Oscillation for higher order superlinear delay differential equations with unstable type

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Abstract

We first show that the $n$-order superlinear delay differential equation with unstable type

$$x^{(n)}(t) = p(t) |x(t - \tau)|^{\alpha-1} x(t - \tau), \quad t \geq t_0,$$

always has an unbounded and nonoscillatory solution, where $n > 1$ is an integer, $p \in C([t_0, \infty), [0, \infty))$, $\tau > 0$ and $\alpha > 1$. Then an almost sharp bounded oscillation and nonoscillation criterion is obtained.

Keywords: Unbounded and nonoscillatory solution; Bounded oscillation; Superlinear; Delay differential equation; Unstable type

1. Introduction

In [1], it is shown that the $n$-order linear delay differential equation with unstable type

$$x^{(n)}(t) = p(t) x(t - \tau), \quad t \geq t_0,$$  \hspace{1cm} (1.1)

always has an unbounded and nonoscillatory solution, where $n \geq 1$ is an even integer, $p \in C([t_0, \infty), [0, \infty))$, $\tau > 0$. Therefore, it is the main subject to find conditions for all bounded solution of Eq. (1.1) to be oscillatory. In recent papers [2,3,6–8], some interesting bounded oscillation criteria have been obtained.
In the present paper, we consider the $n$-order superlinear delay differential equation with unstable type
\[ x^{(n)}(t) = p(t) |x(t - \tau)|^{\alpha - 1} x(t - \tau), \quad t \geq t_0, \] (1.2)
where $\alpha > 1$, $n$, $\tau$ and $p(t)$ are the same as in Eq. (1.1). Usually, the methods used to deal with linear equation (1.1) fail to the superlinear equation (1.2). So, nothing is known about the oscillation of Eq. (1.2).

Very recently, Tang [4,5] studied the oscillation of the first order superlinear delay differential equation and obtained some “almost sharp” oscillation criteria. Motivated by the works of Tang [4,5] and Chen et al. [1], in this paper we first prove that Eq. (1.2) always has either unbounded and nonoscillatory solution as Eq. (1.1), then establish an almost sharp bounded oscillation criterion for Eq. (1.2).

As usual, a solution is said to be oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

2. Existence of unbounded positive solution

In this section, we shall prove Eq. (1.2) always has an unbounded positive solution.

**Theorem 2.1.** Equation (1.2) has a positive solution on $[t_0, \infty)$ which tends to infinity as $t \to \infty$.

**Proof.** Set
\[ \tilde{p}(t) = \begin{cases} p(t), & t \geq t_0, \\ p(t_0), & t < t_0, \end{cases} \] (2.1)
and
\[ H(t) = 2\alpha \tilde{p}(t) + \tau^{-n} n! \ln \alpha. \] (2.2)
Clearly, $\tilde{p}(t)$ and $H(t)$ are continuous on $(-\infty, \infty)$. Now define
\[ y(t) = \exp \left[ \exp \left( \frac{1}{(n - 1)!} \int_{t_0 - 2\tau}^t (t - s)^{n-1} H(s) \, ds \right) \right]. \] (2.3)
Then
\[ y'(t) = y(t) \exp \left( \frac{1}{(n - 1)!} \int_{t_0 - 2\tau}^t (t - s)^{n-1} H(s) \, ds \right) \times \frac{1}{(n - 2)!} \int_{t_0 - 2\tau}^t (t - s)^{n-2} H(s) \, ds. \]
Set
\[ w(t) = \exp \left( \frac{1}{(n - 1)!} \int_{t_0 - 2\tau}^t (t - s)^{n-1} H(s) \, ds \right) \]
and

\[ v(t) = \frac{1}{(n-2)!} \int_{t_0-2\tau}^{t} (t-s)^{n-2} H(s) \, ds. \]

Then

\[ y'(t) = y(t)w(t)v(t). \tag{2.4} \]

It is easy to see that

\[ w^{(i)}(t) > 0 \quad \text{and} \quad v^{(i)}(t) > 0, \quad i = 0, 1, \ldots, n-1. \]

It follows from (2.4) that

\[ y^{(i)}(t) > 0, \quad i = 0, 1, \ldots, n-1, \tag{2.5} \]

and for \( t \geq t_0, \)

\[
\begin{align*}
y^{(n)}(t) & \geq y(t)w(t)v^{(n-1)}(t) \\
& = H(t) \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t} (t-s)^{n-1} H(s) \, ds \right) \\
& \quad \times \exp \left[ \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t} (t-s)^{n-1} H(s) \, ds \right) \right] \\
& \geq H(t) \exp \left[ \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t} (t-s)^{n-1} H(s) \, ds \right) \right] \\
& = H(t) \exp \left[ \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t} (t-s)^{n-1} H(s) \, ds \right) \right] \\
& \quad \times \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t-\tau} (t-s)^{n-1} H(s) \, ds \right) \\
& \geq H(t) \exp \left[ \alpha \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t} (t-s)^{n-1} H(s) \, ds \right) \right] \\
& \geq H(t) \exp \left[ \alpha \exp \left( \frac{1}{(n-1)!} \int_{t_0-2\tau}^{t-\tau} (t-\tau-s)^{n-1} H(s) \, ds \right) \right] \\
& = H(t)y^{\#}(t-\tau) \\
& \geq 2\alpha \tilde{p}(t)y^{\#}(t-\tau).
\end{align*}
\]
That is, 
\[
\frac{1}{2\alpha} y^{(n)}(t) \geq p(t) y^\alpha (t - \tau), \quad t \geq t_0.
\] (2.6)

Integrating (2.6) from \(t_0\) to \(t > t_0\) \(n\) times and using (2.5), we obtain
\[
\frac{1}{2\alpha} y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} p(s) y^\alpha (s - \tau) ds, \quad t \geq t_0.
\] (2.7)

Let BC denote the Banach space of all bounded and continuous functions defined on \((-\infty, \infty)\) with the sup norm. Set
\[
\Omega = \{ z \in BC: 0 \leq z(t) \leq 1, \quad -\infty < t < \infty \}.
\]

Then \(\Omega\) is a bounded, closed and convex subset of BC. Define a mapping \(T : \Omega \to BC\) by
\[
(Tz)(t) = \begin{cases} 
\frac{1}{y(t)} \int_{t_0}^{t} \frac{\ln \alpha}{s-t} (t-s)^{n-1} ds, & t \geq t_0, \\
(Tz)(t_0), & t < t_0.
\end{cases}
\]

In view of (2.2) and (2.3), we have for \(t \geq t_0\),
\[
y(t) = \exp \left[ \exp \left( \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} H(s) ds \right) \right]
\geq \exp \left[ \exp \left( \frac{\ln \alpha}{\tau^n} (t - t_0 + 2\tau)^n \right) \right]
\geq \frac{e^{2 \ln \alpha}}{\tau^n} (t - t_0 + 2\tau)^n
\geq \frac{2e^{2 \ln \alpha}}{\tau^{n-1}} (t - t_0)^{n-1}.
\] (2.8)

By (2.7) and (2.8), we see that \(0 \leq (Tz)(t) \leq 1, \quad t \in (-\infty, \infty)\), which shows that \(T\) maps \(\Omega\) into itself. Next we will show that \(T\) is a contraction on \(\Omega\). In fact, for any \(z_1, z_2 \in \Omega\) and \(t \geq t_0\),
\[
|(Tz_1)(t) - (Tz_2)(t)|
\leq \frac{1}{(n-1)!y(t)} \int_{t_0}^{t} (t-s)^{n-1} p(s) y^\alpha (s - \tau) |z_1^\alpha (s - \tau) - z_2^\alpha (s - \tau)| ds
\leq \frac{\alpha}{(n-1)!y(t)} \int_{t_0}^{t} (t-s)^{n-1} p(s) y^\alpha (s - \tau) |z_1 (s - \tau) - z_2 (s - \tau)| ds
\leq \frac{1}{2} \|z_1 - z_2\|.\]
Hence
\[ \| Tz_1 - Tz_2 \| \leq \frac{1}{2}\| z_1 - z_2 \|, \]
which shows that \( T \) is a contraction on \( \Omega \). Then by Banach contraction principle, \( T \) has a fixed point \( z \in \Omega \), that is
\[ z(t) = \begin{cases} \frac{1}{\gamma(t)} \left[ \frac{\ln \alpha}{\tau n - 1} \right] (t - t_0)^{n-1} \\
\quad + \frac{1}{(n-1)!} \int_{t_0}^{t} (t - s)^{n-1} p(s) y(s - \tau) z(s - \tau) ds, & t \geq t_0, \\
(Tz)(t_0), & t < t_0. \end{cases} \]
Set \( x(t) = y(t)z(t) \). Then \( x(t) \) is positive continuous on \((-\infty, \infty)\) and satisfies
\[ x(t) = \frac{\ln \alpha}{\tau n - 1} (t - t_0)^{n-1} + \frac{1}{(n-1)!} \int_{t_0}^{t} (t - s)^{n-1} p(s) x^{\sigma}(s - \tau) ds, \quad t \geq t_0. \]
It is easy to see that \( x(t) \) is a positive solution of Eq. (1.2) and satisfies that \( x(t) \to \infty \) as \( t \to \infty \). The proof is complete. \( \blacksquare \)

3. Bounded oscillation

In this section, we will present some bounded oscillation criteria for Eq. (1.2).

**Lemma 3.1.** Assume that for large \( t \),
\[ p(s) \neq 0 \quad \text{for} \quad s \in [t, t + \tau). \] (3.1)
Then (1.2) has a bounded eventually positive solution if and only if the corresponding inequality
\[ x^{(n)}(t) \geq p(t) |x(t - \tau)|^{\alpha - 1} x(t - \tau), \quad t \geq t_0, \] (3.2)
has a bounded eventually positive solution.

The proof of Lemma 3.1 is similar to the proof of [2, Theorem 5.1.1] and hence we omit it here.

Associated with (1.2), we consider the inequality
\[ x'(t) + q(t) \left| x(t - \sigma) \right|^{\alpha - 1} x(t - \sigma) \leq 0, \quad t \geq t_0, \] (3.3)
where \( q \in C([t_0, \infty), [0, \infty)), \sigma > 0. \)

**Lemma 3.2.** Assume that \( \sigma < \tau \), and that for large \( t \),
\[ \int_{t}^{t + \tau - \sigma} (s - t)^{n-2} p(s) ds \geq (n - 2)q(t). \] (3.4)
Then (3.3) has no eventually positive solutions implies that every bounded solution of Eq. (1.2) oscillates.
Proof. Assume the contrary, and let $x(t)$ be a bounded positive solution of Eq. (1.2). Then there exists $T > t_0$ such that
\[ (-1)^i x^{(i)}(t) > 0, \quad t \geq T - \tau, \quad i = 0, 1, \ldots, n - 1, \tag{3.5} \]
and
\[ \int_t^{t+\tau-\sigma} (s-t)^{n-2} p(s) ds \geq (n-2)! q(t), \quad t \geq T - \tau. \tag{3.6} \]
Integrating (1.2) from $t$ to $\infty$ $(n-1)$ times and using (3.5), we have
\[ x'(t) + \frac{1}{(n-2)!} \int_t^{\infty} (s-t)^{n-2} p(s) x^{\alpha}(s-\tau) ds \leq 0, \quad t \geq T. \tag{3.7} \]
By the nonincreasing of $x(t)$, it follows from (3.7) that
\[ x'(t) + \frac{1}{(n-2)!} \left[ \int_t^{t+\tau-\sigma} (s-t)^{n-2} p(s) ds \right] x^{\alpha}(t-\sigma) \leq 0, \quad t \geq T. \tag{3.8} \]
From this and (3.6), we have
\[ x'(t) + q(t) x^{\alpha}(t-\sigma) \leq 0, \quad t \geq T. \tag{3.9} \]
This shows that inequality (3.3) has an eventually positive solution. This contradiction completes the proof. \qed

The following lemma is taken from [4,5].

Lemma 3.3. Assume that $\alpha > 1$ and there exists $\lambda > \sigma^{-1} \ln \alpha$ such that
\[ \lim \inf_{t \to \infty} \left[ q(t) \exp(-e^{\lambda t}) \right] > 0; \tag{3.10} \]
then inequality (3.3) has no eventually positive solutions.

Theorem 3.1. Assume that $\alpha > 1$. Then the following conclusions hold:

(i) If there exists $\lambda > \tau^{-1} \ln \alpha$ such that
\[ \lim \inf_{t \to \infty} \left[ p(t) \exp(-e^{\lambda t}) \right] > 0, \tag{3.11} \]
then every bounded solution of Eq. (1.2) oscillates.

(ii) If (3.1) holds and there exists $\mu < \tau^{-1} \ln \alpha$ such that
\[ \lim \sup_{t \to \infty} \left[ p(t) \exp(-e^{\mu t}) \right] < \infty, \tag{3.12} \]
then Eq. (1.2) has a bounded eventually positive solution.
Proof. (i) Let \( \sigma < \tau \) such that \( \lambda > \sigma^{-1} \ln \alpha \) and let
\[
q(t) = \frac{(\tau - \sigma)^{n-1}}{(n-1)!} \min_{t \leq s \leq t + \tau - \sigma} p(s).
\]
Then (3.4) holds and
\[
\liminf_{t \to \infty} \left[ q(t) \exp(-e^{\lambda t}) \right] = \frac{(\tau - \sigma)^{n-1}}{(n-1)!} \liminf_{t \to \infty} \left[ p(t) \exp(-e^{\lambda t}) \right] > 0.
\]
Hence, in view of Lemmas 3.2 and 3.3, every bounded solution of Eq. (1.2) oscillates.

(ii) By (3.11), we may choose \( \mu_1 > \mu \) and \( T > t_0 \) such that
\[
\alpha e^{-\mu \tau} > \alpha e^{-\mu_1 \tau} > 1 \quad (3.12)
\]
and
\[
p(t) \leq \frac{1}{2} \mu_1^n \exp\left[ (\alpha e^{-\mu_1 \tau} - 1) e^{\mu_1 t} + n \mu_1 t \right], \quad t \geq T. \quad (3.13)
\]
Set \( \varphi(t) = e^{\mu_1 t}, x(t) = e^{-\varphi(t)} \). Then, by induction, we have
\[
x^{(n)}(t) = x(t) \left[ (\varphi'(t))^n + \sum_{\beta_1 + 2\beta_2 + \cdots + n\beta_n = n} a_1 (\varphi'(t))^\beta_1 (\varphi''(t))^\beta_2 \cdots (\varphi^{(n)}(t))^\beta_n \right],
\]
where \( a_i, \beta_{ij} \) are integers depending on \( n \) and \( 0 \leq \beta_{ij} \leq n - 1 \). Hence for large \( t \),
\[
x^{(n)}(t) - p(t)x^{\alpha}(t - \tau) = \mu_1^n e^{-\varphi(t)} \left[ e^{n \mu_1 t} + \sum_{\beta_1 + 2\beta_2 + \cdots + n\beta_n = n} a_1 \exp\left( \mu_1 t \sum_{j=1}^n \beta_{ij} \right) \right] - p(t)e^{-\alpha \varphi(t - \tau)}
\geq \frac{1}{2} \mu_1^n e^{-\varphi(t) + n \mu_1 t} - p(t)e^{-\alpha \varphi(t - \tau)}
\geq -e^{-\alpha \varphi(t - \tau)} \left\{ p(t) - \frac{1}{2} \mu_1^n \exp\left[ (\alpha e^{-\mu_1 \tau} - 1) e^{\mu_1 t} \right] + n \mu_1 t \right\}
\geq 0.
\]
This shows that the inequality
\[
x^{(n)}(t) \geq p(t) x^{\alpha}(t - \tau) \left[ x^{(n-1)}(t) \right], \quad t \geq t_0,
\]
has a bounded eventually positive solution. In view of Lemma 3.1, the corresponding equation (1.2) has also a bounded eventually positive solution. The proof is complete. \( \Box \)

Remark 3.1. Note that if \( \alpha > 1 \), it follows that there exists a unique \( \lambda_0 > 0 \) such that \( \alpha e^{-\lambda_0 \tau} = 1 \). Therefore, applying Theorem 3.1 to the following special form of Eq. (1.2),
\[
x^{(n)}(t) = C \exp(e^{\lambda t}) \left| x(t - \tau) \right|^{\alpha - 1} x(t - \tau), \quad t \geq t_0, \quad (3.14)
\]
where \( C > 0 \), we have that every bounded solution of Eq. (3.14) oscillates if \( \lambda > \lambda_0 \) and Eq. (3.14) has a bounded eventually positive solution if \( \lambda < \lambda_0 \). This shows that Theorem 3.1 is an almost sharp criterion of bounded oscillation and nonoscillation for Eq. (1.2).
References