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A new asymptotic approximate model for the Vlasov-Maxwell equations

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Abstract

In this paper, we derive a new asymptotic approximation of the Vlasov-Maxwell equations. This formulation follows the beam in a speed-of-light frame. It is fourth order accurate in the small characteristic velocity of the beam. The formulation is simpler than standard particle-in-cell methods in the lab frame or in the beam frame. It promises to be very powerful in its ability to get an accurate, but fast and easy to implement algorithm.

Keywords: Vlasov-Maxwell, asmyptotic analysis, paraxial model

1. Introduction

Charged particle beams and plasma physics problems are extensively used in Science and Technology (see for instance [1]). If we consider collisionless plasma or non-collisional beams, one of the most complete mathematical models is the time-dependent Vlasov-Maxwell system of equations (cf. [2]). However, the numerical solution of such a model, which is unavoidable in many situations, [3, 4] requires a large computational effort. Therefore, whenever possible, we have to take into account the particularities of the physical problem to derive asymptotic approximate models leading to cheaper simulations (see [5, 6, 7, 8, 9]).

In this work, we consider the case of high energy short beams. A typical example is the transport of a bunch of highly relativistic charged particles in the interior of a perfectly conducting hollow tube. This is usually modeled by the Vlasov equation, where the unknown $f(\mathbf{X}, \mathbf{V}, t)$ represents the distribution function of particles in phase space $(\mathbf{X}, \mathbf{V})$. It is generally coupled with the Poisson or Maxwell equations. Numerical simulations are mostly performed using the particle-in-cell (PIC) method.

In this talk, we will derive a new asymptotic paraxial model as an approximation of the time-dependent Vlasov-Maxwell equations. Following [6], the model is derived by introducing a frame which moves along the optical axis at the speed of light, so that the bunch of particles is evolving slowly in this frame. Then one considers a scaling of the

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equations which reflects the characteristics of the high energy short beam. Finally, we introduce a small parameter and we use asymptotic expansion techniques to obtain a new paraxial model which is accurate up to fourth order.

The simplicity of the so obtained formulation allows to use a finite-difference discretization for the Maxwell equations. Hence, using a particle approximation for the Vlasov equation, a particle-in-cell technique can be easily developed. This approach promises to be very powerful in its ability to get an accurate, but fast and easy to implement algorithm.

2. The Vlasov Maxwell model

Let us consider a beam of charged particles with mass m and charge q which moves in a perfectly conducting cylindrical tube. We define the axis of the tube as the z-axis. We denote by Ω the transverse section of the tube, by Γ its boundary and by $\nu = (\nu_x, \nu_y, 0)$ the unit exterior normal to the tube. We suppose that an external magnetic field $\mathbf{B}^e$ confines the beam in a neighborhood of the z-axis which may be therefore chosen as the optical axis of the beam. We denote by $\mathbf{x}=(x,y,z)$ the position of the particle, by $\mathbf{p}=(p_x,p_y,p_z)$ its momentum and by $\mathbf{v}=(v_x,v_y,v_z)$ its velocity. We assume that the beam is relativistic and noncollisional so that its distribution function f=$f(\mathbf{x},\mathbf{p},t)$ in the phase space $(\mathbf{x},\mathbf{p})$ is a solution of the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_x f + \mathbf{F}.\nabla_p f = 0, \qquad \text{where } \mathbf{p} = \gamma m \mathbf{v},\ \gamma = (1 - \frac{\upsilon^2}{c^2})^{-1/2},\ \upsilon = |\,\mathbf{v}\,|\,. \tag{1}$$

Above $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ denotes the electromagnetic force acting on the particles, whereas the electric field $\mathbf{E}=\mathbf{E}(\mathbf{x},t)$ and the magnetic field $\mathbf{B}=\mathbf{B}(\mathbf{x},t)$ satisfy Maxwell's equations

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{3}$$

$$\nabla.\mathbf{E} = \frac{1}{\varepsilon_0}\rho, \tag{4}$$

$$\nabla.\mathbf{B} = 0. \tag{5}$$

where the charge density ρ and the current density $\mathbf{J}$ are obtained from the distribution function f by

$$\rho = q\int f d\mathbf{p}, \qquad \mathbf{J} = q\int \mathbf{v} f d\mathbf{p}. \tag{6}$$

In the sequel, it will appear more convenient for the analysis to work in the position-velocity phase space $(\mathbf{x},\mathbf{v})$. Denoting again by f=$f(\mathbf{x},\mathbf{v},t)$ the distribution function of the particles, we obtain that f is solution to the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_x f + \nabla_{\mathbf{v}} \cdot (\frac{1}{\gamma m}(\mathbf{I} - \frac{1}{c^2}\mathbf{v} \otimes \mathbf{v})\,\mathbf{F} f) = 0, \tag{7}$$

where $\mathbf{I}$ denotes the unit tensor. From now on, assume that the beam is highly relativistic i.e., satisfies $\gamma >> 1$. Since $\upsilon_z \simeq c$ for any particle in the beam, it is convenient to rewrite the Vlasov-Maxwell equations in the beam frame, i.e., in a frame which moves along z-axis with the light velocity c. Hence we set $\zeta = ct - z$, $\upsilon_\zeta = c - \upsilon_Z$ and perform the change of variables $(x, y, z, v_x, v_y, v_z, t) \to (x, y, \zeta, v_x, v_y, v_\zeta, t)$. Next, following [9], [10], we introduce the notations:

$$\mathbf{x}_\perp = (x, y),\ \mathbf{v}_\perp = (v_x, v_y)$$

$$\mathbf{grad}_\perp \varphi = (\tfrac{\partial\varphi}{\partial x}, \tfrac{\partial\varphi}{\partial y}),\ \mathbf{curl}_\perp \varphi = (\tfrac{\partial\varphi}{\partial y}, -\tfrac{\partial\varphi}{\partial x}),\ \Delta_\perp \varphi = \tfrac{\partial^2\varphi}{\partial x^2} + \tfrac{\partial^2\varphi}{\partial y^2}$$

$$div_\perp \mathbf{A}_\perp = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y},\ curl_\perp \mathbf{A}_\perp = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y},\ div_{\mathbf{v}_\perp} \mathbf{A}_\perp = \frac{\partial A_x}{\partial v_x} + \frac{\partial A_y}{\partial v_y},$$

where $\varphi = \varphi(x,y)$ is a scalar function and $\mathbf{A}_\perp = (A_x, A_y)$ a transverse vector field. If we define $\mathbf{A}_\perp \times \mathbf{e}_z$ by $\mathbf{A}_\perp \times \mathbf{e}_z = (A_y, -A_x)$, we observe that

$$div_\perp(\mathbf{A}_\perp \times \mathbf{e}_z) = curl_\perp \mathbf{A}_\perp,\ curl_\perp(\mathbf{A}_\perp \times \mathbf{e}_z) = -div_\perp \mathbf{A}_\perp,\ curl_\perp \mathbf{curl}_\perp \varphi = -\Delta_\perp \varphi. \tag{8}$$

Moreover, if we denote by $\tau = (-\nu_y, \nu_x)$ the unit tangent along Γ, we have the relation $\mathbf{curl}_\perp \varphi \cdot \tau = -\frac{\partial \varphi}{\partial \nu}$. Using the above notation, the Vlasov equation in the beam frame can be written as

$$\frac{\partial f}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f + v_\zeta \frac{\partial f}{\partial \zeta} + div_{\mathbf{v}_\perp}[\frac{1}{\gamma m}((\mathbf{I} - \frac{1}{c^2}\mathbf{v}_\perp \otimes \mathbf{v}_\perp) \cdot \mathbf{F}_\perp - \frac{1}{c}(1 - \frac{v_\zeta}{c})\mathbf{v}_\perp F_z) f] +$$
$$\frac{\partial}{\partial v_\zeta}[\frac{1}{\gamma mc}((1 - \frac{v_\zeta}{c})\mathbf{v}_\perp \cdot \mathbf{F}_\perp + (2 - \frac{v_\zeta}{c}) v_\zeta F_z) f) = 0\,, \tag{9}$$

where

$$\gamma = (2\frac{v_\zeta}{c} - \frac{1}{c^2}({v_\perp}^2 + {v_\zeta}^2))^{-1/2}, v_\perp = |\mathbf{v}_\perp|. \tag{10}$$

Setting $\mathcal{E}_\perp = (\mathcal{E}_x = E_x - cB_y, \mathcal{E}_y = E_y + cB_x)$ and $J_\zeta = \rho c - J_z = q \int \upsilon_\zeta f d\mathbf{v}$, the Ampere and Poisson equations (2) and (4) give

$$\frac{1}{c^2}\frac{\partial \mathbf{E}_\perp}{\partial t} - \mathbf{curl}_\perp B_z + \frac{1}{c}\frac{\partial \mathcal{E}_\perp}{\partial \zeta} = -\mu_0 \mathbf{J}_\perp, \tag{11}$$

$$\frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{1}{c} div_\perp \mathcal{E}_\perp = -\mu_0 J_\zeta, \tag{12}$$

$$div_\perp \mathbf{E}_\perp - \frac{\partial E_z}{\partial \zeta} = \frac{1}{\epsilon_0}\rho. \tag{13}$$

Similarly, equations (3) and (5) written in the beam frame become

$$\frac{\partial B_\perp}{\partial t} + \mathbf{curl}_\perp E_z + \frac{\partial}{\partial \zeta}(\mathcal{E}_\perp \times \mathbf{e}_z) = 0, \tag{14}$$

$$\frac{\partial B_z}{\partial t} + curl_\perp \mathcal{E}_\perp = 0, \tag{15}$$

$$div_\perp \mathbf{B}_\perp - \frac{\partial B_z}{\partial \zeta} = 0. \tag{16}$$

Finally, the electromagnetic force becomes

$$\mathbf{F}_\perp = q(\mathcal{E}_\perp + (\mathbf{v}_\perp \times \mathbf{e}_z) B_z + v_\zeta(\mathbf{B}_\perp \times \mathbf{e}_z)), \quad F_z = q(E_z + \mathbf{v}_\perp \cdot (\mathbf{B}_\perp \times \mathbf{e}_z)). \tag{17}$$

The treatment of the boundary conditions can be handled in the same way. We refer the reader to [9], [10] for details.

3. A scaling of the equations

At this point, we restrict ourselves to the case of a short beam. Then, one introduces a scaling of the above equations, taking in account the three following properties of the beam:

(i.) the beam dimensions are small compared to the longitudinal length of the device ;
(ii.) the longitudinal velocities v_z of the particles are close to the light velocity c ;
(iii.) the transverse velocities $\mathbf{v}_\perp$ of the particles are small compared to c.

Thus, we introduce two characteristic quantities:
l, the characteristic dimension of the beam,
$\bar{v}$, the transverse characteristic velocity of the particles.

Now, we define a small parameter η with $\eta = \frac{v}{c} \ll 1$. Since the particle velocities are close to c, we have $v^2 = v_\perp^2 + v_z^2 = v_\perp^2 + c^2 - 2cv_\zeta + v_\zeta^2 \simeq c^2$ so that $2cv_\zeta \simeq v_\perp^2$. Hence, v_ζ appears to be of the order $\frac{\overline{v}^2}{c}$ and we choose $\overline{w} = \eta^2 c$ as a characteristic longitudinal velocity of the particles in the beam frame and consequently $\overline{w} = \eta\overline{v}$. We define $\overline{l_\parallel} = \eta l$ as a characteristic longitudinal dimension. Finally one takes $T = \frac{l}{\overline{v}}$ as a characteristic time. We introduce the dimensionless independent variables $\mathbf{x}'_\perp, \zeta', t', \ \mathbf{v}'_\perp, v'_\zeta$ defined by

$$x = lx', y = ly', \zeta = \overline{l_\parallel}\zeta', t = Tt', v_x = \overline{v}v'_x, v_y = \overline{v}v'_y, v_\zeta = \overline{w}v'_\zeta. \tag{18}$$

Remark 1. *We have seen that it was "natural" to define a longitudinal characteristic velocity $\overline{w}$ related to the transverse characteristic velocity $\overline{v}$ by $\overline{w} = \eta\overline{v}$. In the same way, it is "natural" to introduce a longitudinal characteristic dimension $\overline{l_\parallel}$ different from the transverse characteristic dimension l, and satisfying the relation $\overline{l_\parallel} = \eta l$*

Next, choosing the scaling factors $\overline{f}, \overline{\rho}, \overline{J}, \overline{E}, \overline{B}, \overline{F}$ as

$$\left\{ \begin{array}{lll} \overline{f} = \dfrac{\varepsilon_0 m}{q^2 l^2 \overline{\omega}}, & \overline{\rho} = q\overline{f}\overline{v}^2\overline{\omega}, & \overline{J} = \overline{\rho}c, \\ \overline{E} = \dfrac{m\overline{v}^2}{ql}, & \overline{B} = \dfrac{\overline{E}}{c}, & \overline{F} = q\overline{E}, \end{array} \right. \tag{19}$$

we look for the dependent variables f, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{F}$ as functions of the following forms: $f(\mathbf{x}_\perp, \zeta, \mathbf{v}_\perp, v_\zeta, t) = \overline{f}f'(\mathbf{x}'_\perp, \zeta', \mathbf{v}'_\perp, v'_\zeta, t)$, $\mathbf{E}(\mathbf{x}_\perp, \zeta, t) = \overline{E}\mathbf{E}'(\mathbf{x}'_\perp, \zeta', t')'$, $\mathbf{B}(\mathbf{x}_\perp, \zeta, t) = \overline{B}\mathbf{B}'(\mathbf{x}'_\perp, \zeta', t')'$, $\mathbf{F}(\mathbf{x}_\perp, \zeta, \mathbf{v}_\perp, v_\zeta, t) = \overline{F}\mathbf{F}'(\mathbf{x}'_\perp, \zeta', \mathbf{v}'_\perp, v'_\zeta, t)$. Note that $\rho = \overline{\rho}n'$, with $n' = \int f' dv'$, $v' = (\mathbf{v}'_\perp, v_\zeta)$ and $\mathbf{J}_\perp = \eta\overline{J}\mathbf{j}'_\perp, J_\zeta = \eta^2\overline{J}j'_\zeta$.

We are able now to write again the Vlasov-Maxwell equations in the beam frame using these dimensionless variables. Dropping the primes for simplicity, we get

Proposition 2. *The Vlasov equation in dimensionless variables is*

$$\begin{aligned} &\frac{\partial f}{\partial t} + \boldsymbol{v}_\perp \cdot \boldsymbol{grad}_\perp f + v_\zeta \frac{\partial f}{\partial \zeta} + div_{\boldsymbol{v}_\perp}[\frac{1}{\gamma}((\boldsymbol{I} - \eta^2 \boldsymbol{v}_\perp \otimes \boldsymbol{v}_\eta) \cdot \boldsymbol{F}_\perp - \eta(1 - \eta^2 v_\zeta)\boldsymbol{v}_\perp F_z) f] + \\ &\frac{\partial}{\partial v_\zeta}[\frac{1}{\gamma}((1 - \eta^2 v_\zeta)\boldsymbol{v}_\perp \cdot \boldsymbol{F}_\perp + \eta(2 - \eta^2 v_\zeta) v_\zeta F_z) f] = 0\,, \end{aligned} \tag{20}$$

where, with (10), we have $\gamma = \frac{1}{\eta}(2v_\zeta - (v_\perp^2 + \eta^2 v_\zeta^2))^{-\frac{1}{2}}$.

On the other hand, Maxwell's equations (11)- (13) give

$$\eta\frac{\partial \mathbf{E}_\perp}{\partial t} - \mathbf{curl}_\perp B_z + \frac{1}{\eta}\frac{\partial \mathcal{E}_\perp}{\partial \zeta} = -\eta\mathbf{j}_\perp, \tag{21}$$

$$\eta\frac{\partial E_z}{\partial t} + div_\perp \mathcal{E}_\perp = \eta^2 j_\zeta, \tag{22}$$

$$div_\perp \mathbf{E}_\perp - \frac{1}{\eta}\frac{\partial E_z}{\partial \zeta} = n, \tag{23}$$

and equations (14)- (13) also give

$$\eta\frac{\partial \mathbf{B}_\perp}{\partial t} + \mathbf{curl}_\perp E_z + \frac{1}{\eta}\frac{\partial}{\partial \zeta}(\mathcal{E}_\perp \times \mathbf{e}_z) = 0, \tag{24}$$

$$\eta\frac{\partial B_z}{\partial t} + curl_\perp \mathcal{E}_\perp = 0, \tag{25}$$

$$div_\perp \mathbf{B}_\perp - \frac{1}{\eta}\frac{\partial B_z}{\partial \zeta} = 0. \tag{26}$$

In the above equations, the right-hand sides n, j fulfill the charge conservation equation

$$\frac{\partial n}{\partial t} + div_{\perp}\mathbf{j}_{\perp} + \frac{\partial j_{\zeta}}{\partial \zeta} = 0\,.$$

The electromagnetic force F is the same as in [9] or [10]

$$\begin{cases} \mathbf{F}_{\perp} = \mathcal{E}_{\perp} + \eta(\mathbf{v}_{\perp} \times \mathbf{e}_z)B_z + \eta^2 v_{\zeta}(\mathbf{B}_{\perp} \times \mathbf{e}_z), \\ F_z = E_z + \eta(v_x B_y - v_y B_x). \end{cases} \tag{27}$$

The boundary conditions can be handled in the same way.

4. An asymptotic expansion

Starting from the scaled Vlasov-Maxwell equations, we develop asymptotic expansions of f, n, $\mathbf{j}$ and $\mathbf{E}$, $\mathbf{B}$, $\mathcal{E}_{\perp}$, $\mathbf{F}$ in powers of the small parameter η:

$$f = f^0 + \eta f^1 + ..., n = n^0 + \eta n^1 + ..., \mathbf{j} = \mathbf{j}^0 + \eta \mathbf{j}^1 + ...,$$

$$\mathbf{E} = \mathbf{E}^0 + \eta\mathbf{E}^1 + ..., \mathbf{B} = \mathbf{B}^0 + \eta\mathbf{B}^1 + ..., \mathcal{E} = \mathcal{E}^0 + \eta\mathcal{E}^1 + ..., \mathbf{F} = \mathbf{F}^0 + \eta\mathbf{F}^1 + ... \,.$$

The principle of the study consists in replacing formally in the Vlasov-Maxwell equations the functions by their asymptotic expansions; then to identify the coefficients of η^0, η^1 .... Hence, it is proved that for determining the asymptotic expansion $f = f^0 + \eta f^1 + \eta^2 f^2 + \eta^3 f^3$ of the distribution function f up to the order 3 in η, it is enough to know the expansion $\mathbf{F}_{\perp} = \mathbf{F}_{\perp}^0 + \eta\mathbf{F}_{\perp}^1 + \eta^2\mathbf{F}_{\perp}^2$ of the transverse electromagnetic force $\mathbf{F}_{\perp}$ up to the order 2 and the expansion $F_z^0 + \eta F_z^1$ of the longitudinal electromagnetic force F_z up to the order 1 only. Then, using the expressions (27) of the forces, we have

$$\begin{cases} \mathbf{F}_{\perp}^0 = \mathcal{E}_{\perp}^0 \\ F_z^0 = E_z^0 \end{cases} \quad \begin{cases} \mathbf{F}_{\perp}^1 = \mathcal{E}_{\perp}^1 + (\mathbf{v}_{\perp} \times \mathbf{e}_z)B_z^0 \\ F_z^1 = E_z^1 + (v_x B_y^0 - v_y B_x^0) \end{cases} \quad \mathbf{F}_{\perp}^2 = \mathcal{E}_{\perp}^2 + (\mathbf{v}_{\perp} \times \mathbf{e}_z)B_z^1 + v_{\zeta}(\mathbf{B}_{\perp}^0 \times \mathbf{e}_z) \tag{28}$$

Hence, it appears that we need only to know the expansion of $\mathcal{E}_{\perp}$ up to the order 2, the principal part $\mathbf{B}_{\perp}^0$ of $\mathbf{B}_{\perp}$, and the expansions of E_z and B_z up to the order 1.

Next we turn to asymptotic expansions of Maxwell's equations and their boundary conditions. We get the following lemma

Lemma 3. *The transverse component $\mathbf{E}_{\perp}^0$ of $\mathbf{E}^0$ is the unique solution of*

$$\begin{cases} div_{\perp}\boldsymbol{E}_{\perp}^0 & = & n^0 + \dfrac{\partial E_z^1}{\partial \zeta} \ in\ \Omega \\ curl_{\perp}\boldsymbol{E}_{\perp}^0 & = & -\dfrac{\partial B_z^1}{\partial \zeta} \quad in\ \Omega \\ \boldsymbol{E}_{\perp}^0 \cdot \tau & = & 0 \qquad on\ \Gamma \end{cases} \tag{29}$$

On the other hand, the transverse component $\mathbf{B}_{\perp}^0$ of $\mathbf{B}^0$ is characterized by

$$\begin{cases} div_{\perp}\boldsymbol{B}_{\perp}^0 & = & \dfrac{\partial B_z^1}{\partial \zeta} \ in\ \Omega \\ curl_{\perp}\boldsymbol{B}_{\perp}^0 & = & n^0 + \dfrac{\partial E_z^1}{\partial \zeta} \quad in\ \Omega \\ \boldsymbol{B}_{\perp}^0 \cdot \nu & = & \boldsymbol{B}_{\perp}^0 \cdot \nu_{|\zeta=0} \qquad on\ \Gamma \end{cases} \tag{30}$$

Similarly, we can characterize the pseudo-field $\mathcal{E}_{\perp}^0$ as follows

Lemma 4. *The transverse component $\mathcal{E}^0_\perp$ of $\mathcal{E}^0$ is the unique solution of*

$$\left\{\begin{array}{lcl} div_\perp \mathcal{E}^0_\perp & = & 0 \; in\, \Omega \\ curl_\perp \mathcal{E}^0_\perp & = & 0 \quad in\, \Omega \\ \mathcal{E}^0_\perp \cdot \tau & = & \mathbf{B}^0_\perp \cdot \nu_{|\zeta=0} \quad on\, \Gamma \end{array}\right. \tag{31}$$

Remark that up to this order, one can not yet characterize the zero order longitudinal components E^0_z and B^0_z.

We then obtain the following characterization of the zero order longitudinal components E^0_z and B^0_z

Lemma 5. *The zero order longitudinal electric components satisfies*

$$E^0_z = 0$$

whereas the zero order longitudinal magnetic component is independent of ζ, i.e. $B^0_z = B^0_z(\mathbf{x}_\perp, t)$ and solves

$$\Delta_\perp B^0_z = 0$$

Lemma 6. *The first order transverse pseudo field $\mathcal{E}^1_\perp$ is characterized by*

$$\left\{\begin{array}{lcl} div_\perp \mathcal{E}^1_\perp & = & 0 \; in\, \Omega \\ curl_\perp \mathcal{E}^1_\perp & = & -\dfrac{\partial B^0_z}{\partial t} \; in\, \Omega \\ \mathcal{E}^1_\perp \cdot \tau & = & \mathbf{B}^{e,1}_\perp \cdot \nu_{|\zeta=0} \; on\, \Gamma \end{array}\right.$$

Up to now, we have determined $E^0_z, B^0_z, \mathcal{E}^0_\perp, \mathcal{E}^1_\perp$ which correspond to purely external fields.

What we have to determine now are the components $\mathbf{E}^0_\perp, \mathbf{B}^0_\perp, E^1_z, B^1_z, \mathcal{E}^2_\perp$. This is the aim of the three following lemmas.

Lemma 7. *The first order longitudinal electric component E^1_z is characterized by*

$$\left\{\begin{array}{lcl} 2\dfrac{\partial^2 E^1_z}{\partial\zeta\partial t} - \Delta_\perp E^1_z & = & -\dfrac{\partial n^0}{\partial t} + \dfrac{\partial j^0_\zeta}{\partial \zeta} \; in\, \Omega \\ E^1_z & = & 0 \; on\, \Gamma \\ E^1_z(\zeta = 0) & = & 0 \end{array}\right.$$

and satisfies the initial condition: $E^1_z(t = 0)$ is a given function.

Lemma 8. *The first order longitudinal magnetic component B^1_z is characterized by*

$$\left\{\begin{array}{lcl} 2\dfrac{\partial^2 B^1_z}{\partial\zeta\partial t} - \Delta_\perp B^1_z & = & -curl_\perp \mathbf{j}^0_\perp \; in\, \Omega \\ \dfrac{\partial B^1_z}{\partial \nu} & = & \mathbf{j}^0_\perp \cdot \tau + \dfrac{\partial \mathbf{B}^0_\perp}{\partial t} \cdot \nu_{|\zeta=0} \\ B^1_z(\zeta = 0) & & being\ a\ given\ function \end{array}\right.$$

and satisfies the initial condition: $B^1_z(t = 0)$ is a given function.

From the characterizations of E^1_z and B^1_z, one can compute $\mathbf{E}^0_\perp$ and $\mathbf{B}^0_\perp$ as solutions to the transverse boundary values problems (29) and (30) respectively.

Lemma 9. *The second order transverse pseudo field $\mathcal{E}_\perp^2$ is the unique solution to the equation*

$$\left\{\begin{array}{lll} div_\perp \mathcal{E}_\perp^2 & = & j_\zeta^0 - \dfrac{\partial E_z^1}{\partial t} \; in\ \Omega \\ curl_\perp \mathcal{E}_\perp^2 & = & -\dfrac{\partial B_z^1}{\partial t} \; in\ \Omega \\ \mathcal{E}_\perp^2 \cdot \tau & = & \mathbf{B}_\perp^2 \cdot \nu_{|\zeta=0} - \zeta \dfrac{\partial \mathbf{B}_\perp^0}{\partial t} \cdot \nu_{|\zeta=0} \end{array}\right.$$

In order to construct the paraxial model, we need to distinguish the self consistent fields from the external ones, that is

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^s, \quad \mathbf{B} = \mathbf{B}^e + \mathbf{B}^s, \quad \text{and consequently} \quad \mathcal{E} = \mathcal{E}^e + \mathcal{E}^s$$

one easily obtains the same kind of results for self consistent fields.

5. The paraxial model

Using the results of the previous section, we now want to introduce a simplified paraxial model, which gives the same asymptotic expansion of f up to the order three. It is derived in an heuristic way based on the above asymptotic analysis. We will write this model in the beam frame coming back in the physical variables. First note that

$$\mathcal{E}_\perp^s << \mathbf{E}_\perp^s, c\,\mathbf{B}_\perp^s$$

that is, the beam fields are largely greater than the "electromagnetic force" when the beam is ultrarelativistic. Indeed, $\mathcal{E}_\perp^{1,s}, \mathcal{E}_\perp^{2,s}$ are equal to zero, whereas $\mathcal{E}_\perp^{2,s}$ is the first non-vanishing pseudo field. At the contrary, the electric and magnetic fields $\mathbf{E}_\perp^{0,s}, \mathbf{B}_\perp^{0,s}$ are different from zero and depends on the charge density ρ (or n^0 in dimensionless variables). In these conditions, we have $\mathbf{E}_\perp^s - c\,\mathbf{B}_\perp^s \times \mathbf{e}_z \simeq 0$. We deduce with (8) that

$$\begin{array}{lll} div_\perp \mathcal{E}_\perp^s & = & div_\perp \mathbf{E}_\perp^s - c\, curl_\perp \mathbf{B}_\perp^s \simeq 0 \\ curl_\perp \mathcal{E}_\perp^s & = & curl_\perp \mathbf{E}_\perp^s + c\, div_\perp \mathbf{B}_\perp^s \simeq 0 \end{array}$$

We begin to derive the equations satisfied by $\mathbf{E}_\perp^s$. From the equations (29-30), we have

$$div_\perp \mathbf{E}_\perp^s - \frac{\partial E_z^s}{\partial \zeta} = \frac{1}{\varepsilon_0}\rho, \qquad div_\perp \mathbf{B}_\perp^s - \frac{\partial B_z^s}{\partial \zeta} = 0\,.$$

Now, using that $div_\perp \mathbf{B}_\perp^s \simeq -\frac{1}{c} curl_\perp \mathbf{E}_\perp^s$, we then get

$$\left\{\begin{array}{lll} div_\perp \mathbf{E}_\perp^s & = & \dfrac{1}{\varepsilon_0}\rho + \dfrac{\partial E_z^s}{\partial \zeta} \text{ in } \Omega \\ curl_\perp \mathbf{E}_\perp^s & = & -c\dfrac{\partial B_z^s}{\partial \zeta} \text{ in } \Omega \\ \mathbf{E}_\perp \cdot \tau & = & 0 \text{ on } \Gamma\,. \end{array}\right. \tag{32}$$

In another way, we have $\mathbf{B}_\perp^s = -\mathbf{E}_\perp^s \times \mathbf{e}_z$. Hence, we have obtained a quasi-static model for the transverse self consistent fields $\mathbf{E}_\perp^s$ and $\mathbf{B}_\perp^s$. Practically, these fields will be computed after that the longitudinal components E_z^s, B_z^s have been determined.

Let us now deal with these longitudinal self-consistent fields. In this paraxial model, E_z^s is solution to a second order wave-like equation:

$$\left\{\begin{array}{lll} 2\,\dfrac{\partial^2 E_z^s}{\partial \zeta \partial t} - c\,\Delta_\perp E_z^s & = & \dfrac{1}{\varepsilon_0}(-\dfrac{\partial \rho}{\partial t} + \dfrac{\partial J_\zeta}{\partial \zeta})\,, \text{ in } \Omega \\ E_z^s = 0 & & \text{on } \Gamma \\ E^s_{z_{|\zeta=0}} & & \text{a given data} \end{array}\right. \tag{33}$$

with the initial condition: $E^s_{z_{|t=0}}$ being a given data.

Similarly, the longitudinal magnetic part B^s_z solved the following system

$$\begin{cases} 2\,\dfrac{\partial^2 B^s_z}{\partial\zeta\partial t} - c\,\Delta_\perp B^s_z &= \mu_0\, c\, curl_\perp \mathbf{J}_\perp\,,\ \text{in}\ \Omega \\[2ex] \dfrac{\partial B^s_z}{\partial \nu} &= \mu_0 \mathbf{J}_\perp \cdot \tau\ \text{on}\ \Gamma \\[2ex] B^s_{z_{|\zeta=0}} 0 = & 0 \end{cases} \tag{34}$$

with the initial condition: $B^s_{z_{|t=0}}$ being a given data. This allows to compute the longitudinal fields E^s_z, B^s_z then the transverse ones $\mathbf{E}^s_\perp, \mathbf{B}^s_\perp$.

From these quantities, one can also determine the transverse pseudo-field $\mathcal{E}^s_\perp$, by solving the quasi-static system of equations

$$\begin{cases} div_\perp \mathcal{E}^s_\perp &= \mu_0\, c\, J_\zeta - \dfrac{1}{c}\dfrac{\partial E^s_z}{\partial t}\ \text{in}\ \Omega \\[2ex] curl_\perp \mathcal{E}^s_\perp &= -\dfrac{\partial B^s_z}{\partial t}\ \text{in}\ \Omega \\[2ex] \mathcal{E}^s_\perp \cdot \tau &= 0\ \text{on}\ \Gamma \end{cases} \tag{35}$$

As a conclusion, we have derived this model by only assuming that $\mathcal{E}^s_\perp << \mathbf{E}^s_\perp$ and $\mathcal{E}^s_\perp << c\,\mathbf{B}^s_\perp$ and we have

Theorem 10. *The equations (32-35) determine the triple* $(\mathbf{E}^s, \mathbf{B}^s, \mathcal{E}^s_\perp)$ *from* $(\rho, \mathbf{J})$ *in a unique way.*

6. Conclusion

In this paper, a new asymptotic approximate model for the Vlasov-Maxwell equations has been described. It has been constructed from a paraxial approximation of the system of equations, and is adapted to highly relativistic beam. The simplicity of the obtained formulation would allow us to use very simple numerical schemes (like finite-difference discretization and particle-in-cell technique). This approach would be very powerful in its ability to get accurate, but fast and easy to implement algorithm. A more complete study with numerical examples is actually in progress.

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