Uniqueness and Non-uniqueness in the Cauchy Problem for a Class of Operators of Degenerate Type

SHIZUO NAKANE

Department of Mathematics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo, Japan

Received March 1, 1982

INTRODUCTION

In this paper, we consider uniqueness and non-uniqueness of solutions of the non-characteristic Cauchy problem for a class of partial differential operators with $C^\infty$-coefficients whose characteristic roots degenerate on the initial surfaces.

Uryu [12] treated a class of operators $P$ in $\mathbb{R}_x \times \mathbb{R}_t^n$ with real principal symbols and with characteristic roots $\tau = t^j\lambda_j(t, x; \xi)$ ($1 \leq j \leq m$), degenerating on the initial surface $t = 0$. He proved that uniqueness holds for $P$ if there exists an operator $\tilde{P}$ with distinct characteristic roots satisfying

$$t^mP(t, x; D_t, D_x) - \tilde{P}(t, x; tD_t, t^{l+1}D_x).$$

Note that this condition is the so-called Levi condition (see Tahara [11]).

Considering Calderón’s conditions (see [2] or [7]), we extend his result to the case $\tilde{P}$ has non-real double characteristics of constant multiplicity. Roberts [9] also dealt with related topics.

We also consider the necessity of condition (0.1). Zeman [14, 15] showed that Levi type condition implies uniqueness when the characteristics are of constant multiplicity ([14]), or of variable multiplicity and involutive ([15]). On the other hand, Matsumoto [6] and, recently, other mathematicians showed uniqueness for some classes of operators with characteristics of constant multiplicity not satisfying Levi-type conditions. Then the following question arises: Is condition (0.1) necessary for uniqueness?

We answer this question by the following operator in $\mathbb{R}^2$:

$$L = (\partial_t - it^l\partial_x)^p + t^k(i\partial_x)^q - t^m(i\partial_x)^{q-r}.$$  

We show that under some conditions on $p, q, r, l, k$ and $m$ there exist $C^\infty$-functions $u$ and $f$ such that

$$Lu - fu = 0, \quad 0 \in \text{supp } u \subset \{ t \geq 0 \}.$$
Then we get an observation that condition (0.1) is good in a sense (see Remark 3 of Theorem 2).

Furthermore we consider the Gevrey classes to which the null solution $u$ constructed above belongs. Then we obtain a necessary condition for uniqueness in Gevrey classes, which corresponds to the results of Igari [3] and Ivrii [4] on the well-posedness of the Cauchy problem.

In Section 1, we state the main results. In Section 2, we prove the uniqueness result by Carleman-type estimates, which are refinements of those of [12]. Sections 3 and 4 are devoted to the proofs of non-uniqueness results.

1. Statement of Results

Let $U$ be an open neighborhood of 0 in $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n$ and let $P = P(t, x; D_t, D_x)$ be a partial differential operator of order $m$ with $C^\infty$-coefficients in $U$. Here $D_t = \partial / \partial t$, $D_x = \partial / \partial x$.

We assume that the principal symbol $P_m(t, x; \tau, \xi)$ of $P$ is factorized as

$$P_m(t, x; \tau, \xi) = \prod_{j=1}^s (\tau - t^j \lambda_j(t, x; \xi))^2 \prod_{k=s+1}^{m-s} (\tau - t^k \lambda_k(t, x; \xi))^2,$$

where $s$ and $m$ are positive integers, $2s \leq m$, and $\lambda_j(t, x; \xi)$ (1 \( \leq j \leq m - s \)) are $C^\infty$-functions in $U \times (\mathbb{R}^n \setminus 0)$, homogeneous of degree 1 in $\xi$. We require that $\lambda_j$ satisfy Calderón’s conditions there:

$$\lambda_i \neq \lambda_j \quad (i \neq j),$$

$$\text{Im} \lambda_j \neq 0 \quad (1 \leq j \leq s),$$

$$\text{Im} \lambda_k \neq 0 \text{ or } \equiv 0 \quad (s + 1 \leq k \leq m - s).$$

All the conditions above are imposed on the principal part of $P$. Next, we consider the lower order terms of $P$. From (1.1), we can easily see that there exist differential polynomials $R_s$ and $R_{m-2s}$, homogeneous of degree $s$ and $m - 2s$ respectively, having distinct characteristic roots such that

$$P_m(t, x; \tau, \xi) = R_{m-2s}(t, x; \tau, t^s \xi) R_s(t, x; \tau, t^s \xi)^2,$$

(see [10]). Then we can express $P$ as

$$P(t, x; D_t, D_x) = R_{m-2s}(t, x; D_t, t^s D_x) R_s(t, x; D_t, t^s D_x)^2 + \sum_{j=1}^m \frac{m}{j} P_m^{(j)}(t, x; D_t, D_x),$$

where $P_m^{(j)} = \sum_{i=0}^{m-j} \sum_{|\alpha| = j} a_{ij\alpha}(t, x) D_x \alpha D_t^{m-j-i}$, $a_{ij\alpha} \in C^\infty(U)$. 

505/51/1-6
We assume that there exist \( b_{ij} \in C^\infty(U) \) such that
\[
\sum_{i=1}^{m-1} \sum_{\alpha=1}^{s} b_{i1\alpha}(t, x) \xi^\alpha \lambda_j(t, x, \xi)^{m-1-i} \bigg|_{\xi=0} = 0.
\]
(1.6)

Note that, from the assumptions above, there exists a differential polynomial \( \tilde{P} \) of degree \( m \) with characteristic roots \( \lambda_j \) satisfying
\[
t^m P(t, x; D_t, D_x) = \tilde{P}(t, x; tD_t, t^{l+1}D_x).
\]
(1.7)

Furthermore, if \( \tilde{P}_{m-1} \) denotes the subprincipal symbol of \( \tilde{P} \), (1.6) implies \( \tilde{P}_{m-1}(t, x; \lambda_j(t, x; \xi), \xi) \big|_{\xi=0} = 0 \) for double roots \( \lambda_j \) (1 \( \leq j \leq s \)) of \( \tilde{P} \).

**THEOREM 1.** Under assumptions (1.1)–(1.6), there exists an open neighborhood \( U' \) of 0 in \( \mathbb{R}^{n+1} \) such that, if \( u \in C^\infty(U) \) satisfies \( Pu = 0 \) in \( U \), \( (D_t u)(0, x) = 0 \) (0 \( \leq j \leq m - 1 \)), then \( u \equiv 0 \) in \( U' \).

**Remark 1.** Theorem 1 is an extension of the results of Roberts [9] and Uryu [12]. Roberts treated the case \( I < 0 \) (i.e., Fuchsian-type equations), and Uryu treated the case \( s = 0 \). Note that, when \( s = 0 \), assumptions (1.1)–(1.5) implies uniqueness.

**EXAMPLE 1.** Let \( P \) be the operator in \( \mathbb{R}^2 \):
\[
P = (D_t - it D_x)^2 + a(t, x) D_t + t^m b(t, x) D_x + c(t, x),
\]
where \( a, b, c \in C^\infty(U) \) and \( m \geq 0 \). Then \( P \) satisfies our conditions if \( m > l - 1 \).

**EXAMPLE 2.** Let \( P \) be the operator in \( \mathbb{R}^2 \):
\[
P = (D_t - it D_x)^2 - t^k D_x^2 + a(t, x) D_t + t^m b(t, x) D_x + c(t, x),
\]
where \( a, b, c \in C^\infty(U) \) and \( k > l, m \geq 0 \). Though this operator does not satisfy (1.2), we can show that uniqueness holds if \( m > l - 1 \) by the same method.

As for the necessary condition for uniqueness, we consider the following example of a degenerate elliptic operator in \( \mathbb{R}^2 \) with non-real principal symbol:
\[
L = (\partial_r - it \partial_x)^p + t^{k} (i \partial_x)^q - t^m (i \partial_x)^{q-r},
\]
(1.8)
where \( \partial_r = \partial/\partial t, \partial_x = \partial/\partial x, p, q, r, k, l \in \mathbb{N}, r \leq q \leq p, m \in \mathbb{Z}, 0 \leq m < k \).
UNIQUENESS IN THE CAUCHY PROBLEM

THEOREM 2. Suppose one of the following conditions (1.9)–(1.14) is satisfied. Then there exist $C^\infty$-functions $u$ and $f$ in $\mathbb{R}^2$ such that

$$Lu - fu = 0, \quad \{t = 0\} \subset \text{supp } u \subset \{t \geq 0\}.$$  

When $p > q$,

$$k - r(pl - k)/(p - q) \leq m < k - r(k + p)/q, \quad (1.9)$$

$$q > (p + 1)/2, k < q(l + 1) - p, m < k - r(pl - k)/(p - q), \quad (1.10)$$

$$\begin{align*}
q > (p + 1)/2, \quad k & \geq q(l + 1) - p, \\
m & < k + r(pl + l + 1 - p - 2k)/(2q - p - 1), \quad (1.11)
\end{align*}$$

$$\begin{align*}
q & < (p + 1)/2, \\
(k + r(pl + l + 1 - p - 2k)/(2q - p - 1)) & < m < k - r(pl - k)/(p - q). \quad (1.12)
\end{align*}$$

When $p = q$,

$$k \leq pl, m < k - r(k + p)/p, \quad (1.13)$$

$$k > pl, m < k + r(pl + l + 1 - p - 2k)/(p - 1). \quad (1.14)$$

Remark 2. Theorem 2 is a slight modification of Plis [8, Theorem 4]. He treated the case $l = m = 0, r = 1$.

Remark 3. Condition (1.13) with $k = pl$ implies $m < l(p - r) - r$. On the other hand, Theorem 1 with $s = 0$ shows that uniqueness holds in this case if $m > l(p - r) - r$. Hence this necessary condition seems to be the best one and assumption (1.5) in Theorem 1 is indispensable.

Remark 4. Watanabe [13] proved that uniqueness holds for the following degenerate elliptic operator in $\mathbb{R}^2$:

$$P = D_i^2 + i^2D_x^2 + (\text{any lower order terms}).$$

Now we consider the Gevrey classes to which the functions $u$ and $f$ constructed above belong. We denote function spaces $C^\infty(\mathbb{R}; \mathcal{E}^{1,\alpha}(\mathbb{R},))$ by $\mathcal{G}^{1,\alpha}(\mathbb{R},)$ for $\alpha > 1$. Here $\mathcal{E}^{1,\alpha}(\mathbb{R},)$ is as follows:

$$\mathcal{E}^{1,\alpha}(\mathbb{R},) = \{u(x) \in C^\infty(\mathbb{R},); \text{ For any compact set } K \text{ in } \mathbb{R}, \text{ there exist } C \text{ and } \rho > 0 \text{ such that for any } s \text{ } \max_{x \in K} |D_x^s u(x)| \leq Cp^s s!^\alpha \}. $$
We define \( \alpha_0 \) by

\[
\alpha_0 = \frac{(k - m)\rho}{k(q - r) - mq - pr}, \quad \text{if } (1.9) \text{ or } (1.13) \text{ is satisfied}, \quad (1.15)
\]

\[
= \frac{(k - m)(p - 1)}{(2q - p - 1)(k - m) + r(pl + l + 1 - p - 2k)}, \quad \text{otherwise. (1.16)}
\]

**Theorem 3.** Under the assumptions of Theorem 2, we can construct \( u \) and \( f \) in Theorem 2 belonging to \( \gamma(\alpha) \) and \( \gamma(\alpha + 1) \) respectively for any \( \alpha > \alpha_0 \).

**Remark 5.** Leray [5] gave a necessary condition for uniqueness in Gevrey classes for a hyperbolic operator with characteristics of constant multiplicity.

**Remark 6.** When \( p = q = 2, \ r = 1, \ k = 2 \), condition (1.13) implies \( m < l - 1 \) and condition (1.15) means \( \alpha_0 = (2l - m)/(l - 1 - m) \). Theorem 3 shows, in this case, uniqueness does not hold in \( \gamma(\alpha) \) for any \( \alpha > \alpha_0 \). This fact corresponds to the results of Igari [3] and Ivrii [4] on the well-posedness of the Cauchy problem in Gevrey classes for degenerate hyperbolic equations.

**2. Proof of Theorem 1**

In this section, we prove Theorem 1 by a Carleman type estimate for \( P = t^mP \) (see [9] or [12]).

It is easy to see that, if \( u \in C^\infty(U) \) satisfies \( Pu = 0 \) and \( (D^j u)(0, x) = 0 \) \( (0 \leq j \leq m - 1) \), \( u \) is flat on \( t = 0 \). Hence we may assume \( u \equiv 0 \) for \( t \leq 0 \). We make the singular change of variables (see Alinhac and Baouendi [11]):

\[
\begin{align*}
    x &= y, \\
    t &= (r - |y|^2)s, \quad (r \text{ is sufficiently small}).
\end{align*}
\]

Then \( u \) is transformed into

\[
v(s, y) = u((r - |y|^2)s, y) \in C^\infty, \supp v \subset \{s \geq 0, \ |y| \leq \sqrt{r}\},
\]

and \( \bar{P}(t, x; tD_t, t^{l+1}D_x) \) is transformed into

\[
\bar{Q}(s, y; sD_s, f(y) s^{l+1}D_y)
\]

\[
= \bar{P}(r - |y|^2)s, y; sD_s, f(y) s^{l+1}D_y + 2y(r - |y|^2)t s^{l+2}D_s),
\]

where \( f(y) = (r - |y|^2)^{l+1} \).
We can easily see that $0$ satisfies the same properties as $p$ stated in Section 1. Note that $u \equiv 0$ near 0 if $v \equiv 0$ for $s \leq s_0$ for some $s_0 > 0$. We rewrite $(s, y)$ by $(t, x)$ and $v, \Phi$ by $u, \tilde{P}$, respectively. Then we may assume from the beginning $\text{supp } u \subset \{ t \geq 0, |x| \leq \sqrt{r} \}$ for sufficiently small $r$ and

$$i^m \tilde{P}(t, x; tD_x, f(x) t^{i+1} D_x),$$

where $\tilde{P}$ has the properties stated before.

**Lemma 1.** For the above $\tilde{P}$, there exist $T_0, N_0, r > 0$ and a neighborhood $\Omega$ of 0 in $\mathbb{R}^n$ such that, if $0 < T < T_0, N > N_0$ and $u \in C^\infty_0([0, T] \times \Omega)$, then

$$N^{m-1} \int_0^T t^{-2N} \| u \|^2 dt \leq C \int_0^T t^{-2N} \| \tilde{P} u \|^2 dt \quad (\| u \| = \| u(t, \cdot) \|_{L^2(\mathbb{R}^n)}).$$

Theorem 1 follows from Lemma 1 by a standard argument. In order to prove Lemma 1, we need some lemmas. First we show Carleman type estimates for first order pseudo-differential operators of Fuchsian type. Let $S^m$ be the set of symbols of classical pseudo-differential operators of order $m$ with respect to $x$. And we set $\mathcal{B}([0, T], S^m) = C^\infty([0, T], S^m)$. Let $\tilde{\partial}$ be the operator

$$\tilde{\partial} = tD_t - t^k(A(t, x; D_x) + iB(t, x; D_x)), \quad k > 0,$$

where $A(t, x; \xi) = f(x) \tilde{A}(t, x; \xi), B(t, x; \xi) = f(x) \tilde{B}(t, x; \xi), \tilde{A}, \tilde{B} \in \mathcal{B}([0, T], S^1), f \in C^\infty(\mathbb{R}^n)$ and $f, \tilde{A}, \tilde{B}$ are real valued.

**Lemma 2.** Suppose $\tilde{B} \equiv 0$ or $\neq 0$ for any $(t, x, \xi)$. Then, for sufficiently small $T$ and $N^{-1}$, there exists a constant $C > 0$ such that

$$N \int_0^T t^{-2N} \| u \|^2 dt \leq C \int_0^T t^{-2N} \| \tilde{\partial} u \|^2 dt, \quad (2.1)$$

for any $u \in C^\infty_0([0, T] \times \Omega)$. Furthermore, if $\tilde{B} \neq 0$, then we have

$$\int_0^T t^{-2N} (\| tD_t u \|^2 + \sum_{j=1}^n \| t^j f(x) D_x^j u \|^2) dt \leq C N \int_0^T t^{-2N} \| \tilde{\partial} u \|^2 dt. \quad (2.2)$$

**Proof.** We prove only (2.2). The proof of (2.1) is similar and easy. We assume $\tilde{B} \neq 0$. If we set $v = t^{-N} u$, then $t^{-N} \tilde{\partial} u = (\tilde{\partial} - iN)v$. We estimate $I = \int_0^T t^{-2N} \| \tilde{\partial} u \|^2 dt$ from below.
\[ I = \int_0^T \| (tD_t - t^kA)v \|^2 \, dt + \int_0^T \| (t^kB + N)v \|^2 \, dt \]

\[ + 2 \text{Re} \int_0^T (tD_tv, -iNv) \, dt + 2 \text{Re} \int_0^T (-t^kAv, -iNv) \, dt \]

\[ + 2 \text{Re} \int_0^T (-t^kAv, -itkBv) \, dt + 2 \text{Re} \int_0^T (tD_tv, -itkBv) \, dt \]

\[ = I_1 + I_2 + \cdots + I_6. \]

We estimate each \( I_j \) \((j \geq 3)\).

\[ I_3 = -N \int_0^T \| v \|^2 \, dt. \quad (2.3) \]

Since \( A \) and \( B \) are real, it follows from the product and adjoint formulas of pseudo-differential operators:

\[ I_4 = N \int_0^T (it^k[A^* - A]v, v) \, dt \]

\[ \geq -C_1 T^kN \int_0^T \| v \|^2 \, dt, \quad (2.4) \]

\[ I_5 = \int_0^T t^{2k}(i[A^*B - B^*A]v, v) \, dt \]

\[ \geq -C_2 T^k \int_0^T \| t^k f(x) Av \| \| v \| \, dt - C_3 T^{2k} \int_0^T \| v \|^2 \, dt, \]

where \( A^* \) and \( B^* \) are the formal adjoints of \( A \) and \( B \), \( A \) is a pseudo-differential operator with symbol \((1 + |\xi|^2)^{1/2}\).

\[ I_6 - \int_0^T (i[B^*t^{k+1}D_t - D_t t^{k+1}B]v, v) \, dt \]

\[ = \int_0^T (it^{k+1}[B^* - B]D_tv, v) \, dt - (k + 1) \int_0^T (t^{k}Bv, v) \, dt \]

\[ - \int_0^T (t^{k+1}Bv, v) \, dt, \]

where \( B \) belongs to \( \mathcal{B}([0, T], S^1) \) with symbol \( \partial B/\partial t \). Then,

\[ I_6 \geq -C_4 T^k \int_0^T \| tD_tv \| \| v \| \, dt - (k + 1) \int_0^T ([t^{k}B + N]v, v) \, dt \]

\[ + (k + 1)N \int_0^T \| v \|^2 \, dt - C_5 T \int_0^T \| t^k f(x) Av \| \| v \| \, dt \]
These inequalities imply

$$I \geq I_1 + I_2 + \{(k - C_1 T^k) N - C_3 T^{2k}\} \int_0^T \|v\| \, dt - (C_1 T^k + C_2 T) \int_0^T \|t^k f(x) A v\| \, dt$$

$$- C_4 T^k \int_0^T \|(tD_t - t^k A) v\| \|v\| \, dt - (k + 1) \int_0^T \|(t^k B + N) v\| \|v\| \, dt. \quad (2.6)$$

Since $\bar{B}$ is elliptic, there exist $E, R \in \mathcal{D}(\mathbb{R}, S^{-1})$ such that

$$E \bar{B} = Id + R.$$

Then, we have

$$f(x) A = f(x) A E \bar{B} - f(x) AR$$

$$= AE + \text{(zeroth order operator)},$$

where $AE$ is of zeroth order. Hence we have

$$\|t^k f(x) A v\| \leq C_6 \|t^k B v\| + C_7 \|t^k v\|$$

$$\leq C_6 \|(t^k B + N) v\| + C_6 N \|v\| + C_7 \|t^k v\|.$$

This implies for sufficiently small $T$

$$(C_2 T^k + C_4 T^k + C_5 T) \int_0^T \|t^k f(x) A v\| \|v\| \, dt$$

$$\leq C_8 T \int_0^T \|(t^k B + N) v\| \|v\| \, dt + (C_8 NT + C_9 T^{k+1}) \int_0^T \|v\|^2 \, dt.$$

This means

$$I \geq I_1 + I_2 + \{(k - C_1 T^k - C_8 T) N - C_3 T^{2k} - C_9 T^{k+1}\} \int_0^T \|v\|^2 \, dt \quad (2.8)$$

$$- C_4 T^k \int_0^T \|(tD_t - t^k A) v\| \|v\| \, dt - (k + 1 + C_8 T) \int_0^T \|(t^k B + N) v\| \|v\| \, dt$$
We can easily see that
\[ I_{1/2} - C_4 T^k \int_0^T \| (tD_t - t^k A) v \| \| v \| \, dt \geq -\frac{1}{2} C_4^2 T^{2k} \int_0^T \| v \|^2 \, dt, \]
\[ I_{2/2} - (k + 1 + C_8 T) \int_0^T \| (t^k B + N) v \| \| v \| \, dt \geq -\frac{1}{2} (k + 1 + C_8 T)^2 \int_0^T \| v \|^2 \, dt. \]

Hence, if we choose \( T \) and \( N^{-1} \) sufficiently small, we have
\[ I \geq I_{1/2} + I_{2/2} + \frac{k}{2} N \int_0^T \| v \|^2 \, dt \]
(2.9)

Especially we get (2.1).

Next, we have
\[ \int_0^T t^{-2N} \| t^k f(x) D_x^j u \|^2 \, dt = \int_0^T \| t^k f(x) D_x^j v \|^2 \, dt \]
\[ \leq C_{10} \int_0^T \| (t^kB + N)v \|^2 \, dt + C_{10} N^2 \int_0^T \| v \|^2 \, dt \]
\[ + C_{11} T^{2k} \int_0^T \| v \|^2 \, dt \]
\[ \leq CNI. \]
(2.10)

In the same way, we have
\[ \int_0^T t^{-2N} \| tD_t u \|^2 \, dt \leq 2 \int_0^T t^{-2N} \| (tD_t - t^k A) u \|^2 \, dt + 2 \int_0^T t^{-2N} \| t^k A u \|^2 \, dt \]
\[ \leq CNI. \]
(2.11)

Inequalities (2.10) and (2.11) imply (2.2). This completes the proof of Lemma 2.

We may modify the characteristic roots \( \lambda_j(t, x; \xi) \) of \( \hat{P}(t, x; D_t, D_x) \) so that \( \lambda_j \in \mathcal{B}([0, T], S^1) \) and \( \lambda_j \) satisfy assumptions (1.2)–(1.4) in \( [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \). Then we can apply Lemma 2 with \( k = l + 1 \) to \( \partial = \partial_j = tD_t - t^{l+1} f(x) \lambda_j(t, x; D_x) \).

The following lemma is easy (see [12]).

**Lemma 3.** Suppose \( \lambda_l \neq \lambda_j \). Then, for any first order operator of the form:
\[ R = a(t, x) tD_t + t^{l+1} f(x) B(t, x; D_x), a \in C^\infty([0, T] \times \Omega), B \in \mathcal{B}([0, T], S^1), \]
there exist $Q_i, Q_j, Q_{ij} \in \mathcal{H}([0, T], S^0)$ such that

$$R = Q_i \partial_i + Q_j \partial_j + Q_{ij}.$$  

By virtue of Lemmas 2 and 3, we have, by the same argument as in [9],

**Lemma 4.** There exist $C, T_0, N_0 > 0$ such that, if $0 < T < T_0$, $N > N_0$ and $u \in C^\infty_0([0, T] \times \Omega)$, then

$$\sum_{|\alpha| + j \leq m-1} \int_0^T t^{-2N} \left\| (t^{l+1}f(x) D_x)^\alpha (tD_t)^j u \right\|^2 \, dt \leq C \int_0^T t^{-2N} \left\| \Pi_{m-2s} \Pi_s^2 u \right\|^2 \, dt,$$

where

$$\Pi_{m-2s} = \prod_{j=s+1}^{m-s} (tD_t - t^{l+1}f(x) \lambda_j(t, x; D_x)),$$

and $|A|$ is the integral part of $A$.

**Proof of Lemma 1.** (See proof of Lemma 2 of [9].)

Assumptions (1.5) and (1.6) imply that $\tilde{P}$ is factorized as

$$\tilde{P}(t, x; tD_t, t^{l+1}f(x) D_x) = \Pi_{m-2s} \Pi_s^2 + \tilde{P}_{m-1-s}(t, x; tD_t, t^{l+1}f(x) D_x) \Pi_s + t\tilde{P}_{m-1}(t, x; tD_t, t^{l+1}f(x) D_x) + \tilde{P}_{m-2}(t, x; tD_t, t^{l+1}f(x) D_x),$$

where $\tilde{P}_j$ are partial differential operators of order $j$.

Then, it follows from (2.12) that there exists a constant $C > 0$ such that

$$\int_0^T t^{-2N} \left\| \Pi_{m-2s} \Pi_s^2 u \right\|^2 \, dt \leq 4 \int_0^T t^{-2N} \left\| \tilde{P} u \right\|^2 \, dt + 4 \int_0^T t^{-2N} \left\| \tilde{P}_{m-1-s} \Pi_s u \right\|^2 \, dt$$

$$+ 4T^2 \int_0^T t^{-2N} \left\| \tilde{P}_{m-1} u \right\|^2 \, dt + 4 \int_0^T t^{-2N} \left\| \tilde{P}_{m-2} u \right\|^2 \, dt$$

$$\leq 4 \int_0^T t^{-2N} \left\| \tilde{P} u \right\|^2 \, dt$$

$$+ C(N^{-1} + T^2) \int_0^T t^{-2N} \left\| \Pi_{m-2s} \Pi_s^2 u \right\|^2 \, dt.$$
This implies, for small $N^{-1}$ and $T$, there exists a $C > 0$ such that

$$
\int_0^T t^{-2N} \| \Pi_{m-z} \Pi_s^2 u \|^2 \, dt \leq C \int_0^T t^{-2N} \| \tilde{P} u \|^2 \, dt \tag{2.13}
$$

Hence we have from (2.12) and (2.13),

$$
N^{m-1} \int_0^T t^{-2N} \| u \|^2 \, dt \leq C \int_0^T t^{-2N} \| \tilde{P} u \|^2 \, dt.
$$

This completes the proof of Lemma 1.

3. PROOF OF THEOREM 2

Our method of proof of Theorem 2 is essentially due to Plis [8]. First we show a modification of Lemma 3 of [8] for $\lambda(t) = t^n z^{q-r} - t^k z^q$.

**Lemma 5.** We set $\lambda(t) = t^n z^{q-r} - t^k z^q$. Then there exist $C, A > 0$ such that for any $z, c, s$ satisfying

$$
z > 2, \quad c > 0, \quad 0 < s < a = z^{-r/(k-m)},
$$

there exists a $C^\infty$-solution $w(t)$ of the equation

$$
\left( \frac{d}{dt} - t^l z \right)^p w(t) = \lambda(t) w(t) \tag{3.1}
$$

satisfying the conditions

$$
w(s) = c, \tag{3.2}
$$

$$
w(t) > 0 \quad (0 \leq t \leq a), \tag{3.3}
$$

$$w(t) \text{ is non-decreasing for } 0 \leq t \leq a, \tag{3.4}
$$

$$
|w^{(j)}(t)| \leq j! M^j z^j w(t) \quad (j \geq 0, 0 \leq t \leq a), \tag{3.5}
$$

$$
\frac{w(\tau)}{w(T)} \leq z^C \exp \left\{ \frac{z(\tau^{l+1} - T^{l+1})}{l+1} + G^{1/p}(\tau - T) \right\}, \tag{3.6}
$$

$$
\frac{w(\tau)}{w(T)} \geq z^{-C} \exp \left\{ \frac{z(\tau^{l+1} - T^{l+1})}{l+1} + G^{1/p}(\tau - T) \right\}, \tag{3.7}
$$

where $0 \leq T \leq \tau \leq a$ and $0 \leq g \leq \lambda(t) \leq G$ for $T \leq t \leq \tau$. 
Proof. We set \( w(t) = \exp \left( z(t^{i+1} - s^{i+1})/(l + 1) \right) \cdot (y(t)/y(s)) c \), where \( y(t) \) is the solution of the Cauchy problem

\[
\begin{align*}
&y^{(j)}(t) = \lambda(t) y(t), \\
&y^{(j)}(0) = 1 
\end{align*}
\]

Since \( \lambda(t) \geq 0 \) for \( 0 \leq t \leq a \) and \( |\lambda^{(j)}(t)| \leq Cz^j, j \geq 0, 0 \leq t \leq a \), for some \( C \), the proof of Lemma 3 of [8] works for this case.

We introduce the following sequences for \( h > 0 \),

\[
z_n = n^h, \quad a_n = z_n^{-(k-m)} = n^{-r(k-m)},
\]

\[
r_{n} = 7^{-p}(a_n - a_{n+1}) = 7^{-p} \frac{rh}{k-m} n^{-r(k-m)-1} \left( 1 + O \left( \frac{1}{n} \right) \right),
\]

\[
b_n = a_n - 4r_n, \quad s_n = a_n - 2r_n,
\]

\[
y_n = 7^{-1}(rh)^{1/p} n^{k(k-qm-rk)-k+m}/(k-m) \quad (n \geq 1).
\]

Let \( w_n(t) \) be the solution from Lemma 5 for \( z = z_n, a = a_n, s = s_n, \lambda(t) = \lambda_n(t) = t^mz_n^{q-r} - t^kz_n^q, c = 1 \) \((n = 1)\), \( c = w_{n-1}(s_n) \) \((n > 1)\), i.e.,

\[
w_n(s_n) = 1, \quad n = 1
\]

\[
= w_{n-1}(s_n), \quad n > 1. \tag{3.8}
\]

Lemma 6. Suppose we set

\[
G_n = \max \{ \lambda_n(t); b_n \leq t \leq a_n \}, \quad g_n = \min \{ \lambda_{n-1}(t); b_n \leq t \leq a_n \}.
\]

Then we have for sufficiently large \( n \)

\[
G_n^{1/p} \leq 5y_n, \tag{3.9}
\]

\[
g_n^{1/p} \geq 6y_n. \tag{3.10}
\]

Proof. Since \( \lambda_n(a_n) = 0 \), we have for some \( \theta, 0 < \theta < 1 \),

\[
\lambda_n(t) = (t - a_n) \lambda_n'(a_n + \theta(t - a_n))
\]

\[
= (t - a_n) \{ a_n + \theta(t - a_n) \}^{m-1} z_n^{q-r} [m - k z_n^q (a_n + \theta(t - a_n))^{k-m}].
\]

We can easily see that for \( b_n \leq t \leq a_n \),

\[
m - k z_n^r (a_n + \theta(t - a_n))^{k-m} = m - k \left( 1 + \theta \left( \frac{t}{a_n} - 1 \right) \right)^{k-m}
\]

\[
= m - k + O \left( \frac{1}{n} \right).
\]
Then we have for \( b_n \leq t \leq a_n \),
\[
\lambda_n(t) = (k - m)(a_n - t) a_n^{m-1} z_n^{q-r} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]
and
\[
= 4(k - m) r_n a_n^{m-1} z_n^{q-r} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]
\[
= 4 \cdot 7^{-p} r_n n^{-1 + \frac{h(qk - qm - rk)}{(k - m)}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

This implies for sufficiently large \( n \),
\[
\lambda_n(t)^{1/p} \leq 5y_n, \quad b_n \leq t \leq a_n.
\]

Next, it is easy to see that, for large \( n \), \( \lambda_{n-1}(t) \) is monotone decreasing on \( b_n \leq t \leq a_n \). Hence we have for some \( \theta, 0 < \theta < 1 \), the following on \( b_n \leq t \leq a_n \),
\[
\lambda_{n-1}(t) \geq \lambda_{n-1}(a_n)
\]
\[
= (a_n - a_{n-1}) \lambda'_{n-1}(a_{n-1} + \theta(a_n - a_{n-1}))
\]
\[
= (a_n - a_{n-1})(a_{n-1} + \theta(a_n - a_{n-1}))^{m-1} z_n^{q-r(m-k)} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]
\[
= r_n n^{-1 + \frac{h(qk - qm - rk)}{(k - m)}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

Then we have for sufficiently large \( n \),
\[
\lambda_{n-1}(t)^{1/p} \geq 6y_n.
\]

This completes the proof.

From (3.6) and (3.9), we get (\( T = s_n, \tau = t \)),
\[
\frac{w_n(t)}{w_n(s_n)} \leq n^c \exp \left\{ \frac{z_n(t^{l+1} - s_n^{l+1})}{l+1} + 5y_n(t - s_n) \right\}, \quad s_n \leq t \leq a_n,
\]
(3.11)
and, from (3.7) and (3.10), we get
\[
\frac{w_{n-1}(t)}{w_{n-1}(s_n)} \geq n^{-c} \exp \left\{ \frac{z_{n-1}(t^{l+1} - s_n^{l+1})}{l+1} + 6y_n(t - s_n) \right\}, \quad s_n \leq t \leq a_n,
\]
(3.12)
From (3.8), (3.11) and (3.12), we have for \( s, < t < a, \)
\[
\frac{w_n(t)}{w_{n-1}(t)} \leq n^{2C} \exp \left\{ \left( \frac{z_n - z_{n-1}}{l+1} \right) t^{l+1} - s^{l+1} - y_n(t-s) \right\}. \tag{3.13}
\]

There exists a \( B > 0, \) independent of \( n, \) such that for \( s, < t, < a, \)
\[
\frac{(z_n - z_{n-1})(t^{l+1} - s^{l+1})}{l+1} = \frac{(z_n - z_{n-1})(t-s)}{l+1} \sum_{j=0}^{l} t^{l-j}s_j
\]
\[
\leq B(t-s)n^{h-1-rhl/(k-m)}.
\]

Hence, if we assume
\[
|h(qk - qm - rk) - k + m|/p(k-m) > h - 1 - rh/(k-m), \tag{3.14}
\]
we have for \( s, t, \leq a, \)
\[
\frac{w_n(t)}{w_{n-1}(t)} \leq n^{2C} \exp \{-B(t-s)n^{h(qk-qm-rk) - k + m}/p(k-m)\}.
\]

Then, if we assume
\[
\delta = \frac{h(qk - qm - rk) - k + m}{p(k-m)} - 1 - rh/(k-m) > 0, \tag{3.15}
\]
we have for large \( n, \) and for \( a, - r, \leq t, \leq a, \)
\[
\frac{w_n(t)}{w_{n-1}(t)} \leq n^{2C} \exp(-Bn^\delta) < 1/2. \tag{3.16}
\]

In the same way, we have for large \( n, \) and for \( b, \leq t, \leq b, + r, \)
\[
\frac{w_{n-1}(t)}{w_n(t)} \leq n^{2C} \exp(-Bn^\delta). \tag{3.17}
\]

Furthermore, from (3.4), (3.8) and (3.16), we have for large \( n, \) and for \( 0, \leq t, \leq a, \)
\[
w_n(t) \leq n^{2C} \exp(-Bn^\delta). \tag{3.18}
\]

Denote by \( n_0, \) a positive integer such that (3.16) and (3.17) are satisfied for \( n, \geq n_0. \)

Now we define \( u \) and \( f. \) The functions
\[
u_n(t, x) = w_n(t) \exp(-ix_n x), \quad n \geq 1, \tag{3.19}
\]
satisfy $Lu_n = 0$. Then we set
\[
    u(t, x) = u_n(t, x) = A \left( \frac{t - b_n}{4r_n} \right) u_{n-1}(t, x) + B \left( \frac{t - b_n}{4r_n} \right) u_n(t, x)
\]
\[
    = A \left( \frac{t - b_n}{4r_n} \right) u_{n-1}(t, x) + B \left( \frac{t - b_n}{4r_n} \right) u_n(t, x)
\]
\[
    = 0 \quad (a_{n+1} \leq t \leq a_n, n \geq n_0),
\]
\[
    = 0 \quad (t \leq 0),
\]
\[
    f(t, x) = Lu/u \quad (a_{n+1} \leq t \leq b_n + r_n \text{ or } a_n - r_n \leq t \leq a_n, n \geq n_0),
\]
\[
    = 0 \quad (t > a_{n_0} \text{ or } b_n + r_n < t < a_n - r_n, n \geq n_0 \text{ or } t \leq 0),
\]
where $A(s)$ and $B(s)$ are $C^\infty$-functions satisfying
\[
    A(s) = 0 \text{ for } s < 1/6, \quad A(s) = 1 \text{ for } s > 1/5, \quad 0 \leq A(s) \leq 1, \quad (3.22)
\]
\[
    B(s) = 1 \text{ for } s < 4/5, \quad B(s) = 0 \text{ for } s > 5/6, \quad 0 \leq B(s) \leq 1. \quad (3.23)
\]
Then, as in the proof of Lemma 4 of [8] with
\[
    v = u_{n-1}, \quad z = u_n, \quad a = a_n, \quad b = b_n, \quad e = a_{n+1}, \quad r = r_n,
\]
\[
    e = n^2 \exp(-Bn^\delta),
\]
we conclude that $u$ and $f$ are $C^\infty$ under assumptions (3.14) and (3.15).

Condition (3.14) is equivalent to
\[
    (p - q)(k - m) \leq r(pl - k) \quad (3.24)
\]
or
\[
    (p - q)(k - m) > r(pl - k), \quad h < \frac{(p - 1)(k - m)}{(p - q)(k - m) + r(k - pl)}. \quad (3.25)
\]
Condition (3.15) is equivalent to
\[
    m < k - r(k + p)/q, \quad h > \frac{(p + 1)(k - m)}{q(k - m) - r(k + p)}. \quad (3.26)
\]
First we assume $p > q$. Then, in order that (3.14) and (3.15) are satisfied, the following condition is necessary and sufficient:
\[
    \left\{ \begin{array}{l}
    k - r(pl - k)/(p - q) \leq m < k - r(k + p)/q, \\
    h > \frac{(p + 1)(k - m)}{q(k - m) - r(k + p)},
\end{array} \right. \quad (3.27)
\]
or
\[
\begin{aligned}
m &< k - r\frac{(p - q)}{(p - q)} \frac{m}{r}, \quad (3.29) \\
\frac{(p + 1)(k - m)}{q(k - m) - r(k + p)} < h < \frac{(p - 1)(k - m)}{r(k - p)}.
\end{aligned}
\]

It is easy to see that the necessary and sufficient condition for us to be able to choose such \( h > 0 \) is (1.9)-(1.12).

In the same way, when \( p = q \), the condition "(3.14) and (3.15)" is equivalent to
\[
\begin{aligned}
k &\leq p, \\
m &< k - r\frac{(p + p)}{p}, \\
h &> \frac{(p + 1)(k - m)}{p(k - m) - r(k + p)},
\end{aligned}
\]
or
\[
\begin{aligned}
k &> p, \\
m &< k - r\frac{(p + p)}{p}, \\
\frac{(p + 1)(k - m)}{p(k - m) - r(k + p)} < h < \frac{(p - 1)(k - m)}{r(k - p)}.
\end{aligned}
\]

The necessary and sufficient condition for us to be able to choose such \( h \) is equivalent to (1.13) or (1.14). This completes the proof of Theorem 2.

4. Proof of Theorem 3

In order to prove Theorem 3, we have to estimate the derivatives of \( u \) and \( f \) more precisely than in Section 3. First we estimate \( \partial_t \partial_x u(t, x) \).

In virtue of (3.18), (3.19) and (3.20), there exists a \( C_1 > 0 \) such that
\[
|\partial_x u(t, x)| \leq C_1 n^{h s + 2c} \exp(-Bn^6), 
\]
for \( a_{n+1} \leq t \leq a_n, n \geq n_0 \). We can easily show that, for any \( n \),
\[
n^{h s + 2c} \exp(-Bn^6) \leq \left( \frac{h s + 2c}{eB\delta} \right)^{(h s + 2c)/\delta} 
\]
From (4.1), (4.2) and Stirling's formula, there exists an \( M_0 > 0 \) independent of \( s \) such that the following holds
\[
|\partial_x u(t, x)| \leq M_0 s^{h s / \delta} \quad (t \leq a_n). 
\]
In the same way, by using (3.5), we can show
\[ |\partial_t^j \partial_x^s u(t, x)| \leq M_j s!^{\delta/\beta} \quad (t \leq a_{n_0}), \tag{4.4} \]
where \( M_j \) \((j \geq 1)\) are other constants. This implies \( u \in \gamma^{(h/\delta)} \).

From (3.21), we have only to estimate \( f \) only on \( a_{n+1} \leq t \leq b_n + r_n \) or on \( a_n - r_n \leq t \leq a_n \). From (3.22) and (3.23), we have for \( a_n - r_n \leq t \leq a_n \),
\[ f = L(Bu_n)/u \]
and from (3.16), (3.18) and (3.3), \(|u| > |u_{n-1}|/2 \geq |u| > 0\) there. It is easy to see that
\[ \partial_x^j(1/u) = \sum_{i=1}^{j} (-1)^i u^{-i-1} \sum_{p_1 + \cdots + p_i = j} \frac{j!}{p_1! \cdots p_i!} \partial_x^{p_1} u \cdots \partial_x^{p_i} u. \]

Then,
\[
|\partial_x^j(1/u)| \leq \frac{1}{|u|} \sum_{i=1}^{j} \sum_{p_1 + \cdots + p_i = j} \frac{j!}{p_1! \cdots p_i!} \frac{|\partial_x^{p_1} u|}{u} \cdots \frac{|\partial_x^{p_i} u|}{u} \\
\leq \frac{C z_n^j}{|u|} \sum_{i=1}^{j} \sum_{p_1 + \cdots + p_i = j} \frac{j!}{p_1! \cdots p_i!} i^j \\
\leq \frac{C z_n^j}{|u|} j^j.
\]

Hence, we have for \( a_n - r_n \leq t \leq a_n \),
\[
|\partial_x^s f(t, x)| \leq \sum_{j=0}^{s} \binom{s}{j} |\partial_x^{s-j} L(Bu_n)| |\partial_x^j(1/u)| \\
\leq C' \sum_{j=0}^{s} \binom{s}{j} \frac{2 \partial_x^{s-j} L(Bu_n)}{|u_{n-1}|} z_n^j \\
\leq C'' \sum_{j=0}^{s} \binom{s}{j} j^j \frac{w_n(t)}{w_{n-1}(t)} \\
\leq C'' n^{s+2} \exp(-Bn^\delta) \sum_{j=0}^{s} \binom{s}{j} j^j \\
\leq C'' \rho^s s!^{1+\delta/\beta},
\]
where \( C, C', C'' \) and \( \rho > 0 \). The same holds for \( a_{n+1} \leq t \leq b_n + r_n \). This implies \( f \in \gamma^{(1 + h/\delta)} \).

Recall that

\[
\frac{h}{\delta} = \frac{ph(k - m)}{h(qk - qm - rk - rp) - (p + 1)(k - m)}
\]

and \( h \) must satisfy (3.28), (3.30), (3.33) or (3.36) in each case. Hence we have

\[
\frac{h}{\delta} > \frac{p(k - m)}{k(q - r) - qm - rp}, \quad \text{if (1.9) or (1.13) is satisfied,}
\]

\[
> \frac{(k - m)(p - 1)}{(2q - p - 1)(k - m) + r(pl + l + 1 - p - 2k)}, \quad \text{otherwise.}
\]

This completes the proof of Theorem 3.

REFERENCES

4. V. JA. IVRII, Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity for Gevrey class, Sib. Mat. Zh. 17 (1976), 1256–1270.
13. K. Watanabe, Sur l'unicité du prolongement des solutions des équations elliptiques

equations with characteristics of constant multiplicity, *J. Differential Equations* 24

15. M. Zeman, Uniqueness of solutions of the Cauchy problem for linear partial differential
equations with characteristics of variable multiplicity, *J. Differential Equations* 27
(1978), 1–18.