Locally (Soluble-by-Finite) Groups of Finite Rank

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1. INTRODUCTION

A group $G$ is said to have finite rank $r$ if every finitely generated subgroup of $G$ is at most $r$-generator. If no such integer $r$ exists then we say that $G$ is of infinite rank. There are numerous papers in the literature concerning the theory of groups the abelian subgroups of which satisfy some finiteness condition on their ranks. The highlights of this theory include the theorem of Šunkov [14] stating that a locally finite group with all abelian subgroups of finite rank itself has finite rank and the theorem of Merzljakov [8] which says that a locally soluble group with its abelian subgroups of bounded rank also has finite rank. We should point out that Šunkov also shows that locally finite groups with finite rank are almost locally soluble; here we use the word “almost” to indicate that the group has a locally soluble normal subgroup of finite index.

Much of the work that has been done in this area has been concerned with showing that the rank of the whole group is heavily influenced by the
ranks of the abelian subgroups, at least when some sort of generalized solubility is present. Some of the best results have been obtained by Baer and Heineken [1] who showed, for example, that a radical group with all abelian subgroups of finite rank itself has finite rank. For locally soluble groups things are not quite so nice since Merzljakov [8] has constructed an example of a locally soluble group with all its abelian subgroups of finite but unbounded ranks. Consequently there is no analogue of the Baer–Heineken theorem for locally soluble groups. Much of this work has been summarized and extended in Robinson’s excellent survey paper [12].

In the present paper we consider the class of locally (soluble-by-finite) groups $G$. We shall show that if certain subgroups of $G$ have finite rank then $G$ itself has finite rank and is almost locally soluble. Our main result is the following.

**Theorem.** Let $G$ be a locally (soluble-by-finite) group with all locally soluble subgroups of finite rank. Then $G$ has finite rank and is almost locally soluble.

This theorem is dependent upon, but clearly generalizes, the theorem of Šunkov mentioned above.

We show in Section 3 that our result enables us to extend Merzljakov’s theorem to the class of locally (soluble-by-finite) groups. Indeed we show that in Merzljakov’s theorem the main obstruction is presented by the torsion free abelian subgroups; provided they have bounded ranks and the torsion abelian subgroups have finite rank then the whole group has finite rank and is almost locally soluble. We already know from the main theorem of [4] that a locally (soluble-by-finite) group of finite rank is almost locally soluble. The present work is mainly concerned with establishing that the rank of a locally (soluble-by-finite) group is finite if certain subgroups have finite (or bounded) rank.

We remark, in passing, that using a result of Ol’šanskii [9, Theorem 35.1] it is easy to give a remarkable example of a 2-generator group of infinite rank all of whose proper subgroups have finite rank.

It turns out that much of the work needed in proving our theorem is concerned with showing that there does not exist an infinite sequence of finite simple groups satisfying certain properties, more fully explained in Section 2. Consequently our work is dependent upon the classification of finite simple groups. In Section 2 we obtain the necessary results concerning finite simple groups. In Section 3 we prove our main results and obtain a number of interesting consequences. We remark also that our work would be considerably easier if the soluble-by-finite groups occurring were known to have their finite factor groups of bounded order since [6, 1.K.2] shows that a locally (soluble-by-finite) group with this property is almost...
locally soluble. Our notation is that in standard use and we refer the reader to [11] for this. We thank the referee for drawing our attention to Černikov's paper.

2. CONCERNING FINITE SIMPLE GROUPS

We shall let $r(A)$ denote the rank of a group $A$. Since we shall be interested in local systems of soluble-by-finite groups we shall naturally be interested in infinite sequences of finite semisimple groups. Here we call a group semisimple if it has no non-trivial abelian normal subgroups. Our first result shows that, in certain cases that occur naturally in Section 3, it suffices to consider sequences of finite simple groups.

PROPOSITION 2.1. Suppose that $(F_i)_{i \geq 1}$ is an infinite sequence of distinct finite semisimple groups with the following properties.

(i) There is a positive integer $r$ such that, for all $i$, the socle $N_i$ of $F_i$ has rank $r$.

(ii) For each $i$, $F_i$ is isomorphic to a section $A_{i+1}/B_{i+1}$ of $F_{i+1}$ where $B_{i+1}$ is soluble.

Then there is an infinite subsequence $G_1, G_2, \ldots$ of $(F_i)$ and an infinite sequence $S_1, S_2, \ldots$ of non-abelian simple groups such that for all $i$,

(a) $r(S_i) = r(S_{i+1})$,

(b) $S_i$ is a subgroup of $G_i$,

(c) $|S_i| < |S_{i+1}|$, and

(d) $S_i$ is a section of $S_{i+1}$.

We begin the proof of this proposition with the observation that every infinite subsequence of $(F_i)$ inherits properties (i) and (ii). Now for each $i$ write $N_i = M_{i,1} \times \cdots \times M_{i,k_i}$ where each $M_{i,j}$ is a direct product of pairwise isomorphic non-abelian simple groups and the simple factors of $M_{i,j_1}$ and $M_{i,j_2}$ are non-isomorphic if $j_1 \neq j_2$. Since every abelian 2-subgroup of $N_i$ has rank at most $r$, applying the Feit–Thompson Theorem allows us to assume that $k_i = k_{i+1} = k \leq r$ say, for all $i$ (by passing to a suitable subsequence if necessary). Since $C_{F_i}(N_i) = 1$ for all $i$ we have $|F_i/N_i| \leq |\text{Aut} N_i|$ and so the orders of the $N_i$ are unbounded. We may suppose that $|N_i| < |N_{i+1}|$ for all $i$ and hence assume that each $N_i$ has a simple direct factor of order greater than $s = (r)!$. For each $i$ write $N_i = K_i \times L_i$, where $K_i$ is the direct product of the simple direct factors of
LEMMA 2.2. For each i there is a normal subgroup $E_i$ of $F_i$ such that $N_i \leq E_i$, $E_i/N_i$ is soluble and $F_i/E_i$ has order at most s.

Proof. For ease of notation we shall drop the subscript i and write $N = M_1 \times \cdots \times M_k$, where for each $j$, $M_j$ is a direct product of $n_j$ copies of the simple group $C_j$. Then for each $j$ we have $\text{Aut } M_j \cong \text{Aut } C_j \times \Sigma_{n_j}$, where this wreath product is taken with respect to the natural permutation representation of the symmetric group $\Sigma_{n_j}$. Further we have $\text{Aut } N = \text{Aut } M_1 \times \text{Aut } M_2 \times \cdots \times \text{Aut } M_k$ (see p. 87 of [13]). Clearly $C_{\Sigma}(N) = 1$ and we shall identify $F$ with its image in $\text{Aut } N$. Let $B_j$ denote the base group of $\text{Aut } M_j$ for each $j$ and write $A = B_1 \times \cdots \times B_k$, $E = F \cap A$. Then $E/N \leq A/N$ which is a direct product of groups each isomorphic to some $\text{Aut } C_j/C_j$. From the classification theorem we know that $\text{Out } C_j$ is soluble (this being the well-known Schreier conjecture). It follows that $E/N$ is soluble. Now $F/E \cong FA/A \leq (\text{Aut } N)/A \cong \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} = \Sigma$, say. Since $n_j \leq r$ for all $j$ we certainly have $|\Sigma| \leq (r^r)^s = s$ and the lemma is proved.

We let $\text{rad } A$ denote the soluble radical of the finite group $A$.

LEMMA 2.3. For each i there is a perfect subgroup $P_{i+1}$ of $L_{i+1}$ such that $P_{i+1}/\text{rad } P_{i+1} \cong L_i$.

Proof. Our notation is simplified by assuming that $i = 1$. By hypothesis, $F_1$ is isomorphic to some section $A/B$ of $F_2$ where $B$ is soluble and we have $L_1 \cong T/B$ for some normal subgroup $T$ of $A$. Clearly $B = \text{rad } T$. Let $P$ denote the first perfect term of the derived series of $T$. Since $T/PB$ is an image of $T/P$ and of $L_1$ we have $T = PB$ and hence $L_1 \cong P/P \cap B$. But $P \cap B = \text{rad } P$ and so $L_1 \cong P/\text{rad } P$. It remains to show that $P \leq L_2$. If $Q$ is a maximal normal subgroup of $P$ then, since $P$ is perfect, $P/Q$ is isomorphic to a simple direct factor of $L_1$ and hence has order greater than $s$. With the notation as in Lemma 2.2, we have $PE_2/E_2 \cong P/P \cap E_2$, which therefore has order at most $s$ and it follows that $P \leq E_2$. Again by Lemma 2.2, $PN_2/N_2$ is soluble and hence trivial. Thus $P \leq N_2 = K_2 \times L_2$.
and since $P$ has no simple images of order at most $s$, we see that $P \leq L_2$, as required.

**Proof of Proposition 2.1.** The sequence $(S_i)$ whose existence is asserted by the proposition will consist of direct factors of suitably chosen subgroups $L_i$. For each $i$ let us now write $L_i = C_{i,1} \times \cdots \times C_{i,d}$, where each $C_{i,j}$ is simple.

We claim the following: For each $i$ there exists a permutation $f_i$ of $(1, \ldots, d)$ such that $C_{i,j}$ is isomorphic to a section of $C_{i+1,f_i(j)}$, for each $j = 1, \ldots, d$.

Before establishing this claim, we shall show that the statement of the proposition follows from it. Assuming the claim valid, a suitable relabelling (if necessary) allows us to assume that $C_{i,j}$ is isomorphic to a section of $C_{i+1,j}$ for all $i$ and all $j$. Then at least one of the sequences $C_{1,j}, C_{2,j}, C_{3,j}, \ldots$ contains infinitely many non-isomorphic subgroups and an appropriate refinement of this sequence gives the result.

We now proceed to verify the claim. We may as well assume (as before) that $i = 1$. Write $L_2 = C_1 \times \cdots \times C_d$ and, as in Lemma 2.3, let $P$ be a perfect subgroup of $L_2$ such that $P/R \cong L_1$, where $R = \ker P$. Also write $P/R = U_1/R \times \cdots \times U_d/R$, where each $U_j/R$ is non-abelian simple.

For each $j = 1, \ldots, d$ let $\pi_j: L_2 \to C_j$ be the natural projection. If for some $j$ we have $P\pi_j = 1$ then $P$ embeds in $C_1 \times \cdots \times C_{j-1} \times C_{j+1} \times \cdots \times C_d$ and $L_2$ has as a section the group $L_3 \times C_j$. But $\text{ssr}(L_2) > \text{ssr}(L_1) = \text{ssr}(L_2)$, a contradiction which shows that $P\pi_j \neq 1$. Since $P = U_1 \cdots U_d$ is perfect it follows that $U_j\pi_j$ is insoluble for some $i$ and hence that $\ker \pi_j \cap U_i \leq R$, since $U_i$ has only one insoluble composition factor. Now take some fixed $i$ and consider the subgroups $U_i\pi_j$, $j = 1, \ldots, d$. If each of these is soluble then so is $U_j$, a contradiction. Thus $U_i\pi_j$ is insoluble for some $j$. Now suppose for a contradiction that $U_i\pi_i$ and $U_i\pi_j$, are both insoluble for some distinct $i, j$. There is nothing lost by assuming $l_1 = 1$ and $l_2 = 2$. Let $U = U_1U_2$. If $\ker \pi_j \cap U \leq R$ then it follows easily that $C_j$ has a section isomorphic to $U_i/R \times U_j/R$. But, for all $i \neq j$, $C_j$ has an insoluble subgroup $U_i\pi_j$, for some $i$, and it follows that $\text{ssr}(L_2) > \text{ssr}(L_1) + 1$, another contradiction. So $\ker \pi_j \cap U \nleq R$ and there is an element $u = u_1u_2$ of $\ker \pi_j \setminus R$, where $u_1 \in U_1$, $u_2 \in U_2$, and without loss of generality, $u_1 \notin R$. Since $U_i/R$ is non-abelian there exists $z \in U_1$ such that $[u, z] \notin R$. But $[u, z] = [u_1, z] \mod R$ and so $[u, z] \in \ker \pi_j \cap U_1 \leq R$, our final contradiction. We have shown that for each $j$ there is exactly one $U_j$ such that $U_j\pi_j$ is insoluble. Then $U_j/R$ is isomorphic to a simple direct factor of $L_2$ and to a section of $C_j$ and the claim follows easily, thus concluding the proof of the proposition.
In the remainder of this section we show that a sequence of finite simple groups satisfying the conclusion of Proposition 2.1 cannot exist. Specifically we have:

**Proposition 2.4.** Let \( \{S_i\}_{i \geq 1} \) be an infinite sequence of finite non-abelian simple groups with the following properties:

(a) \( S_i \) is a section of \( S_{i+1} \).

(b) There exists \( r \in \mathbb{N} \) such that the rank of \( S_i \) is at most \( r \) for all \( i \geq 1 \).

Then there exists an integer \( N \) such that \( S_i = S_{j+1} \) for all \( i \geq N \).

Our proof of this proposition depends upon the classification of finite simple groups. At various points in the proof we shall use results from the structure theory of finite groups of Lie type. We do not wish to presume that the reader is familiar with this theory but neither do we wish to reproduce a large amount of technical material that is already available in the literature. Accordingly, we shall take certain fundamental concepts of the theory as given (root systems, Dynkin diagrams, root subgroups, etc.) and support our arguments with detailed references to [3]. Our notation and terminology are those used in [3].

The classification of finite simple groups implies that the non-abelian finite simple groups are of the following types:

1. 26 sporadic groups.
2. Alternating groups \( \text{Alt}(k) \), for \( k \geq 5 \).
3. Chevalley groups: \( A_l(q) \) (\( l \geq 1 \)), \( B_l(q) \) (\( l \geq 2 \)), \( C_l(q) \) (\( l \geq 3 \)), \( D_l(q) \) (\( l \geq 4 \)), \( G_2(q) \), \( F_4(q) \), \( E_6(q) \), \( E_7(q) \), \( E_8(q) \).
4. Steinberg groups: \( ^2A_l(q^2) \) (\( l \geq 2 \)), \( ^2D_l(q^2) \) (\( l \geq 4 \)), \( ^2E_6(q^2) \), \( ^3D_4(q^3) \).
5. Suzuki groups: \( ^2B_2(2^{2m+1}) \).
6. Ree groups: \( ^2G_2(3^{2m+1}) \), \( ^2F_4(2^{2m+1}) \).

We shall use the notation \( ^*X_l(q) \) for an arbitrary group of Lie type where, of course, \( q = p^n \) is a power of the prime \( p \), \( X \) is one of \( A, B, \ldots, G \), and \( ^* \) is blank, 2, or 3. We shall often write \( A_l(q) \), \( B_l(q) \), etc., without mentioning the range of the parameter \( l \). In all cases \( l \) satisfies the relevant inequality from the list above. We shall use this notation in several results below.

We show that the Lie ranks and the value of \( n \) for the groups in our sequence \( \{S_i\} \) are bounded in terms of \( r \).

**Lemma 2.5.** Suppose \( H = ^*X_l(q) \) is a finite simple group of Lie type and rank at most \( r \). Then the Lie rank \( l \) and the number \( n \) are bounded in terms of \( r \) only.
Proof. Let the order of a Sylow $p$-subgroup of $H$ be $q^m = p^{mn}$ for some $m > 0$. The precise value of $q^m$ is given in [3, Theorem 8.6.1 and Theorem 14.3.1].

If $H = \Gamma_X(q)$ is of type (v) or (vi) then clearly the value of the Lie rank $l$ is bounded for such groups. The groups $A_i(q), B_i(q), C_i(q), D_i(q)$ are all images of linear groups of degree $y + 1$ over a field of order $q$ where $y$ is dependent only upon $l$. This follows from [3, Theorem 11.3.2]. Also $\Gamma_2(q^2) \leq A_i(q^2)$ and $\Gamma_2(q^2) \leq D_i(q^2)$. Similarly each of the Chevalley groups $D_i(q^3), E_i(q), E_{17}(q), C_i(q), D_i(q), D_2(q^2)$ is a group of linear transformations of a vector space of dimension at most 248, using the table on page 43 of [3]. Also $\Gamma_2(q^2) \leq E_i(q^2)$ and $\Gamma_2(q^3) \leq D_i(q^3)$. The Suzuki groups and Ree groups are also subgroups of their respective untwisted versions. Hence in any case $H$ is an image of a linear group of degree $y + 1$ for some function $y$ of $l$.

Since a Sylow $p$-subgroup of $GL(y + 1, q)$ consists of upper triangular matrices it is clear that a Sylow $p$-subgroup of $H$ has a series of length $y$ with elementary abelian factors. Since the order of a Sylow $p$-subgroup of $H$ is $p^{mn}$ it follows that $H$ has an elementary abelian $p$-section of rank at least $mn/y$. Hence $mn/y \leq r$.

If $H$ is one of the groups $A_i(q), 2A_i(q^2), B_i(q), C_i(q), D_i(q), 2D_i(q^2)$ then using the table on page 43 of [3] we see that $m/y$ is at least $(l - 1)/2$ so we have

$$\frac{n(l - 1)}{2} \leq \frac{mn}{y} \leq r.$$  

In particular for these groups $n$ and $l$ are bounded in terms of $r$.

If $H$ is one of the remaining groups the Lie rank parameters are bounded and the inequality $mn/y \leq r$ implies that $n$ is also bounded in terms of $r$, as required.

Observe that every infinite subsequence of $\{S_i\}$ inherits properties (a) and (b). Consequently we are at liberty to delete terms of the sequence $\{S_i\}$ and prove Proposition 2.4 for the resulting subsequence.

**Lemma 2.6.** We may assume that all of the groups $S_i$ are of the form $\Gamma_X(p^n)$ for some fixed $*, X, l,$ and $n$. We may also assume that $*X \neq 2B, 2G$, or $2F$.

**Proof.** Clearly we can delete all occurrences of sporadic groups in our sequence. Moreover the rank of $\text{Alt}(k)$ is at least $[k/3]$, the integer part of $k/3$, as is easily seen by considering the group generated by $(123), (456), (789), \ldots$. Consequently only finitely many of the $S_i$ can be alternating groups and so we may delete these. Lemma 2.5 shows that for the Ree and Suzuki groups the value of $m$ is bounded in terms of $r$ so that only finitely many Ree and Suzuki groups occur in the sequence and we may delete these also. Finally, Lemma 2.5 shows that $n$ and $l$ are bounded.
in terms of $r$ and, since there are only finitely many families $\ast X$ of type (iii) or (iv), it is clear that $\{S_i\}$ has a subsequence with the desired properties.

Next we examine the structure of the Sylow $p$-subgroup of the groups $S_i$. We prove the following, presumably well-known fact.

**Lemma 2.7.** Let $G$ be a group of type (iii) or (iv) over a finite field of characteristic $p$. If the Sylow $p$-subgroups of $G$ are abelian then $G \cong A_4(q) \cong \text{PSL}(2, q)$ where $q = p^n$.

**Proof.** Let $G$ be of type (iii) and suppose that $G \neq A_4(q)$. The unipotent subgroup $U$ of $G$ generated by the root subgroups corresponding to positive roots of $G$ is a Sylow $p$-subgroup of $G$. Let $r$ and $s$ be fundamental roots of $G$ corresponding to adjacent nodes of the Dynkin diagram of $G$ (such roots exist since $G \neq A_4(q)$). An easy calculation using Chevalley's commutator formula [3, Theorem 5.2.2] shows that $[X_r, X_s] \neq 1$. Thus $U$ is non-abelian in this case.

Let $G \cong 2A_{2k-1}(q^3)$, $3D_4(q^2)$, $3D_4(q^3)$, or $2E_6(q^2)$ and let $H$ denote the corresponding Chevalley group; thus $H \cong 2A_{2k-1}(q^3)$, $3D_4(q^2)$, $3D_4(q^3)$, and $E_6(q^2)$ in the respective cases. Also let $\Pi$ denote a fundamental root system of $H$. If $G \cong 2D_4(q^2)$ or $2E_6(q^2)$ then there exist $r, s \in \Pi$ corresponding to adjacent nodes of the Dynkin diagram of $H$ and such that $S_1 = \{r\}$, $S_2 = \{s\}$ are $p$-orbits of $\Pi$. Using [3, Proposition 13.6.3(i)] and Chevalley's commutator formula we obtain that $[X^1_{s}, X^2_{s}] \neq 1$. If $G \cong 2A_{2k-1}(q^3)$ then there exists a unique $p$-orbit $S_1 = \{r\}$ of $\Pi$ of type $A_1$ and a unique $p$-orbit $S_2$ of $\Pi$ of type $A_1 \times A_1$ that contains an element of $\Pi$ adjacent to $r$ in the Dynkin diagram of $H$. Using [3, Proposition 13.6.3(ii)] and the Chevalley commutator formula we see that $[X^1_{s}, X^2_{s}] \neq 1$. If $G \cong 3D_4(q^3)$ then there are only two $p$-orbits of $\Pi$, $S_1$ and $S_2$, and again $[X^1_{s}, X^2_{s}] \neq 1$. Thus in all of the cases considered in this paragraph the Sylow $p$-subgroups of $G$ are non-abelian.

Finally, if $G \cong 2A_{2k}(q^2)$, there exists a $p$-orbit $S$ of the set of fundamental roots of $A_{2k}(q^2)$ which generates a root system of type $A_2$. We have seen in the proof of Lemma 2.5 that $X^1_S$ is non-abelian.

That $G \cong \text{PSL}(2, q)$ now follows from [3, Theorem 11.3.2]. The proof is complete.

The following lemma is a special case of [5, Theorem 1.4B].

**Lemma 2.8.** Let $K$ be a finite field and $s$ a natural number. Let $p$ be a prime that is different from the characteristic of $K$ and such that $p > s$. Then the Sylow $p$-subgroups of $\text{GL}(s, K)$ are abelian.

**Proof of Proposition 2.4.** By Lemma 2.6, we may assume that $S_i = \ast X^i(p_n^s)$ for some fixed $\ast$, $X$, $I$, and $n$. Moreover $\ast X \neq B$, $2G$, or $2F$. As in
the proof of Lemma 2.5 there exists an integer \( y \) such that each \( S_i \) is an image of a linear group of degree \( y + 1 \) over \( GF(p^n) \).

We suppose for a contradiction that the sequence \( \{ |S_i| \}_{i \geq 1} \) is unbounded. It follows that the sequence \( \{ p_i \}_{i \geq 1} \) is also unbounded and we may further assume, on deleting certain \( S_i \) from our sequence if necessary, that \( y + 1 \leq p_1 < p_2 < \cdots \).

If \( P_i \) is a non-abelian Sylow \( p_i \)-subgroup of \( S_i \) then Lemmas 2.7 and 2.8 imply that \( p_i = p_{i+1} \), a contradiction. Hence the Sylow \( p_i \)-subgroup of \( S_i \) is abelian and \( S_i \cong PSL(2,q_i) \). For all primes \( p \) and natural numbers \( n \) a complete list of the subgroups of \( PSL(2,GF(p^n)) \) is known and can be found, for instance, in [16, Theorem 6.26]. This list shows that each simple section of \( S_i \) is either of the form \( A_5(p^n) \) for some \( m \) dividing \( n_i \) or is isomorphic to \( Alt(5) \). Now \( S_1 \cong A_5(p_1^{n_1}) \) is a section of \( S_2 \equiv A_5(p_2^{n_2}) \) and, as \( |S_1| > 60 \), it follows that \( A_5(p_1^{n_1}) \cong A_5(p_2^{n_2}) \) for some \( m \) dividing \( n_2 \).

However, since \( p_1, p_2 \geq y + 1 \), it follows from [16, 6.27] that \( p_1 = p_2 \). This is a contradiction since \( p_1 < p_2 \). The proof is now complete.

3. PROOF OF THE MAIN THEOREM

Suppose now that \( G \) is a locally (soluble-by-finite) group and that all its locally soluble subgroups have finite rank. Suppose we can show that every countable subgroup of \( G \) has finite rank. Then it follows that \( G \) has finite rank, since the property of “having finite rank” is a property of countably recognizable character. Hence in order to prove that \( G \) has finite rank, we may assume that \( G \) is countable.

The following result shows that our group \( G \) has a locally soluble radical. Of course such a result is not true for arbitrary groups as an example of \( P \). Hall shows see [11, Theorem 8.19.1]. In fact we have:

**Lemmas 3.1.** Let \( G \) be a group and suppose that all locally soluble subgroups of \( G \) have finite rank. Then \( G \) has a unique maximal normal locally soluble subgroup which is hyperabelian of finite rank.

**Proof.** Let \( K \) be a locally soluble normal subgroup of \( G \). Then \( K \) has finite rank. Thus \( K \) is hyperabelian by [11, Lemma 10.39] and has an ascending characteristic abelian series, by the same result. Suppose that \( H \) is the product of all normal locally soluble subgroups of \( G \). It follows that \( H \) is hyperabelian and its locally soluble subgroups have finite rank. A theorem of Baer and Heineken [1] shows that in this case \( H \) has finite rank. That \( H \) is locally soluble follows from [11, p. 179].

A further consequence of the Baer–Heineken theorem is that if \( N \) is a normal locally soluble subgroup of our group \( G \) and \( H/N \) is a locally
soluble subgroup of $G/N$ then $H$ is also locally soluble. For if $F$ is a finitely generated subgroup of $H$ then $F \cap N$ is locally soluble of finite rank, so is hyperabelian and $FN/N$ is soluble. Hence $F$ is hyperabelian with all abelian subgroups of finite rank so $F$ is locally soluble, and hence soluble. Consequently the hypotheses on $G$ are inherited by $G/N$ so that we may assume that in our group $G$ the locally soluble radical is trivial.

Since $G$ is countable we can write $G = \bigcup_{i \geq 1} H_i$ where $H_i$ is a finitely generated soluble-by-finite group, with $H_i \leq H_{i+1}$ for all $i$. It follows from a theorem of Robinson [11, Theorem 10.38] that the group $H_i$ is a minimax group. Let $R_i$ be the soluble radical of $H_i$ for each $i$ so that $R_i \triangleleft H_i$ and $F_i = H_i/R_i$ is a finite group containing no non-trivial abelian normal subgroups. Hence $F_i$ is semisimple for all $i$ and $R_i$ is a soluble minimax group.

We note the following:

**Lemma 3.2.** The group $R = \langle R_i | i \geq 1 \rangle$ is a locally soluble group of finite rank and if $G$ is infinite then the indices $|H_i : R_i|$ are unbounded.

**Proof.** Note that for each $i$, $R_i \triangleleft H_i$ so that we can perform the product $\prod_{i \geq 1} R_i$. It is then clear that $R$ is locally soluble. Since $G$ has all its locally soluble subgroups of finite rank it follows that $R$ has finite rank. Suppose there is an integer $n$ such that $|H_i : R_i| \leq n$ for all $i$ and let $g_1, \ldots, g_{n+1}$ be arbitrary elements of $G$. Then there exists $i$ such that $g_j \in H_i$ for $j = 1, \ldots, n + 1$ and it follows that $g_k = g_i \mod R_i$, for some distinct $k, l$. Hence $|G : R| \leq n$ and $G$ is finite since $\text{core}_G R = 1$.

Throughout the rest of this paper we shall assume that the rank of the group $R$ is $r$. Hence each of the groups $R_i$ has rank at most $r$.

We recall from [17, 9.33] that the periodic subgroups of $GL(r, \mathbb{Q})$ are all finite of order bounded by some function $f(r)$. This fact is crucial in the proof of the next proposition. We also use the fact that some term of the derived series of a locally soluble group of rank $r$ is a periodic hyper-central group (see [11, Lemma 10.39]). Furthermore this term of the derived series depends only on $r$.

**Proposition 3.3.** Suppose that $M$ is a finitely generated soluble-by-finite minimax group such that the soluble radical $N$ of $M$ has rank $r$. Suppose that $N^{(d)} = N^{(d-1)}$ is periodic and let $k = f(r)$ denote the order of a largest finite subgroup of $GL(r, \mathbb{Q})$. Suppose that $M/N$ has socle of rank $n$, where $n > kd$. Then

(i) $M/N^{(d)}$ has a normal Černikov subgroup with an image which is a direct product of simple groups and of rank at least $n - kd$.

(ii) $M$ contains a normal locally finite subgroup of rank at least $n - kd$.

**Proof.** It suffices to prove (i). We argue by induction on $d$, the case $d = 0$ being clear. Let $D = N^{(d-1)}$, and $E = N^{(d)}$. Let $T/E$ be the torsion
subgroup of $D/E$, so that $T/E$ is Černikov. The conditions are satisfied by $M/D$ so we may assume inductively that $M/D$ has a normal Černikov subgroup $W/D$ with an image, $W/K$ say, which is a direct product of simple groups and of rank at least $n - k(d - 1)$. Now there is a homomorphism $\theta$ from $W/D$ to $\text{Aut}(D/T)$ since $D/T$ is abelian. Then $\text{Aut}(D/T) \leq \text{GL}(r, \mathbb{Q})$ implies $\ker \theta = C_{W/D}(D/T) = V/D \leq M/D$ satisfies $[W/V] \leq k$. Now $V/(V \cap K) \cong VK/K$ is a direct product of simple groups since $VK/K \triangleleft W/K$. Moreover $|W/K : VK/K| = |W : VK| \leq k$ so that $V/(V \cap K)$ has rank at least $n - kd$.

Now $V < M$ and $V/D$ is Černikov, being a subgroup of $W/D$. Therefore $V/T$ is centre-by-Černikov and it follows that $(V/T)' = V'T/T$ is Černikov, by [11, Theorem 4.23]. Since $VK/K$ is perfect $V'T/(V'T \cap K) \cong V'VK/K = VK/K$ is a direct product of simple groups and of rank at least $n - kd$. Thus $V'T/T$ is Černikov with an image which is a direct product of simple groups and of rank at least $n - kd$. Since $T/E$ is Černikov it follows that $V'T/E$ has the desired property. 

By discarding certain of the $H_i$ and renumbering the remaining ones we may assume that the orders of the semisimple factor groups $F_i = H_i/R_i$ increase with $i$. Clearly $F_i$ is a section of $F_{i+1}$ so that Proposition 2.1 and Theorem 2.4 imply that there is no bound on the ranks of the socles of the $F_i$. Suppose that soc $F_i$ has rank $n_i$ and that $n_i < n_{i+1}$ for all $i$. By deleting and reindexing the subgroups if necessary we may also assume that for each $i$, $n_i > kd$, where $k$ is defined in Proposition 3.3 and $d$ is such that $R^{(d)}$ is periodic. Then Proposition 3.3 implies that $H_i$ contains a locally finite normal subgroup $A_i$ of rank at least $n_i - kd$. Let $A$ denote the product of all the groups $A_i$. Then $A$ is locally finite and its locally soluble subgroups have finite rank. Hence by Šunkov's theorem [14], $A$ is almost locally soluble and hence has finite rank. However, this is a contradiction since $r(A) \geq n_i - kd$. We have therefore proved the following theorem:

**Theorem 3.4.** Suppose that $G$ is a locally (soluble-by-finite) group and that all locally soluble subgroups of $G$ have finite rank. Then $G$ has finite rank.

Next we note that the property of being “almost locally soluble” is also a property of countably recognizable character. Specifically we have:

**Lemma 3.5.** Suppose $G$ is a group and all countable subgroups of $G$ are almost locally soluble. Then $G$ is almost locally soluble.

**Proof.** Suppose $G$ satisfies the hypotheses of the lemma and for each subgroup $H$ of $G$ let $R(H)$ denote the locally soluble radical of $H$ assuming such exists. If $H \leq K$ and both $H$ and $K$ are almost locally soluble then we have $H \cap R(K) \leq R(H)$ and hence $|H : R(H)| \leq |K : R(K)|$, while $|H : R(H)| = |K : R(K)|$ implies $R(H) \leq R(K)$. 


Assume first that there is no bound for the indices $|H:R(H)|$ as $H$ ranges over all finitely generated subgroups of $G$. If $A, B$ are two such subgroups satisfying $|A : R(A)| < |B : R(B)|$ then, by the above remarks, $|A : R(A)| < |B : R(A)|$. We may therefore construct an infinite ascending chain $X_1 < X_2 < \ldots$ of finitely generated subgroups such that, for each $i$, $|X_i : R(X_i)| < |X_{i+1} : R(X_{i+1})|$. The union $X$ of these subgroups is countable and so $|X : R(X)| = m$, say, is finite. But then $|X_i : R(X_i)| \leq m$ for all $i$, a contradiction which implies that there is a largest integer $n$ such that $|H : R(H)| = n$ for some finitely generated subgroup $H$ of $G$.

Let $\mathcal{L} = \{H \leq G | |H : R(H)| = n\}$. Again from our remarks, $\mathcal{L}$ forms a local system for $G$ and if $H, K \in \mathcal{L}$ are such that $H \leq K$ then $R(H) \leq R(K)$. It follows that $J = \langle R(H) | H \in \mathcal{L} \rangle$ is a normal locally soluble subgroup of $G$. If $g_1, \ldots, g_{n+1}$ are arbitrary elements of $G$ then there exists an $H \in \mathcal{L}$ such that $g_i \in H$ for $i = 1, \ldots, n + 1$ and then $g_i = g_j \mod R(H)$ for some distinct $i, j$. Thus $|G : J| = n$ and the lemma is proved.

Our next result is a special case of the main theorem of [4]; we present a proof based on our previous discussion. It is clear that the theorem quoted in our Introduction follows from this result in conjunction with Theorem 3.4.

**Theorem 3.6 (Černikov [4]).** Suppose that $G$ is a locally (soluble-by-finite) group of finite rank. Then $G$ is almost locally soluble.

**Proof.** If the result is true for all countable such $G$ then Lemma 3.5 implies the result in general. So suppose $G$ is a countable locally (soluble-by-finite) group of finite rank. The locally soluble radical $N$ of $G$ exists and $G/N$ has trivial locally soluble radical, by Lemma 3.1, and the remarks following Lemma 3.1 show that we may assume that $G$ has trivial locally soluble radical. Let $G = \bigcup_{i \geq 1} H_i$, where $H_i$ is a finitely generated soluble-by-finite group with $H_i \leq H_{i+1}$ for all $i$. If $R_i$ is the soluble radical of $H_i$ then $F_i = H_i/R_i$ is semisimple of bounded rank. Proposition 2.1 and Theorem 2.4 now show that the sequence $\{F_i\}_{i \geq 1}$ is finite and hence so is $G$. The theorem is proved.

It is not difficult to show directly that a locally (soluble-by-finite) group of finite rank is necessarily countable. We note the following corollary of our work which generalizes Merzljakov's theorem mentioned in the Introduction.

**Corollary 3.7.** Suppose that $G$ is a locally (soluble-by-finite) group with all abelian torsion subgroups of finite rank and all torsion free abelian subgroups of bounded rank. Then $G$ has finite rank and is almost locally soluble.
Proof. By Theorems 3.4 and 3.6 we only need to prove that all locally soluble subgroups of $G$ have finite rank, so we suppose that $H$ is such a subgroup. Let $r$ be the bound on the ranks of the torsion free abelian subgroups of $G$. We may assume that $H$ is countable since the property of having finite rank is countably recognizable.

Therefore, let $H = \bigcup_{i \geq 1} X_i$, where $X_i \leq X_{i+1}$ and $X_i$ is a finitely generated soluble group, for all $i$. Suppose $T_i$ is the maximal normal torsion subgroup of $X_i$. Then $T = \prod_{i \geq 1} T_i$ is a locally finite group and by Sunkov’s theorem $T$ has finite rank, $s$ say. Thus each of the groups $T_i$ has finite rank at most $s$. Then, by [12, Theorem 8], $X_i$ has finite rank, $k_i$, say.

It now follows from [12, Theorem 10] that for each $i$, $X_i/T_i$ has a series of length at most a function of $r$ only each of whose factors has rank a function of $r$ only. Hence $X_i/T_i$ has rank bounded by a function $g(r)$ of $r$ only, for each $i$, and $X_i$ has rank bounded by $g(r) + s$, which is a function of $r$ and $s$ only. Therefore $H = \bigcup_{i \geq 1} X_i$ has rank at most $g(r) + s$, as required. □

This result shows that the obstructions to proving rank theorems for locally soluble groups in general lie with the torsion free subgroups. Note here that we cannot replace the hypotheses with all abelian subgroups of finite rank since Merzljakov [8] has constructed a locally soluble group with all abelian subgroups torsion free but of unbounded rank.

An easy consequence of Černikov’s main theorem is that a locally (residually finite) group of finite rank is almost locally soluble. This provides a generalization of the theorem of Lubotzky and Mann [7] concerning residually finite groups of finite rank. There seems to be no way of generalizing the Lubotzky–Mann theorem further along the lines of this paper. The abelian subgroups of a free group freely generated by at least two elements have bounded rank but such a free group is neither of finite rank nor almost locally soluble.

We note that the class of locally (soluble-by-finite) groups with all locally soluble subgroups of finite rank coincides with the class of locally (radical-by-finite) groups with all locally soluble subgroups of finite rank. For, if $H$ is a finitely generated radical-by-finite group with all abelian subgroups of finite rank and $K < H$ is a radical subgroup with $H/K$ finite then $K$ has finite rank, by the theorem of Baer and Heineken [1]. Hence $K$ is locally soluble and finitely generated so $H$ is actually soluble-by-finite.

Finally, we remark that Robinson [12] studies the class of generalized radical groups; this is the class of groups with an ascending series all of whose factors are finite or locally nilpotent. Černikov’s theorem and the Baer–Heineken theorem now show among other things that the class of locally (soluble-by-finite) groups of finite rank is precisely the class of generalized radical groups of finite rank.
REFERENCES


