Some Aspects of the Development of Linear Algebra in the Last Sixty Years*

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ABSTRACT

This after-dinner talk is mainly a brief discussion of interactions between linear algebra and some other mathematical branches.

This is not a history of the development of linear algebra. What I intend to do is to present my personal view of some aspects of its development during the last sixty years.

Sixty years ago, four-dimensional space was a subject of science fiction. It was also a fascinating topic in many popular writings on relativity theory. In widely used textbooks (e.g., [14, 24]), the name "linear algebra" had not yet made its debut. Linear transformations were treated as changes of variables without mentioning the structure of a vector space. In fact, the word "vector" cannot be found in the subject index of [14] or [24]. The first modern texts on linear algebra, such as Schreier and Sperner [59, 60] and van der Waerden [64], were published about sixty year ago, but in the beginning were not widely distributed outside Germany. Although Peano's axioms for vector spaces were given in his 1888 book [51], one had to wait until 1923–32 for these axioms to appear in books such as Weyl [65, 66], van der Waerden [64], Banach [12], and von Neumann [48].

As late as 1931, Weyl wrote in the introduction of his book [66]: "It is somewhat distressing that the theory of linear algebras must again and again

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be developed from the beginning, for the fundamental concepts of this branch of mathematics crop up everywhere in mathematics and physics, and a knowledge of them should be as widely disseminated as the elements of differential calculus."

Today, linear algebra crops up in various branches of mathematics, physics, engineering, statistics, operations research, economics, etc. Problems in these fields often lead to new problems and significant results in linear algebra. The modern development of linear algebra has not only greatly benefited by its manifold interactions with other mathematical and nonmathematical fields, but it has also deeply influenced the development of these fields. It is perhaps not out of place to quote from a second-hand source [23, p. 426] a statement by Stanislaw Jerzy Lec: "Ideas hop like fleas from one human being onto the other. But they do not bite all of them." In the following lines, I shall discuss this hopping of ideas in the form of interactions between linear algebra and some other mathematical branches.

COMBINATORIAL MATHEMATICS AND MATRIX THEORY

The area of intersection of linear algebra and combinatorial mathematics has now become a well developed discipline known as combinatorial matrix theory [18]. One direction of this theory is the study of block designs, (0, 1) matrices, latin squares, Hadamard matrices, etc. [33, 57]. Another direction is the interplay between linear algebra and graph theory, in particular the study of graph spectra [20–22] and the graph-theoretical methods for studying inclusion regions and estimates for the eigenvalues of matrices [17].

Closely related to graph theory is the theory of matroids [1, 19], which was originally introduced in the 1930s by G. Birkhoff, Mac Lane, and Whitney to generalize basic concepts such as linear dependence, span, and basis in linear algebra. Now a large part of combinatorial mathematics (e.g., graph theory, combinatorial lattice theory, transversal theory) can be unified into the realm of the theory of matroids, which has its roots in linear algebra.

FROM LINEAR ALGEBRA TO CONVEX AND NONLINEAR ANALYSIS

The far-reaching influence of linear algebra on linear functional analysis is well known. In nonlinear analysis, Lusternik and Schnirelmann's theory of category and critical points [42, 41] was inspired by Fischer's minimax characterization of the eigenvalues of Hermitian matrices (after Courant's generalization to linear analysis). A main result in game theory and mathematical economics is von Neumann's minimax theorem, which is a theorem in linear algebra concerning bilinear forms. It was von Neumann's ingenious proof of this theorem [47, 50] that started the fixed-point theorems for set-valued mappings. These fixed-point theorems and general minimax theorems (no longer restricted to bilinear forms) have now become part of nonlinear analysis. They have wide applications in various branches of mathematical analysis (e.g., potential theory, partial differential equations, monotone operators, variational inequalities, optimization problems), game theory, and mathematical economics [7–9], [23, 26, 35, 69]. The theory of games has stimulated interest in systems of linear inequalities [40] and linear programming, which should be regarded as a chapter of linear algebra. Nonlinear programming [10, 44] and convex analysis [55, 27, 13] are only natural extensions in this development. On the other hand, work on matrix computation for problems in linear and nonlinear programming has led to the research on computational methods for approximation of fixed points and economic equilibria [58, 29].

LINEAR ALGEBRA AND GROUP REPRESENTATIONS

In noncommutative harmonic analysis, two main problems are the determination, up to equivalence, of all irreducible linear representations of a locally compact group, and the decomposition of a given representation into irreducible ones [34, 46]. The theory of finite-dimensional representations and the geometry of classical groups [67, 25] may be considered as a branch of linear algebra, which is substantially enriched by the additional ingredients from group theory. This is particularly true for compact groups, for every irreducible representation of any compact group is finite-dimensional. Since all continuous complex-valued functions on any compact group are almost periodic, an essentially algebraic treatment (involving very little topological consideration) of representations of compact groups is possible, as von Neumann's theory of almost periodic functions [49, 43] has made clear.

For many important groups, finite-dimensional irreducible unitary representations have been ingeniously constructed. The detailed calculations of matrix elements of these representations have also made it possible to use representation theory to unify the theory of important classes of special functions such as the polynomials of Legendre, Jacobi, Laguerre, Hermite, and Tchebycheff, Bessel functions, gamma functions, and hypergeometric functions. Thus, the modern theory of special functions [63] is closely related to linear algebra.

It should also be mentioned that a substantial part of Lie algebras can be treated purely algebraically as a branch of linear algebra [37].

INTERACTIONS WITH MATHEMATICAL PHYSICS

It is well known that the theory of group representations plays an important role in many parts of mathematics as well as in quantum mechanics and in elementary particle physics. The special functions mentioned above are also basic tools of mathematical physics.

Many inequalities for eigenvalues or singular values of compact operators in Hilbert spaces are natural extensions of finite-dimensional results in linear algebra. These inequalities have also become useful tools in mathematical physics [2, 30, 52, 61].

Another important area of linear algebra, the geometry of indefinite inner product spaces and the spectral theory of operators on these spaces, has its origin in Dirac's work on quantum theory. The mathematical foundation was laid by Pontrjagin, M. G. Krein and his school. Now there is a comprehensive theory of indefinite inner product spaces [3, 11, 16, 32, 36, 68]. Many important applications have been made in differential equations, spectral theory of polynomial operator pencils, mechanics, scattering theory, quantum field theory, systems theory, etc.

Investigations of vibrations of mechanical systems in the nineteenth century have directly led to certain area of classical mathematical analysis (e.g., the works of Sturm, Liouville, Routh, Hurwitz, Liapunov). Motivated by problems on small vibrations, F. R. Gantmacher and M. G. Krein [28] developed the theory of totally positive matrices, or more generally oscillatory matrices, beginning in 1935. This beautiful matrix analysis has applications not only in mechanics and differential equations, but also in interpolation by spline functions, stochastic processes, and statistical decision theory [38].

RELATIONSHIP WITH COMPLEX ANALYSIS

The intersection of linear algebra (or more generally, operator theory) and complex analysis is an extremely fertile ground for interactions, and continues to expand with rapid advances. Interplay with complex analysis brings powerful techniques to operator theory and increases its depth. The earliest appearance of complex analysis in operator theory is in functional calculus. Now the theory of operator-valued analytic functions of a complex variable is also well developed [6, 39, 53, 54, 56, 62]. Many classical results on bounded analytic functions on the open unit disc or half plane have inspired generalizations or analogous results for operator-valued analytic functions (in particular, results on interpolation, factorization, integral representation, etc.). Thus the familiar names in classical complex analysis—Schwarz, Pick, Fatou, Carathéodory, Hardy, Blaschke, and Nevanlinna—appear also in the contemporary literature of operator theory. For some classical results in complex analysis, one might even think that their appropriate setting is the C^* algebra of bounded linear operators on a complex Hilbert space (or the algebra of all $n \times n$ complex matrices) rather than the complex plane. Sometimes it is not clear whether a theorem belongs to operator theory or to complex analysis (perhaps to non-commutative complex analysis). Even in the central core of operator theory, such as the spectral analysis of operator-valued analytic functions or operator polynomials [45], the methods depend heavily on results from complex analysis. This "symbiotic relationship" (a phrase borrowed from Dick Brualdi's opening talk at this Conference) of operator theory and complex analysis is not only mutually beneficial, it has also numerous applications in differential equations, engineering, prediction theory, and stochastic processes.

The above is but a glimpse, through my eyes, into the manifold interactions between linear algebra and other mathematical branches. It is only due to the limitation of my knowledge that many important areas (e.g., the relationship between linear algebra and various algebraic fields) have to be left untouched.

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APPENDIX. WORDS OF THANKS BY FRANK UHLIG

Dear friends and matricians: It is time to acknowledge Ky Fan's life contribution to us with a special and symbolic gift.

As we learnt today, "God gave us matrices."[1] However, we must equally acknowledge the spiritual fact that matrices, or our mind, our thoughts, our emotions, and our work, give us God.

Much of Ky Fan's life here was spent giving us of his mind, his thoughts, and his problems in mathematics. We have shared in them and have been enriched by them for many decades.

May I then offer Ky Fan my laurel, a wreath of cotton from the rich Alabama soil, which like the laurel of antiquity is a trophy for poets of the soul.

The rest of this package contains a symbolic gift. Please, Ky, share with us the contents of this bowl, and do symbolically again what you have done for us mathematicians over the years: share the mints.

[1] Referring to Nick Trefethen's talk.