $V(t_0, \phi) = u_0$, then from condition (iv) we have

$$V(t_0, \phi) = u_0 \leq z(\tilde{h}_0(t_0, \phi)) < z(u)$$

which implies that $u(t, t_0, u_0) < \beta(v)$, $t \geq t_0 \geq \tau(u, v)$.

Since the conditions ofLemma 1 hold, then

$$V(t, x(t)) \leq r(t, t_0, u_0), \quad t \geq t_0,$$

where $r(t, t_0, u_0)$ is the maximal solution of (2), this together with condition (iv), we have the following contradiction:

$$\beta(v) \leq \beta(h(s_1, x(s_1))) \leq V(s_1, x(s_1)) \leq r(s_1, t_0, u_0) < \beta(v).$$

So inequality (3) holds. So system (1) is $(\tilde{h}_0, h)$—uniformly eventually practically stable with respect to $(u, v)$.

(b) For given $(u, v, T)$ with $0 < u < v, T > 0$, since the comparison system (2) is strongly uniformly eventually practically stable with respect to $(z(u), \beta(v))$, by what we have proved in (a), system (1) is $(\tilde{h}_0, h)$—uniformly eventually practically stable. Thus for any $t_0 \in R^+$, there exists a $\tau_1(u, v) > 0$, such that $\tilde{h}_0(t_0, \phi) < u$ implies

$$h(t, x(t)) < v, \quad t \geq t_0 \geq \tau_1(u, v).$$  \hspace{1cm} (4)

Since the comparison system (2) is uniformly eventually practically quasistable with respect to $(z(u), \beta(v))$, then for any $t_0 \in R^+$, there exists a $\tau_2(u, v) > 0$ such that $u_0 < z(u)$ implies $u(t, t_0, u_0) < \beta(v)$, $t \geq t_0 + T, t_0 \geq \tau_2(u, v)$, where $u(t, t_0, u_0)$ is any solution of system (2).

Let $\tau(u, v) = \max\{\tau_1(u, v), \tau_2(u, v)\}$, we will prove that

$$h(t, x(t)) < v, \quad t \geq t_0 + T, \quad t_0 \geq \tau(u, v).$$  \hspace{1cm} (5)

Let $V(t_0, \phi) = u_0$, then from condition (iv) we have

$$V(t_0, \phi) = u_0 \leq z(\tilde{h}_0(t_0, \phi)) < z(u)$$

which implies that $u(t, t_0, u_0) < \beta(v)$, $t \geq t_0 \geq \tau_2(u, v)$.

Since the conditions of Lemma 1 hold, then

$$V(t, x(t)) \leq r(t, t_0, u_0), \quad t \geq t_0,$$

where $r(t, t_0, u_0)$ is the maximal solution of system (2), this together with condition (iv) we have

$$\beta(h(t, x(t))) \leq V(t, x(t)) \leq r(t, t_0, u_0) < \beta(v), \quad t \geq t_0 + T, \quad t_0 \geq \tau(u, v)$$

from which we know that inequality (5) holds.

Thus, system (1) is $(\tilde{h}_0, h)$—strongly uniformly eventually practically stable with respect to $(u, v)$.  \hspace{1cm} $\square$

**Theorem 2.** Let condition (i), (iii) and (iv) of Theorem 1 be satisfied and

(iv) $(u, v, T)$ with $u > 0, v > 0, T > 0$ are given;

(v) $\phi(u) < v$.  \hspace{1cm}
Then if system (2) is uniformly eventually practically quasistable with respect to \((u), (v)\), then system (1) is \((\tilde{h}_0, h)\)-uniformly eventually practically quasistable with respect to \((u, v)\).

**Proof.** For given \((u, v, T)\) with \(u > 0, v > 0, T > 0\), since the comparison system (2) is uniformly eventually practically quasistable with respect to \((u), (v)\), then for any \(t_0 \in R^+\), there exists a \(\tau(u, v) > 0\) such that \(u_0 < \alpha(u)\) implies \(u(t, t_0, u_0) < \beta(v), t \geq t_0 + T, t_0 \geq \tau(u, v)\), where \(u(t, t_0, u_0)\) is any solution of system (2).

For any \((t_0, \varphi) \in R^+ \times PC([-\tau, 0], R^n)\) such that \(\tilde{h}_0(t_0, \varphi) < u\), we have from condition (iii) of Theorem 1 and (v)

\[ h(t_0, \varphi) \leq \phi(\tilde{h}_0(t_0, \varphi)) \leq \phi(u) < v. \]

We then have

\[ h(t, x(t)) < v \quad t \geq t_0 + T, \quad t_0 \geq \tau(u, v). \quad (6) \]

Let \(V(t_0, \varphi) = u_0\), then from condition (iv) we have

\[ V(t_0, \varphi) = u_0 \leq \alpha(\tilde{h}_0(t_0, \varphi)) < \alpha(u) \]

which implies that \(u(t, t_0, u_0) < \beta(v), t \geq t_0 + T, t_0 \geq \tau(u, v)\).

Since the conditions of Lemma 1 hold, then

\[ V(t, x(t)) \leq r(t, t_0, u_0), \quad t \geq t_0, \]

where \(r(t, t_0, u_0)\) is the maximal solution of the system (2), this together with condition (iv) of Theorem 1, we have the following inequality:

\[ \beta(h(t, x(t))) \leq V(t, x(t)) \leq r(t, t_0, u_0) < \beta(v), \quad t \geq t_0 + T, \quad t_0 \geq \tau(u, v). \]

So inequality (6) holds. System (1) is \((\tilde{h}_0, h)\)—uniformly eventually practically quasistable with respect to \((u, v)\). □

4. Conclusion

In this paper, we have introduced a new stability—eventual practical stability for impulsive differential equations with time delay. By using Lyapunov functions and comparison principle, we have obtained some results for eventual practical stability, eventual practical quasistability and strong eventual practical stability of this system. We can see that impulses do contribute to the system’s practical stability behavior.

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