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Near minimally normed spline quasi-interpolants on uniform partitions

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Abstract

Spline quasi-interpolants (QIs) are local approximating operators for functions or discrete data. We consider the construction of discrete and integral spline QIs on uniform partitions of the real line having small infinity norms. We call them near minimally normed QIs: they are exact on polynomial spaces and minimize a simple upper bound of their infinity norms. We give precise results for cubic and quintic QIs. Also the QI error is considered, as well as the advantage that these QIs present when approximating functions with isolated discontinuities.

Keywords: B-splines; Discrete quasi-interpolants; Integral quasi-interpolants; Infinity norm

1. Introduction

Usually, the construction of spline approximants requires the solution of linear systems. Spline quasi-interpolants (QIs) are local approximants avoiding this problem, so they are very convenient in practice. For a given nondecreasing biinfinite sequence $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$ such that $|t_i| \to +\infty$ as $i \to \pm\infty$, and $t_i < t_{i+k}$ for all i, let $N_{i,k}$ be the ith B-spline of order $k \in \mathbb{N}$, and $S_{k,\mathbf{t}}$ the linear space spanned by these B-splines (see e.g. [28]). In [4,6], QIs $Qf = \sum_{i\in\mathbb{Z}} \lambda_i(f)N_{i,k}$ were constructed, where the linear form λ_i uses both functional and derivative values (see [5, Chapter XII; 13, Chapter 5; 28, Chapter 6], as well as [21,27] for

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the uniform case). In [17], another B-spline QI method is defined with λ_i involving functional values in a divided difference scheme. A discrete QI (dQI) is obtained when $\lambda_i(f)$ is a finite linear combination of functional values. This problem has been considered in [10] (see also [21–24]). When $\lambda_i(f)$ is the inner product of f with a linear combination of B-splines, Q is an integral QI (iQI). For details on this topic, see for example [12,14,20,24–26] (and references therein).

All these QIs are exact on the space $\mathbb{P}_{k-1}$ of polynomials of degree at most $k-1$ (or a subspace of $\mathbb{P}_{k-1}$). In [16], one can find a general construction of univariate spline QIs reproducing the spline space $S_{k,\mathbf{t}}$.

Once the QI is constructed, a standard argument (see e.g., [13, p. 144]) shows that, if $\mathscr{S}$ is the reproduced space, then $\|f - Qf\|_\infty \leqslant (1 + \|Q\|_\infty)\mathrm{dist}(f, \mathscr{S})$. This leads us to the construction of QIs with minimal infinity norm (see [7, p. 73, 21] in the box-spline setting). This problem has been considered in the discrete case in [1,15].

Here, we are interested in spline QIs Q on uniform partitions of the real line such that (a) $\lambda_i(f) := \lambda(f(\cdot + i))$ is either a linear combination of values of f at points lying in a neigbourhood of the support of the ith B-spline, or the inner product of f with a linear combination of B-splines; (b) Q reproduces the polynomials in the spline space; and (c) a simple upper bound of the infinity norm of Q is minimized.

The paper is organized as follows. In Section 2, we establish the exactness conditions on polynomials of the dQIs or iQIs. In Section 3, we define a minimization problem whose solution will be called near minimally normed (NMN) dQI or iQI. In Sections 4 and 5, we describe some NMN cubic and quintic dQIs and iQIs respectively. In Section 6, we establish some error bounds for cubic dQIs and iQIs. In Section 7, we show that these QIs diminish the overshoot when applying them to the Heaviside function, so seem suitable for the approximation of functions with isolated discontinuities.

Only the even order B-splines are considered here, because the results for the others are similar.

2. Discrete and integral QIs on uniform partitions

Consider the sequence $\mathbf{t} = \mathbb{Z}$ of integer knots. Let $M := M_{2n}$ be the B-spline of even order $2n$, $n \geqslant 2$, with support $[-n, n]$ (see e.g., [27,28]). We deal with discrete and integral spline QI operators

$$Qf = \sum_{i \in \mathbb{Z}} \lambda_i(f) M_i,$$

where $M_i := M(\cdot - i)$, and the linear form λ_i has one of the two following forms

(i) $\lambda_i(f) = \sum_{j=-m}^{m} \gamma_j f(i-j)$ when Q is a dQI, and
(ii) $\lambda_i(f) = \sum_{j=-m}^{m} \gamma_j \langle f, M_{i-j}\rangle$ when Q is an iQI, with $\langle f, g\rangle := \int_{\mathbb{R}} fg$

for $m \geqslant n$ and $\gamma_j \in \mathbb{R}$, $-m \leqslant j \leqslant m$.

Defining the fundamental function

$$L := L_{2n,m} = \sum_{j=-m}^{m} \gamma_j M_j, \tag{1}$$

then Qf can be written in the form

$$Qf := Q_{2n,m}f = \sum_{i\in\mathbb{Z}} \chi_i(f)L(\cdot - i), \tag{2}$$

where $\chi_i(f)$ is equal to $f(i)$ or $\langle f, M_i\rangle$. It is clear that L is symmetric with respect to the origin if and only if $\gamma_{-j} = \gamma_j$, $j = 1, 2, \ldots, m$. In the sequel we only consider *symmetric* dQIs

$$Qf = \sum_{i\in\mathbb{Z}} \left(\gamma_0 f(i) + \sum_{j=1}^{m} \gamma_j (f(i+j) + f(i-j))\right) M_i \tag{3}$$

and *symmetric* iQIs

$$Qf = \sum_{i\in\mathbb{Z}} \left(\gamma_0 \langle f, M_i\rangle + \sum_{j=1}^{m} \gamma_j (\langle f, M_{i-j}\rangle + \langle f, M_{i+j}\rangle)\right) M_i. \tag{4}$$

Usually, the coefficients γ_j are determined in such a way that the operator Q be exact on the space $\mathbb{P}_{2n-1}$ of polynomials of degree at most $2n-1$. In order to determine the linear constraints that are equivalent to the exactness of Q on $\mathbb{P}_{2n-1}$, we need the expressions of the monomials $e_k(x) := x^k$, $k=0, 1, \ldots, 2n-1$, as linear combinations of the integer translates of the B-spline M (see [9, Theorem 6.2.1, p. 464]).

Proposition 1. *For $k = 0, 1, \ldots, n-1$, there hold*

$$e_{2k} = \sum_{i\in\mathbb{Z}} \left\{ i^{2k} + \sum_{l=1}^{k} \frac{(2k)!}{(2k-2l)!} \beta(2l, 2n) i^{2k-2l} \right\} M_i, \tag{5}$$

$$e_{2k+1} = \sum_{i\in\mathbb{Z}} \left\{ i^{2k+1} + \sum_{l=1}^{k} \frac{(2k+1)!}{(2k+1-2l)!} \beta(2l, 2n) i^{2k+1-2l} \right\} M_i, \tag{6}$$

where the numbers $\beta(2l, 2n)$ are provided by the expansion

$$\left(\frac{x/2}{\sin(x/2)}\right)^{2n} = \sum_{l=0}^{\infty} (-1)^l \beta(2l, 2n) x^{2l}. \tag{7}$$

Remark 2. In [27], the coefficients in (7) are denoted by $\gamma_{2l}^{(2n)}$, and the fact that $(-1)^l 2^l (2l)!\, \gamma_{2l}^{(2n)}$ is a polynomial at n of degree l is indicated. On the other hand, in [9, Prop. 6.2.1, p. 464], it was proved that these coefficients are related to the central factorial numbers $t(r, s)$ of the first kind (see [8,9]). More specifically, for $0 \leqslant l \leqslant n-1$ we have (cf. [9, p. 469]).

$$\beta(2l, 2n) = \frac{(2n-1-2l)!}{(2n-1)!} t(2n, 2n-2l).$$

Tables 1 and 2 show some values of $t(2n, 2l)$ and $\beta(2l, 2n)$ respectively.

Table 1
Some values of the central factorial numbers of the first kind $t(2n, 2l)$

l	n = 0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	1	−1	4	−36	576	−14 400	518 400	−25 401 600	1 625 702 400
2	0	0	1	−5	49	−820	21 076	−773 136	38 402 064	−2 483 133 696
3	0	0	0	1	−14	273	−7645	296 296	−15 291 640	1 017 067 024
4	0	0	0	0	1	−30	1023	−44 473	2 475 473	−173 721 912
5	0	0	0	0	0	1	−55	3003	−191 620	14 739 153
6	0	0	0	0	0	0	1	−91	7462	−669 188
7	0	0	0	0	0	0	0	1	−140	16 422
8	0	0	0	0	0	0	0	0	1	−204
9	0	0	0	0	0	0	0	0	0	1

Table 2
Values of $\beta(2l, 2n)$ for $2 \leqslant n \leqslant 9$

n	l = 0	1	2	3	4	5	6	7	8
2	1	$\frac{-1}{6}$							
3	1	$\frac{-1}{4}$	$\frac{1}{30}$						
4	1	$\frac{-1}{3}$	$\frac{7}{120}$	$\frac{-1}{140}$					
5	1	$\frac{-5}{12}$	$\frac{13}{144}$	$\frac{-41}{3024}$	$\frac{1}{630}$				
6	1	$\frac{-1}{2}$	$\frac{31}{240}$	$\frac{-139}{6048}$	$\frac{479}{151\,200}$	$\frac{-1}{2772}$			
7	1	$\frac{-7}{12}$	$\frac{7}{40}$	$\frac{-311}{8640}$	$\frac{37}{6480}$	$\frac{-59}{79\,200}$	$\frac{1}{12\,012}$		
8	1	$\frac{-2}{3}$	$\frac{41}{180}$	$\frac{-67}{1260}$	$\frac{2473}{259\,200}$	$\frac{-4201}{2\,993\,760}$	$\frac{266\,681}{1\,513\,512\,000}$	$\frac{-1}{51\,480}$	
9	1	$\frac{-3}{4}$	$\frac{23}{80}$	$\frac{-757}{10\,080}$	$\frac{2021}{134\,400}$	$\frac{-4679}{1\,900\,800}$	$\frac{3\,739\,217}{10\,897\,286\,400}$	$\frac{-63\,397}{1\,513\,512\,000}$	$\frac{1}{218\,790}$

In order to express the exactness conditions of Q on $\mathbb{P}_{2n-1}$ in terms of the parameters γ_j, we define

$$\Gamma_0 = \gamma_0 + 2\sum_{j=1}^{m} \gamma_j,$$

$$\Gamma_{2l} = 2\sum_{j=1}^{m} j^{2l}\gamma_j, \quad 1 \leqslant l \leqslant n-1.$$

Lemma 3. *For each* $d \leqslant n-1$, *the dQI* Q *given by* (3) *reproduces the monomials* e_{2k}, $0 \leqslant k \leqslant d$, *if and only if*

$$\Gamma_{2l} = (2l)!\,\beta(2l, 2n), \quad l = 0, 1, \ldots, d. \tag{8}$$

Proof. We have

$$\begin{aligned} Qe_{2k} &= \sum_{i \in \mathbb{Z}} \left\{ \gamma_0 i^{2k} + \sum_{j=1}^{m} \gamma_j ((i+j)^{2k} + (i-j)^{2k}) \right\} M_i \\ &= \sum_{i \in \mathbb{Z}} \left\{ \Gamma_0 i^{2k} + \sum_{l=1}^{k} \binom{2k}{2l} \Gamma_{2l} i^{2k-2l} \right\} M_i. \end{aligned}$$

According to (5) of Proposition 1, Q reproduces $e_0, e_2, \ldots,$ and e_{2d} if and only if

$$\binom{2k}{2l} \Gamma_{2l} = \frac{(2k)!}{(2k-2l)!} \beta(2l, 2n), \quad l = 0, 1, \ldots, d,$$

and the claim follows. □

The exactness relations for the iQI Q given by (4) are more involved. They need the moments of the B-spline M, denoted by

$$\mu_l(2n) := \int_{\mathbb{R}} e_l(x) M(x)\, \mathrm{d}x, \quad l \geqslant 0.$$

In particular, $\mu_0(2n) = 1$, and $\mu_{2l+1}(2n) = 0$ for $l \geqslant 0$.

Lemma 4. *For each* $d \leqslant n-1$, *the iQI* Q *given by* (4) *reproduces the monomials* e_{2k}, $0 \leqslant k \leqslant d$, *if and only if*

$$\sum_{r=0}^{k} \binom{2k}{2r} \Gamma_{2r} \mu_{2k-2r}(2n) = (2k)!\,\beta(2k, 2n), \quad k = 0, 1, \ldots, d. \tag{9}$$

Proof. Let $d = 0$. Since

$$\lambda_i(e_0) = \gamma_0 \langle e_0, M_i \rangle + \sum_{j=1}^{m} \gamma_j \langle e_0, M_{i-j} + M_{i+j} \rangle = \Gamma_0,$$

Q reproduces e_0 if and only if $\Gamma_0 = 1$. Let us assume that Q reproduces the monomials $e_0, e_2, \ldots,$ and e_{2d} if and only if (9) holds. We will prove that $Qe_{2k} = e_{2k}$ for $k = 0, 1, \ldots, d+1$ if and only if

$$\sum_{r=0}^{k} \binom{2k}{2r} \Gamma_{2r} \mu_{2k-2r}(2n) = (2k)!\beta(2k, 2n), \quad k = 0, 1, \ldots, d+1.$$

Let us assume that $Qe_{2k}=e_{2k}$, $k=0,1,\ldots,d+1$. By hypothesis, (9) holds, and moreover $Qe_{2d+2}=e_{2d+2}$. We have

$$Qe_{2d+2}=\sum_{i\in\mathbb{Z}}\lambda_i(e_{2d+2})M_i,$$

where

$$\lambda_i(e_{2d+2})=\gamma_0\langle e_{2d+2},M_i\rangle+\sum_{j=1}^{m}\gamma_j\langle e_{2d+2},M_{i-j}+M_{i+j}\rangle.$$

Since

$$\langle e_{2d+2},M_i\rangle=\int_{\mathbb{R}}x^{2d+2}M_i\,\mathrm{d}x=\int_{\mathbb{R}}(x+i)^{2d+2}M(x)\,\mathrm{d}x$$

and

$$\begin{aligned}\langle e_{2d+2},M_{i-j}+M_{i+j}\rangle&=\int_{\mathbb{R}}x^{2d+2}(M_{i-j}(x)+M_{i+j}(x))\,\mathrm{d}x\\&=\int_{\mathbb{R}}\{(x+i+j)^{2d+2}+(x+i-j)^{2d+2}\}M(x)\,\mathrm{d}x\\&=\sum_{l=0}^{d+1}\binom{2d+2}{2l}2j^{2l}\gamma_j\int_{\mathbb{R}}(x+i)^{2d+2-2l}M(x)\,\mathrm{d}x,\end{aligned}$$

we deduce that

$$\lambda_i(e_{2d+2})=\Gamma_0\int_{\mathbb{R}}(x+i)^{2d+2}M(x)\,\mathrm{d}x+\sum_{l=1}^{d+1}\binom{2d+2}{2l}\Gamma_{2l}\int_{\mathbb{R}}(x+i)^{2d+2-2l}M(x)\,\mathrm{d}x.$$

By using the moments of M, we get

$$\int_{\mathbb{R}}(x+i)^{2d+2}M(x)\,\mathrm{d}x=\sum_{r=0}^{d+1}\binom{2d+2}{2r}\mu_{2r}(2n)i^{2d+2-2r}$$

and

$$\int_{\mathbb{R}}(x+i)^{2d+2-2l}M(x)\,\mathrm{d}x=\sum_{s=0}^{d+1-l}\binom{2d+2-2l}{2s}\mu_{2s}(2n)i^{2d+2-2l-2s}.$$

As $\Gamma_0 = 1$, we obtain

$$\begin{aligned}
\lambda_i(e_{2d+2}) &= i^{2d+2} + \sum_{r=1}^{d+1} \binom{2d+2}{2r} \mu_{2r}(2n) i^{2d+2-2r} \\
&\quad + \sum_{l=1}^{d+1} \binom{2d+2}{2l} \Gamma_{2l} \sum_{s=0}^{d+1-l} \binom{2d+2-2l}{2s} \mu_{2s}(2n) i^{2d+2-2l-2s} \\
&= i^{2d+2} + \sum_{r=1}^{d+1} \binom{2d+2}{2r} \mu_{2r}(2n) i^{2d+2-2r} \\
&\quad + \sum_{r=0}^{d} \left\{ \sum_{s=0}^{r} \binom{2d-2s}{2r-2s} \binom{2d+2}{2s+2} \mu_{2r-2s}(2n) \Gamma_{2s+2} \right\} i^{2d-2r} \\
&= i^{2d+2} + \sum_{r=0}^{d} \left\{ \binom{2d+2}{2r} \mu_{2r+2}(2n) \right. \\
&\quad \left. + \sum_{s=0}^{r} \binom{2d-2s}{2r-2s} \binom{2d+2}{2s+2} \mu_{2r-2s}(2n) \Gamma_{2s+2} \right\} i^{2d-2r}.
\end{aligned}$$

Taking into account that

$$\binom{2d-2s}{2r-2s} \binom{2d+2}{2s+2} = \binom{2d+2}{2r+2} \binom{2r+2}{2s+2},$$

we have

$$\lambda_i(e_{2d+2}) = i^{2d+2} + \sum_{r=0}^{d} \binom{2d+2}{2r+2} T_r i^{2d-2r}$$

with

$$T_r = \mu_{2r+2}(2n) + \sum_{s=0}^{r} \binom{2r+2}{2s+2} \mu_{2r-2s}(2n) \Gamma_{2s+2}.$$

By hypothesis,

$$\begin{aligned}
\lambda_i(e_{2d+2}) &= i^{2d+2} + \sum_{r=0}^{d-1} \binom{2d+2}{2r+2} (2r+2)! \beta(2r+2, 2n) i^{2d-2r} \\
&\quad + \left\{ \mu_{2d+2}(2n) + \sum_{s=0}^{d} \binom{2d+2}{2s+2} \mu_{2d-2s}(2n) \Gamma_{2s+2} \right\} \\
&= i^{2d+2} + \sum_{r=1}^{d} \frac{(2d+2)!}{(2d+2-2r)!} \beta(2r, 2n) i^{2d+2-2r} \\
&\quad + \left\{ \mu_{2d+2}(2n) + \sum_{s=0}^{d} \binom{2d+2}{2s+2} \mu_{2d-2s}(2n) \Gamma_{2s+2} \right\}.
\end{aligned}$$

From (5), we find that

$$e_{2d+2} = \sum_{i\in\mathbb{Z}} \left\{ i^{2d+2} + \sum_{l=1}^{d+1} \frac{(2d+2)!}{(2d+2-2l)!}\beta(2l, 2n) i^{2d+2-2l} \right\} M_i .$$

Hence, $Qe_{2d+2} = e_{2d+2}$ implies that

$$\mu_{2d+2}(2n) + \sum_{s=0}^{d} \binom{2d+2}{2s+2} \mu_{2d-2s}(2n) \Gamma_{2s+2} = (2d+2)!\beta(2d+2, 2n).$$

This is the equality obtained by taking $k = d+1$ in (9).

The proof of the converse implication is similar. □

As a consequence of the symmetry properties, the dQIs and iQIs also reproduce the monomials of odd degree.

Lemma 5. *Let Q be the dQI given by* (3) *or the iQI given by* (4). *Then, for each $d < n$, the fact that $Qe_{2k} = e_{2k}$, $k = 0, 1, \ldots, d$, implies that $Qe_{2d+1} = e_{2d+1}$.*

Proof. Let Q be the dQI given by (3). Then,

$$\begin{aligned} Qe_{2d+1} &= \sum_{i\in\mathbb{Z}} \left\{ \gamma_0 i^{2d+1} + \sum_{j=1}^{m} \gamma_j ((i+j)^{2d+1} + (i-j)^{2d+1}) \right\} M_i \\ &= \sum_{i\in\mathbb{Z}} \left\{ \Gamma_0 i^{2d+1} + \sum_{l=1}^{d} \binom{2d+1}{2l} \Gamma_{2l} i^{2d+1-2l} \right\} M_i . \end{aligned}$$

According to (8), we have

$$Qe_{2d+1} = \sum_{i\in\mathbb{Z}} \left\{ i^{2d+1} + \sum_{l=1}^{d} \frac{(2d+1)!}{(2d+1-2l)!} \beta(2l, 2n) i^{2d+1-2l} \right\} M_i .$$

Finally, by (6), $Qe_{2d+1} = e_{2d+1}$.

Now, let Q be the iQI given by (4). As in the proof of Lemma 4,

$$Qe_{2d+1} = \sum_{i\in\mathbb{Z}} \left\{ i^{2d+1} + \sum_{r=0}^{d-1} \binom{2d+1}{2r+2} T_r i^{2d-2r-1} \right\} M_i .$$

Using (9), we get

$$\begin{aligned} Qe_{2d+1} &= \sum_{i\in\mathbb{Z}} \left\{ i^{2d+1} + \sum_{r=0}^{d-1} \binom{2d+1}{2r+2} (2r+2)!\beta(2r+2, 2n) i^{2d-2r-1} \right\} M_i \\ &= \sum_{i\in\mathbb{Z}} \left\{ i^{2d+1} + \sum_{r=1}^{d} \frac{(2d+1)!}{(2d+1-2r)!} \beta(2r, 2n) i^{2d+1-2r} \right\} M_i \end{aligned}$$

and $Qe_{2d+1} = e_{2d+1}$ follows from (6). □

Remark 6. The moments of a B-spline can be expressed in terms of central factorial numbers $T(p, q)$ of the second kind (cf. [9, p. 423]). For the even order B-spline, this relationship is

$$\mu_{2l}(2n) = \frac{T(2l + 2n, 2n)}{\binom{2l+2n}{2n}}.$$

Tables 3 and 4 show some values of $T(2n, 2l)$ and $\mu_{2l}(2n)$, respectively.

Let $\mathbf{\Gamma} := (\Gamma_0, \Gamma_2, \ldots, \Gamma_{2n-2})^{\mathrm{T}}$, and

$$\boldsymbol{\beta} := (1, 2!\beta(2, 2n), \ldots, (2n-2)!\beta(2n-2, 2n))^{\mathrm{T}}.$$

Table 3
Some values of the central factorial numbers of the second kind $T(2n, 2l)$

l	n										
	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1	1
2	0	0	1	5	21	85	341	1365	5461	21 845	87 381
3	0	0	0	1	14	147	1408	13 013	118 482	1 071 799	9 668 036
4	0	0	0	0	1	30	627	11 440	196 053	3 255 330	53 157 079
5	0	0	0	0	0	1	55	2002	61 490	1 733 303	46 587 905
6	0	0	0	0	0	0	1	91	5278	251 498	10 787 231
7	0	0	0	0	0	0	0	1	140	12 138	846 260
8	0	0	0	0	0	0	0	0	1	204	25 194
9	0	0	0	0	0	0	0	0	0	1	285
10	0	0	0	0	0	0	0	0	0	0	1

Table 4
Values of $\mu(2l, 2n)$ for $2 \leqslant n \leqslant 9$

n	l								
	0	1	2	3	4	5	6	7	8
2	1	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{17}{42}$	$\frac{31}{45}$	$\frac{15}{11}$	$\frac{54\,612}{1820}$	$\frac{257}{36}$	$\frac{1533}{85}$
3	1	$\frac{1}{2}$	$\frac{7}{10}$	$\frac{32}{21}$	$\frac{13}{3}$	$\frac{651}{44}$	$\frac{63\,047}{1092}$	$\frac{7483}{30}$	
4	1	$\frac{2}{3}$	$\frac{19}{15}$	$\frac{80}{21}$	$\frac{457}{30}$	$\frac{2455}{33}$	$\frac{164\,573}{390}$		
5	1	$\frac{5}{6}$	2	$\frac{215}{28}$	$\frac{713}{18}$	$\frac{11\,095}{44}$			
6	1	1	$\frac{29}{10}$	$\frac{569}{49}$	$\frac{2569}{30}$				
7	1	$\frac{7}{6}$	$\frac{119}{30}$	$\frac{131}{6}$					
8	1	$\frac{4}{3}$	$\frac{26}{5}$						
9	1	$\frac{3}{2}$							

Let $A := (a_{ij})_{1\leqslant i,j\leqslant n}$ be the lower triangular matrix given by

$$a_{r+s,s} = \binom{2r+2s-2}{2s-2}\mu_{2r}(2n), \quad r = 0, 1, \ldots, n-1, \; s = 1, 2, \ldots, n-r.$$

It is clear that $a_{ii} = 1$, $i = 1, 2, \ldots, n$. Its inverse $B = (b_{i,j})_{1\leqslant i,j\leqslant n}$ is also a lower triangular matrix, and $b_{ii} = 1$.

Proposition 7. (1) *The dQI Q given by* (3) *is exact on* $\mathbb{P}_{2n-1}$ *if and only if*

$$\gamma_0 + 2\sum_{j=1}^{m}\gamma_j = 1 \quad \textit{and} \quad 2\sum_{j=1}^{m} j^{2l}\gamma_j = (2l)!\beta(2l, 2n), \quad 1\leqslant l\leqslant n-1. \tag{10}$$

(2) *The iQI Q given by* (4) *is exact on* $\mathbb{P}_{2n-1}$ *if and only if*

$$\gamma_0 + 2\sum_{j=1}^{m}\gamma_j = 1,$$

$$2\sum_{j=1}^{m} j^{2l}\gamma_j = \sum_{r=1}^{l+1}(2r-2)!b_{l+1,r}\beta(2r-2, 2n), \quad 1\leqslant l\leqslant n-1. \tag{11}$$

Proof. Eqs. (10) follow from (8) of Lemma 3 with $d = n-1$. According to Eqs. (9) given in Lemma 4, the iQI (4) is exact on $\mathbb{P}_{2n-1}$ if and only if $A\boldsymbol{\Gamma} = \boldsymbol{\beta}$. Equivalently, $\boldsymbol{\Gamma} = B\boldsymbol{\beta}$, and (11) follows. □

Eqs. (10) and (11) show that the exactness on $\mathbb{P}_{2n-1}$ of the dQIs or iQIs is expressed by linear systems having a common fixed matrix while the right-hand side depends on the type of the QI considered. Therefore, we jointly study the construction of NMN dQIs and iQIs.

3. Near minimally normed discrete and integral QIs

Once the exactness conditions have been established for a symmetric dQI or iQI Q, we consider a simple upper bound of its infinity norm.

Let $\boldsymbol{\gamma} := (\gamma_0, \gamma_1, \ldots, \gamma_m)^{\mathrm{T}}$, and $v(\boldsymbol{\gamma}) := v_{2n,m}(\boldsymbol{\gamma}) = |\gamma_0| + 2\sum_{j=1}^{m}|\gamma_j|$. Let f be a function such that $\|f\|_\infty \leqslant 1$. From (2), we have

$$|Qf| \leqslant \sum_{i\in\mathbb{Z}} |\chi_i(f)||L(\cdot - i)|.$$

As, $|\chi_i(f)| \leqslant 1$, we deduce that

$$|Qf| \leqslant \Lambda,$$

where Λ is the Lebesgue function

$$\Lambda := \Lambda_{2n,m} = \sum_{i\in\mathbb{Z}} |L(\cdot - i)|.$$

Hence, from (1), we conclude that

$$\|Q\|_\infty \leqslant \nu(\boldsymbol{\gamma}).$$

We propose the construction of dQIs and iQIs that minimize the bound $\nu(\boldsymbol{\gamma})$ under the linear constraints consisting of reproducing all monomials in $\mathbb{P}_{2n-1}$. In general, it is difficult to minimize the infinity norm of Q, which is equal to the Chebyshev norm of the Lebesgue function.

Let Q be the QI given by (3) (resp. (4)). Then, let us define

$$V := V_{2n,m} = \{\boldsymbol{\gamma} \in \mathbb{R}^{m+1} : \boldsymbol{\gamma} \text{ satisfies (10) (resp. (11))}\}. \tag{12}$$

Problem 8. Solve $\mathrm{Min}\{\nu(\boldsymbol{\gamma}), \boldsymbol{\gamma} \in V\}$.

Definition 9. Let $\boldsymbol{\gamma}$ be a solution of Problem 8. Then the corresponding discrete (resp. integral) QI given by (3) (resp. (4)) is said to be a NMN dQI (resp. iQI) of order $2n$ relative to m and exact on $\mathbb{P}_{2n-1}$.

Solving system (10) (resp. (11)) in $\gamma_0, \gamma_{m-n+2}, \ldots, \gamma_m$ by Cramer's rule yields

$$\gamma_i = \gamma_i^* - \sum_{j=1}^{m-n+1} c_{i,j}\gamma_j, \quad i \in \{0, m-n+2, \ldots, n\} \quad \text{and} \quad c_{i,j} \in \mathbb{R}, \tag{13}$$

where $(\gamma_0^*, \gamma_{m-n+2}^*, \ldots, \gamma_m^*)^{\mathrm{T}} \in \mathbb{R}^n$ is the unique solution of (10) (resp. (11)) when $\gamma_1 = \gamma_2 = \cdots = \gamma_{m-n+1} = 0$.

Let $\boldsymbol{\gamma}^* = (\gamma_0^*, 0, \ldots, 0, 2\gamma_{m-n+2}^*, \ldots, 2\gamma_m^*)^{\mathrm{T}} \in \mathbb{R}^{m+1}$. From (13),

$$\begin{aligned}\nu(\boldsymbol{\gamma}) &= \left|\gamma_0^* - \sum_{j=1}^{m-n+1} c_{0,j}\gamma_j\right| + \sum_{i=1}^{m-n+1} |2\gamma_i| + 2\sum_{k=n-m+2}^{m}\left|\gamma_i^* - \sum_{j=1}^{m-n+1} c_{i,j}\gamma_j\right| \\ &= \|\boldsymbol{\gamma}^* - G\widetilde{\boldsymbol{\gamma}}\|_1\end{aligned}$$

with $\widetilde{\boldsymbol{\gamma}} = (\gamma_1, \gamma_2, \ldots, \gamma_{m-n+1})^{\mathrm{T}}$, and

$$G = \begin{pmatrix} c_{0,1} & c_{0,2} & \cdots & c_{0,m-n+1} \\ 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \\ 2c_{m-n+2,1} & 2c_{m-n+2,2} & \cdots & 2c_{m-n+2,m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 2c_{m,1} & 2c_{m,2} & \cdots & 2c_{m,m-n+1} \end{pmatrix}. \tag{14}$$

Proposition 10. *$\boldsymbol{\gamma}$ is a solution of Problem 8 if and only if $G\widetilde{\boldsymbol{\gamma}}$ is a best linear l_1-approximation of $\boldsymbol{\gamma}^*$ in $\{G\mathbf{c} : \mathbf{c} \in \mathbb{R}^{m-n+1}\}$. Therefore, Problem 8 has at least one solution.*

The existence of NMN dQIs and iQIs is guaranteed, but not the uniqueness. A solution can be calculated by using the simplex method, because the minimization problem is equivalent to a linear programming one. Other methods can be used to determine a solution of the minimization problem (cf. [18]). In practice, a moderate value of n is used and moreover the higher decrease of the infinity norm occurs for small values of m. So, the minimization problem is of small size and it demands little computational effort. The computational cost of the resulting QI is not much bigger than the one of the classical QI.

4. Some examples of near minimally normed discrete QIs

In this section, we give some examples of the dQIs studied in the preceding sections. Concretely, we will consider the cubic and quintic cases.

4.1. Cubic discrete QIs

Let us consider the cubic case $n=2$. Relations (5) and (6) give

$$e_0=\sum_{i\in\mathbb{Z}} M_i,\quad e_1=\sum_{i\in\mathbb{Z}} iM_i,\quad e_2=\sum_{i\in\mathbb{Z}}\left(i^2-\frac{1}{3}\right)M_i,\quad \text{and}\quad e_3=\sum_{i\in\mathbb{Z}}(i^3-i)M_i.$$

Then, for $m\geqslant 2$, the dQI $Q_{4,m}$ is exact on $\mathbb{P}_3$ if and only if

$$\gamma_0+2\sum_{j=1}^{m}\gamma_j=1\quad\text{and}\quad \sum_{j=1}^{m} j^2\gamma_j=-\frac{1}{6}.$$

$\boldsymbol{\gamma}$ is a solution of this linear system if and only if

$$\gamma_0=1+\frac{1}{3m^2}-\sum_{j=1}^{m-1}2\left(1-\frac{j^2}{m^2}\right)\gamma_j\quad\text{and}\quad \gamma_m=-\frac{1}{6m^2}-\sum_{j=1}^{m-1}\frac{j^2}{m^2}\gamma_j.$$

Proposition 11. *Let $m\geqslant 1$ and $\widetilde{Q}_{4,m}$ be the cubic dQI given by*

$$\widetilde{Q}_{4,m}f=\sum_{i\in\mathbb{Z}}\left\{\left(1+\frac{1}{3m^2}\right)f(i)-\frac{1}{6m^2}(f(i+m)+f(i-m))\right\}M_i. \tag{15}$$

Then

(1) *For $m=1$, $\widetilde{Q}_{4,1}$ is the unique dQI exact on $\mathbb{P}_3$ among all the cubic dQIs given by* (3).
(2) *For each $m\geqslant 2$, $\widetilde{Q}_{4,m}$ is a NMN cubic dQI exact on $\mathbb{P}_3$.*
(3) *When $m\to+\infty$, $(\widetilde{Q}_{4,m})_{m\geqslant 1}$ converges in the infinity norm to the Schoenberg's operator $S_4: C(\mathbb{R})\to S_{4,\mathbb{Z}}$ defined by $S_4f=\sum_{i\in\mathbb{Z}}f(i)M_i$.*
(4) *The following equalities hold: $\|\widetilde{Q}_{4,1}\|_\infty=\frac{11}{9}\simeq 1.2222$, $\|\widetilde{Q}_{4,2}\|_\infty=\frac{41}{36}\simeq 1.1389$, and*

$$\|\widetilde{Q}_{4,m}\|_\infty=1+\frac{2}{3m^2}\quad for\ all\ m\geqslant 3.$$

Proof. (1) For $m=1$, the dQI

$$Qf=\sum_{i\in\mathbb{Z}}(\gamma_0 f(i)+\gamma_1(f(i+1)+f(i-1)))M_i$$

is exact on $\mathbb{P}_3$ if and only if $\gamma_0=\frac{4}{3}$ and $\gamma_1=-\frac{1}{6}$. Thus, $\widetilde{Q}_{4,1}$ is obtained.

(2) We will prove that $(1+\frac{1}{3m^2},0,\ldots,0,-\frac{1}{6m^2})^{\mathrm{T}} \in \mathbb{R}^{m+1}$ is a solution of Problem 8 (its associated cubic dQI is $\widetilde{Q}_{4,m}$). Consider the dQI given by (3). The objective function $v_{4,m}(\boldsymbol{\gamma}) = |\gamma_0| + 2\sum_{i=1}^{m} |\gamma_i|$ coincides on the affine subspace $V_{4,m}$ with the polyhedral convex function $\|g(\widetilde{\boldsymbol{\gamma}})\|_1$, where $g(\widetilde{\boldsymbol{\gamma}}) := \boldsymbol{\gamma}^* - G\widetilde{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma}^* = \left(1+\frac{1}{3m^2},0,\ldots,0,-\frac{1}{3m^2}\right)^{\mathrm{T}}$, and the matrix G given by (14) is

$$G = 2\begin{pmatrix} 1-\dfrac{1}{m^2} & 1-\dfrac{2^2}{m^2} & \cdots & 1-\dfrac{(m-1)^2}{m^2} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \dfrac{1}{m^2} & \dfrac{2^2}{m^2} & \cdots & \dfrac{(m-1)^2}{m^2} \end{pmatrix}. \tag{16}$$

Hence, $\left(1+\frac{1}{3m^2},0,\ldots,0,-\frac{1}{6m^2}\right)^{\mathrm{T}}$ is a solution of Problem 8 if and only if $\mathbf{0} \in \mathbb{R}^{m-1}$ minimizes $\|g(\widetilde{\boldsymbol{\gamma}})\|_1$. Let $\mathbf{v} = (v_0, v_1, \ldots, v_m)^{\mathrm{T}}$ be the vector of components $v_0 = 1$, $v_m = -1$ and, for $j = 1, \ldots, m-1$,

$$v_j = 2\,\frac{j^2}{m^2} - 1.$$

We have $g(\mathbf{0}) = \boldsymbol{\gamma}^*$. It is clear that $|v_j| \leqslant 1$, $0 \leqslant j \leqslant m-1$, $\operatorname{sgn} v_0 = \operatorname{sgn} \gamma_0^*$, $\operatorname{sgn} v_m = \operatorname{sgn} \gamma_m^*$, and $\mathbf{v}^{\mathrm{T}} G = 0$. Therefore (cf. [29, Theorem 1.7, p. 16]), g attains its minimum at $\mathbf{0} \in \mathbb{R}^{m-1}$, and this completes the proof of (2).

(3) Let $f \in C(\mathbb{R})$ such that $\|f\|_\infty \leqslant 1$. We have

$$\begin{aligned} |\widetilde{Q}_{4,m} f - S_4 f| &\leqslant \sum_{i\in\mathbb{Z}} \left| -\frac{1}{6m^2}\, f(i-m) + \frac{1}{3m^2}\, f(i) - \frac{1}{6m^2}\, f(i+m) \right| M_i \\ &\leqslant \sum_{i\in\mathbb{Z}} \left(\frac{2}{3m^2} \|f\|_\infty \right) M_i \\ &\leqslant \frac{2}{3m^2}. \end{aligned}$$

Hence, $\|\widetilde{Q}_{4,m} - S_4\| \to 0$ as $m \to +\infty$, and $(\widetilde{Q}_{4,m})_{m\geqslant 1}$ converges to S_4.

(4) For each $m \geqslant 4$, the supports of M_{-m}, M, and M_m have pairwise disjoint interiors. Therefore, the Lebesgue function $\widetilde{\Lambda}_{4,m}$ corresponding to $\widetilde{Q}_{4,m}$ can be written as

$$\widetilde{\Lambda}_{4,m} = \sum_{i\in\mathbb{Z}} \left(\frac{1}{6m^2}\, M_{i+m} + \frac{3m^2+1}{3m^2}\, M_i + \frac{1}{6m^2}\, M_{i-m} \right),$$

i.e., $\widetilde{\Lambda}_{4,m}$ is a constant function equal to $1 + \frac{2}{3m^2}$.

Let $m=1$. By using the Bernstein–Bézier representation (see e.g., [11, Chapter 1]), one can prove that

$$\|\widetilde{Q}_{4,1}\|_\infty = \widetilde{\Lambda}_{4,1}\left(\frac{1}{2}\right).$$

Thus, we obtain

$$\|\widetilde{Q}_{4,1}\|_\infty = 2\left(\left|\widetilde{L}_{4,1}\left(\frac{1}{2}\right)\right| + \left|\widetilde{L}_{4,1}\left(\frac{3}{2}\right)\right| + \left|\widetilde{L}_{4,1}\left(\frac{5}{2}\right)\right|\right).$$

Since M is a symmetric function supported on $[-2,2]$, $M(\frac{1}{2}) = \frac{13}{48}$, and $M(\frac{3}{2}) = \frac{1}{48}$, we get

$$\widetilde{L}_{4,1}\left(\frac{1}{2}\right) = \frac{5}{9},\ \widetilde{L}_{4,1}\left(\frac{3}{2}\right) = -\frac{5}{96}, \quad \text{and} \quad \widetilde{L}_{4,1}\left(\frac{5}{2}\right) = -\frac{1}{288}.$$

The claim follows.

The values of $\|\widetilde{Q}_{4,2}\|_\infty$ and $\|\widetilde{Q}_{4,3}\|_\infty$ can be computed in a similar way. □

Remark 12. A detailed study of $\|\boldsymbol{\gamma}^* - G\widetilde{\boldsymbol{\gamma}}\|_1$ shows that this function attains its absolute minimum at $\mathbf{0} \in \mathbb{R}^{m-1}$. Then, for $m \geqslant 2$ fixed, $\underset{\sim}{Q}_{4,m}$ is the unique NMN dQI exact on $\mathbb{P}_3$.

The sequence $(\|\widetilde{Q}_{4,m}\|_\infty)_{m\geqslant 1}$ is strictly decreasing, and the new cubic dQIs are also better than the classical $\widetilde{Q}_{4,1}$ with respect to the infinity norm.

4.2. Quintic discrete QIs

The quintic case ($n=3$) can be studied in a similar way. Now, we have

$$e_0 = \sum_{i\in\mathbb{Z}} M_i, \quad e_1 = \sum_{i\in\mathbb{Z}} iM_i, \quad e_2 = \sum_{i\in\mathbb{Z}}\left(i^2 - \frac{1}{2}\right)M_i,$$
$$e_3 = \sum_{i\in\mathbb{Z}}\left(i^3 - \frac{3}{2}i\right)M_i, \quad e_4 = \sum_{i\in\mathbb{Z}}\left(i^4 - 3i^2 + \frac{4}{5}i\right)M_i, \quad e_5 = \sum_{i\in\mathbb{Z}}(i^5 - 5i^3 + 4i)M_i.$$

For each $m \geqslant 3$, the dQI

$$Q_{6,m}f = \sum_{i\in\mathbb{Z}}\left(\gamma_0 f(i) + \sum_{j=1}^{m}\gamma_j(f(i+j) + f(i-j))\right)M_i$$

is exact on $\mathbb{P}_5$ if and only if

$$a_0 + 2\sum_{j=1}^{m}\gamma_j = 1, \quad \sum_{j=1}^{m} j^2\gamma_j = -\frac{1}{4} \quad \text{and} \quad \sum_{j=1}^{m} j^4\gamma_j = \frac{2}{5}.$$

The general solution of these equations satisfies

$$\gamma_{m-1} = \gamma_{m-1}^* - \sum_{j=1}^{m-2}\frac{j^2(m^2-j^2)}{(m-1)^2(2m-1)}\gamma_j, \quad \gamma_m = \gamma_m^* - \sum_{j=1}^{m-2}\frac{j^2(j^2-(m-1)^2)}{m^2(2m-1)}\gamma_j$$

and

$$\gamma_0 = \gamma_0^* - 2\sum_{j=1}^{m-2}\left(1 - \frac{j^2}{(m-1)^2m^2}\right)\gamma_j,$$

where

$$\gamma_0^* = 1 + \frac{8+5(m^2+(m-1)^2)}{10(m-1)^2m^2}, \quad \gamma_{m-1}^* = -\frac{5m^2+8}{20(m-1)^2(2m-1)}$$

and

$$\gamma_m^* = \frac{8+5(m-1)^2}{20m^2(2m-1)}.$$

So, minimizing the objective function

$$\nu_{6,m}(\boldsymbol{\gamma}) = |\gamma_0| + 2\sum_{j=1}^{m}|\gamma_j|$$

subject to the exactness conditions is equivalent to the unconstrained minimization of

$$\begin{aligned} g(\widetilde{\boldsymbol{\gamma}}) = {} & \left|\gamma_0^* - 2\sum_{j=1}^{m-2}\left(1-\frac{j^2}{(m-1)^2m^2}\right)\gamma_j\right| + 2\sum_{j=1}^{m-2}|\gamma_j| \\ & + 2\left|\gamma_{m-1}^* - \sum_{j=1}^{m-2}\frac{j^2(m^2-j^2)}{(m-1)^2(2m-1)}\gamma_j\right| + 2\left|\gamma_m^* - \sum_{j=1}^{m-2}\frac{j^2(j^2-(m-1)^2)}{m^2(2m-1)}\gamma_j\right|. \end{aligned}$$

The following result can be proved as in the cubic case.

Proposition 13. *Let* $m \geqslant 2$ *and* $\widetilde{Q}_{6,m} f = \sum_{i\in\mathbb{Z}} \widetilde{\lambda}_i(f) M_i$ *be the quintic dQI given by*

$$\widetilde{\lambda}_i(f) = \gamma_0^* f(i) + \gamma_{m-1}^*(f(i+m-1) + f(i-m+1)) + \gamma_m^*(f(i+m) + f(i-m)).$$

Then

(1) *For* $m = 2$, $\widetilde{Q}_{6,2}$ *is the unique dQI exact on* $\mathbb{P}_5$ *among all the quintic dQIs given by* (3).
(2) *For each* $m \geqslant 3$, $\widetilde{Q}_{6,m}$ *is the unique NMN quintic dQI exact on* $\mathbb{P}_5$.
(3) *When* $m \to +\infty$, $(\widetilde{Q}_{6,m})_{m\geqslant 1}$ *converges in the infinity norm to the Schoenberg's operator* $S_6 : C(\mathbb{R}) \to S_{6,\mathbb{Z}}$ *defined by* $S_6 f = \sum_{i\in\mathbb{Z}} f(i) M_i$.
(4) *The following equalities hold*: $\|\widetilde{Q}_{6,2}\|_\infty = \frac{37\,183}{28\,800} \simeq 1.2911$, $\|\widetilde{Q}_{6,3}\|_\infty = \frac{61}{48} \simeq 1.2708$, $\|\widetilde{Q}_{6,4}\|_\infty = \frac{23\,152\,727}{19\,353\,600} \simeq 1.1963$, $\|\widetilde{Q}_{6,5}\|_\infty = \frac{853}{720} \simeq 1.1847$, *and*

$$\|Q_{6,m}\|_\infty = 1 + \frac{8+5m^2}{5(m-1)^2(2m-1)} \quad \mathit{for\ all}\ m \geqslant 6.$$

Once again, for each $m \geqslant 3$ the corresponding NMN quintic dQI is better than the classical one w.r.t. the infinity norm.

5. Some examples of near minimally normed integral QIs

In this section, we briefly describe some results on NMN cubic and quintic iQIs.

5.1. Cubic integral QIs

In the cubic case, Eqs. (11) become

$$\gamma_0 + 2\sum_{j=1}^{m} \gamma_j = 1 \quad \text{and} \quad \sum_{j=1}^{m} j^2\gamma_j = -\frac{1}{3}.$$

A NMN cubic iQI relative to m and exact on $\mathbb{P}_3$ corresponds to a $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_m)^{\mathrm{T}}$ such that $\widetilde{\boldsymbol{\gamma}} = (\gamma_1, \ldots, \gamma_{m-1})^{\mathrm{T}}$ minimizes $g(\widetilde{\boldsymbol{\gamma}}) = \|\boldsymbol{\gamma}^* - G\widetilde{\boldsymbol{\gamma}}\|_1$, where the matrix G is given by (16), and $\boldsymbol{\gamma}^* = (1 + \frac{2}{3m^2}, 0, \ldots, 0, -\frac{2}{3m^2})^{\mathrm{T}}$. A similar proof as that of Proposition 11 shows that g achieves its minimum at $\mathbf{0} \in \mathbb{R}^{m-1}$. Hence, we have the following result.

Proposition 14. *Let $m \geqslant 1$ and $\widetilde{Q}_{4,m}$ be the cubic iQI given by*

$$\widetilde{Q}_{4,m} f = \sum_{i\in\mathbb{Z}} \left\{ \left(1 + \frac{2}{3m^2}\right) \langle f, M_i\rangle - \frac{1}{3m^2} \langle f, M_{i-m} + M_{i+m}\rangle \right\} M_i. \tag{17}$$

Then

(1) *For $m = 1$, $\widetilde{Q}_{4,1}$ is the unique iQI exact on $\mathbb{P}_3$ among all the cubic iQIs given by* (4).
(2) *For each $m \geqslant 2$, $\widetilde{Q}_{4,m}$ is the unique NMN cubic iQI relative to m and exact on $\mathbb{P}_3$.*
(3) *When $m \to +\infty$, $(\widetilde{Q}_{4,m})_{m\geqslant 1}$ converges in the infinity norm to the Schoenberg's operator defined by*

$$S_4 f = \sum_{i\in\mathbb{Z}} \langle f, M_i\rangle M_i.$$

(4) *The following equalities hold*: $\|\widetilde{Q}_{4,1}\|_\infty = \frac{55}{36} \simeq 1.5278$, $\|\widetilde{Q}_{4,2}\|_\infty = \frac{23}{18} \simeq 1.2778$, $\|\widetilde{Q}_{4,3}\|_\infty = \frac{31}{27} \simeq 1.1481$, *and*

$$\|\widetilde{Q}_{4,m}\|_\infty = 1 + \frac{4}{3m^2} \quad \textit{for all } m \geqslant 4.$$

5.2. Quintic integral QIs

The quintic case is qualitatively different. The iQI is exact on $\mathbb{P}_5$, if and only if

$$\Gamma_0 = 1, \quad \mu_2(6) + \Gamma_2 = 2!\beta(2, 6), \quad \text{and} \quad \mu_4(6) + 6\mu_2(6)\Gamma_2 + \Gamma_4 = 4!\beta(4, 6),$$

that is,

$$\gamma_0 + 2\sum_{j=1}^{m} \gamma_j = 1, \quad \sum_{j=1}^{m} j^2\gamma_j = -\frac{1}{2} \quad \text{and} \quad \sum_{j=1}^{m} j^4\gamma_j = \frac{31}{20}.$$

Proposition 15. (1) *When* $m=2$, *the unique iQI exact on* $\mathbb{P}_5$ *among all the quintic iQIs given by* (4) *is* $\widetilde{Q}_{6,2}f=\sum_{i\in\mathbb{Z}}\langle f,C_{i,2}\rangle M_i$, *with*

$$C_{i,2}=\frac{111}{35}\,M_i-\frac{71}{60}\,(M_{i-1}+M_{i+1})+\frac{41}{240}\,(M_{i-2}+M_{i+2}).$$

(2) *When* $m=3$ *or* 4, *there is a unique NMN iQI* $\widetilde{Q}_{6,3}f=\sum_{i\in\mathbb{Z}}\langle f,C_{i,m}\rangle M_i$ *exact on* $\mathbb{P}_5$, *given by*

$$C_{i,3}=\frac{531}{360}\,M_i-\frac{121}{400}\,(M_{i-2}+M_{i+2})+\frac{71}{900}\,(M_{i-3}+M_{i+3}),$$
$$C_{i,4}=\frac{1721}{1440}\,M_i-\frac{191}{1260}\,(M_{i-3}+M_{i+3})+\frac{121}{2240}\,(M_{i-4}+M_{i+4}).$$

(3) *When* $m=5$, *for each* $\alpha\in[-\frac{281}{2880},0]$ *the iQI given by*

$$\widetilde{Q}_{6,5}f=\sum_{i\in\mathbb{Z}}\langle f,C_{i,5}\rangle M_{i,6},$$

with

$$C_{i,5}=\frac{4441-2240\alpha}{4000}\,M_i-\frac{281+2880\alpha}{2880}\,(M_{i-4}+M_{i+4})+\frac{191+1260\alpha}{4500}\,(M_{i-5}+M_{i+5})$$

is a NMN quintic iQI.

(4) *When* $m=6$, *there is a unique NMN quintic iQI, given by*

$$\widetilde{Q}_{6,6}f=\sum_{i\in\mathbb{Z}}\langle f,C_{i,6}\rangle M_i$$

with

$$C_{i,6}=\frac{49\,534}{47\,385}\,M_i-\frac{14\,479}{568\,620}\,(M_{i-3}+M_{i+3})+\frac{317}{113\,724}\,(M_{i-6}+M_{i+6}).$$

Observe that $(\frac{49\,534}{47\,385},0,0,-\frac{14\,479}{568\,620},0,0,\frac{317}{113\,724})^{\mathrm{T}}$ is the unique solution of the integral Problem 8 when $n=3$ and $m=6$.

Remark 16. As in the cubic case, it is easy to check the following results.

$$\|\widetilde{Q}_{6,2}\|_\infty\simeq 2.0313.$$

$$\|\widetilde{Q}_{6,3}\|_\infty=\frac{719}{432}\simeq 1.6643,\quad\text{and}\quad \|\widetilde{Q}_{6,4}\|_\infty=\frac{27\,520\,099}{19\,353\,600}\simeq 1.4220.$$

$$\|\widetilde{Q}_{6,5}\|_\infty=\frac{1721}{1440}\simeq 1.1951\quad\text{for }\alpha=-\frac{281}{2880},\ \text{and}\ \|\widetilde{Q}_{6,5}\|_\infty=\frac{88\,469\,567}{69\,120\,000}\simeq 1.2799\ \text{for }\alpha=0.$$

$$\|\widetilde{Q}_{6,6}\|_\infty=\frac{70\,529\,153}{57\,396\,000}\simeq 1.2288.$$

6. Some error bounds for cubic discrete and integral QIs

In this section, we give some bounds for the error $f-\widetilde{Q}_{4,m}f$ associated with the cubic QIs given by (15) and (17). The exactness of $\widetilde{Q}_{4,m}$ on $\mathbb{P}_3 \subset S_{4,\mathbb{Z}}$ yields

$$\|f-\widetilde{Q}_{4,m}f\|_\infty \leqslant (1+\|\widetilde{Q}_{4,m}\|_\infty)\mathrm{dist}(f, S_{4,\mathbb{Z}}).$$

We observe that the class of the approximated function f does not take place in the construction of the NMN QI. Then, as we will show below, a better bound is obtained if we take into account the regularity of f.

Proposition 17. *Let $m \geqslant 1$ and $\xi \in [0,1]$. For any $f \in C^4(\mathbb{R})$, the quasi-interpolation error for the dQI given by* (15) (*resp. the iQI given by* (17)) *satisfies*

$$|f(\xi)-\widetilde{Q}_{4,m}f(\xi)| \leqslant \widetilde{C}_m \|f^{(4)}\|_{\infty,[0,1]} \tag{18}$$

with $\widetilde{C}_1 = \frac{37}{1728} \simeq 0.0214$, *and* $\widetilde{C}_m = (96m^3-105m^2+120m-38)/3456m^2$ *for* $m \geqslant 2$ (*resp.* $\widetilde{C}_1 = \frac{135\,937}{2\,903\,040} \simeq 0.0468$, $\widetilde{C}_2 = \frac{151\,213}{1\,935\,360} \simeq 0.0781$, $\widetilde{C}_3 = \frac{70\,031}{580\,608} \simeq 0.1206$, *and* $\widetilde{C}_m = (80\,640m^3 - 118\,281m^2 + 181\,440m - 81\,374)/145\,1520m^2$ *for* $m \geqslant 4$).

Proof. Let $\widetilde{Q}_{4,m}$ be the dQI given by (15). Let $\mathscr{L}$ be the linear form defined as $\mathscr{L}f = f(\xi) - \widetilde{Q}_{4,m}f(\xi)$. The Peano's Theorem (see e.g., [19, Chapter 4]) gives

$$f(\xi)-\widetilde{Q}_{4,m}f(\xi) = \int_0^1 k_{4,m}(\xi,t) f^{(4)}(t)\,\mathrm{d}t,$$

where

$$3!k_{4,m}(\xi,t) = \mathscr{L}(\cdot - t)_+^3 = (\xi-t)_+^3 - Q_{4,m}(\cdot - t)_+^3(\xi).$$

Since $(\xi-t)_+^3 = 0$ for $\xi < t$, in this case it follows easily that

$$k_{4,m}(\xi,t) = \sum_{i=-1}^{2} \alpha_i(t) M_4(\xi - i)$$

with

$$\alpha_{-1}(t) = \begin{cases} 0 & \text{if } m=1,\\ \dfrac{1}{36}m^2(-1+m-t)^2 & \text{if } m \geqslant 2,\end{cases}$$

$$\alpha_0(t) = \frac{1}{36m^2}(m-t)^3,$$

$$\alpha_1(t) = \frac{1}{36m^2}(-2(3m^2+1)(1-t)^3 + (1+m-t)^3),$$

$$\alpha_2(t) = \begin{cases} \dfrac{1}{36}((1-t^3) - 8(2-t)^3 + (3-t)^3) & \text{if } m=1,\\ \dfrac{1}{36m^2}(-2(3m^2+1)(2-t)^3 + (2+m-t)^3) & \text{if } m \geqslant 2.\end{cases}$$

For $-1\leqslant i\leqslant 2$, $M_4(\xi-i)$ is a cubic polynomial on [0, 1]. Let $b_i^3(t)=\binom{3}{i}(1-t)^3t^{3-i}$, $0\leqslant i\leqslant 3$, be the ith Bernstein polynomial. It is well known (see e.g., [11, p. 13]) that

$$M_4(\xi+1)=\frac{1}{6}\,b_0^3(\xi),$$

$$M_4(\xi)=\frac{1}{6}\,(4b_0^3(\xi)+4b_1^3(\xi)+2b_2^3(\xi)+b_3^3(\xi)),$$

$$M_4(\xi-1)=\frac{1}{6}\,(b_0^3(\xi)+2b_1^3(\xi)+4b_2^3(\xi)+4b_3^3(\xi)),$$

$$M_4(\xi-2)=\frac{1}{6}\,b_3^3(\xi).$$

Thus, we get

$$k_{4,m}(\xi,t)=\sum_{i=0}^{3}\omega_i(t)b_i^3(\xi), \tag{19}$$

where

$$\boldsymbol{\omega}=W\mathbf{b} \tag{20}$$

with $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3,\omega_3)^{\mathrm{T}}$, $\mathbf{b}=(b_0^3,b_1^3,b_2^3,b_3^3)^{\mathrm{T}}$, and $W=(W_{ij})_{1\leqslant i,j\leqslant 4}$ is given by

$$W=\frac{1}{216}\begin{pmatrix} 4 & 4 & 2 & 1\\ 4 & 6 & 4 & 2\\ 2 & 16 & 8 & 4\\ -35 & 2 & 4 & 4\end{pmatrix}$$

if $m=1$, and

$$W_{11}=W_{12}=W_{21}=W_{34}=W_{43}=W_{44}=\frac{1}{108m^2}(-1+3m-3m^2+3m^3),$$

$$W_{13}=W_{2,4}=W_{3,1}=W_{4,2}=\frac{1}{108m^2}\,(-2+6m-6m^2+3m^3),$$

$$W_{2,2}=W_{3,3}=\frac{1}{108m}\,(1+3m^2),$$

$$W_{2,3}=W_{3,2}=\frac{1}{108m}\,(2-3m+3m^2),$$

$$W_{1,4}=\frac{1}{108m^2}\,(-2+4m-3m^2+m^3),$$

$$W_{4,1}=\frac{1}{36m^2}\,(-2+4m-3m^2+m^3)$$

for $m\geqslant 2$.

All the coefficients in the matrix W are positive for $m \geqslant 9$. Hence, in these cases the kernel $k_{4,m}$ is positive in $[0, 1] \times [0, 1]$. For the other values of m, the element $W_{4,1}$ is negative. Therefore, we will analyze carefully the kernel in the triangle T_1 of vertices $A_1 = (0, 0)$, $A_2 = (1, 1)$, and $A_3 = (0, 1)$.

For $m = 1$, let (u, v, w) be the barycentric coordinates of (ξ, t) with respect to T_1, i.e.,

$$uA_1 + vA_2 + wA_3 = (\xi, t), \quad u + v + w = 1.$$

The kernel $k_{4,1}$ being a polynomial in (ξ, t) of total degree 6, it has a unique Bézier representation (see e.g., [11, Chapter 5])

$$k_{4,1}(\xi, t) = \sum_{|l|=6} a_l b_l^6(u, v, w),$$

where $b_l^6(u, v, w) = \frac{6!}{l_1!l_2!l_3!} u^{l_1} v^{l_2} w^{l_3}$, $0 \leqslant l_1, l_2, l_3$ and $|l| := l_1 + l_2 + l_3 = 6$, are the Bernstein polynomials of degree 6, and the Bernstein–Bézier coefficients a_l are as follows:

$$\begin{array}{ccccccc} a_{0,0,6} & a_{0,1,5} & a_{0,2,4} & a_{0,3,3} & a_{0,4,2} & a_{0,5,1} & a_{0,6,0} \\ a_{1,0,5} & a_{1,1,4} & a_{1,2,3} & a_{1,3,2} & a_{1,4,1} & a_{1,5,0} & \\ a_{2,0,4} & a_{2,1,3} & a_{2,2,2} & a_{2,3,1} & a_{2,4,0} & & \\ a_{3,0,3} & a_{3,1,2} & a_{3,2,1} & a_{3,3,0} & & & \\ a_{4,0,2} & a_{4,1,1} & a_{4,2,0} & & & & \\ a_{5,0,1} & a_{5,1,0} & & & & & \\ a_{6,0,0} & & & & & & \end{array} = \frac{1}{2592} \begin{array}{ccccccc} 20 & 30 & 44 & 59 & 72 & 80 & 80 \\ 30 & 46 & 69 & 89 & 96 & 80 & \\ 44 & 69 & 106 & 129 & 112 & & \\ 59 & 89 & 129 & 146 & & & \\ 72 & 96 & 112 & & & & \\ 80 & 80 & & & & & \\ 80 & & & & & & \end{array}$$

We then get the positivity of $k_{4,1}(\xi, t)$ on T_1. The same conclusion can be drawn for $k_{4,m}$, $2 \leqslant m \leqslant 8$.

Now, let us determine the Peano kernel when $\xi \geqslant t$. Since $(\xi - t)_+^3 = (\xi - t)^3$, and $\widetilde{Q}_{4,m}$ is exact on $\mathbb{P}_3$, we have

$$k_{4,m}(\xi, t) = \widetilde{Q}_{4,m}((\cdot - t)^3 - (\cdot - t)_+^3)(\xi) = -\widetilde{Q}_{4,m}(t - \cdot)_+^3(\xi).$$

After some calculations, we find that

$$k_{4,m}(\xi, t) = \sum_{i=0}^{3} \omega_{3-i}(1 - t) b_i^3(\xi). \tag{21}$$

Therefore, $k_{4,m}$ is also positive on the triangle T_2 of vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

As $k_{4,m}$ does not changes sign, there exists $\tau \in [0, 1]$ such that

$$f(\xi) - \widetilde{Q}_{4,m} f(\xi) = f^{(4)}(\tau) K_{4,m}(\xi),$$

where

$$K_{4,m}(\xi) := \int_0^1 k_{4,m}(t)\,\mathrm{d}t.$$

Let $X = \xi(1 - \xi)$. From (19)–(21), we obtain for $m = 1$

$$K_{4,1}(\xi) = \frac{1}{864}(36X^2 + 21X + 11)$$

and for $m \geqslant 2$

$$K_{4,m}(\xi) = \frac{1}{1728m^2}\,(X^2 + 12(7 - 12m + 9m^2)X + 4(12m^3 - 21m^2 + 24m - 10)).$$

For all $m \geqslant 1$, the function $K_{4,m}$ attains its maximum value at $\xi = \frac{1}{2}$, and $\widetilde{C}_m = K_{4,m}(\frac{1}{2})$.

The proof for the iQI given by (17) runs as before. $\square$

From (18), the QI error for the scaled QI $\widetilde{Q}^h_{m,4}$, $h > 0$, associated with $\widetilde{Q}_{m,4}$ satisfies

$$\|f - \widetilde{Q}^h_{4,m} f\|_\infty \leqslant \widetilde{C}_m h^4 \|f^{(4)}\|_\infty. \tag{22}$$

7. Application to functions with discontinuities

When the classical scaled dQI $\widetilde{Q}^h_{4,1}$ is used to approximate the Heaviside function H defined as

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0, \end{cases}$$

the QI $\widetilde{Q}^h_{4,1}H$ oscillates near the discontinuity point with an overshoot independent of f and equal to $\frac{11}{1800}(6 + \sqrt{11}) \simeq 5.7\%$. Since the scaled Schoenberg's dQI $S^h_4 H$ does not oscillate near zero and the sequence $(\widetilde{Q}^h_{4,m})_{m \geqslant 1}$ converges to S_4 in the infinity norm when $m \to +\infty$, it seems natural to use a NMN cubic dQI in order to obtain another QI $\widetilde{Q}^h_{4,m}H$ giving a smaller overshoot. For $m = 2$ (resp. $m = 3$) this equals $(762 + 13\sqrt{26})/22\,500 \simeq 3.7\%$ (resp. 1.85%). However, the constant $\widetilde{C}_m$ in (22) given by (18) increases with m. Thus it is better to choose a low value of m. Table 5 shows the higher decrease of the upper bound $\widetilde{v}_{4,m}$ and the infinity norm of $\widetilde{Q}_{4,m}$ occurs for $m = 2$ and 3. Fig. 1 shows how the overshoot is damped and shifted when the dQIs $\widetilde{Q}_{4,2}$ and $\widetilde{Q}_{4,3}$ are used.

For a function with isolated discontinuities, when $h \to 0$ also happens the oscillatory behaviour near each discontinuity point. $\widetilde{Q}_{4,2}$ and $\widetilde{Q}_{4,3}$ behave better than $\widetilde{Q}_{4,1}$ in a neigbourhood of each discontinuity point. Far from them (22) holds. For cubic iQI a similar result holds.

The behaviour of the cubic dQIs $\widetilde{Q}_{4,m}$ w.r.t. the approximation of functions with isolated discontinuities can be exploited to refine locally the error produced in approximating a given function f on an interval

Table 5
Values of $\widetilde{v}_{4,m}$, $\|\widetilde{Q}_{4,m}\|_\infty$, and $\widetilde{C}_m$ for some cubic NMN QIs $\widetilde{Q}_{4,m}$

m	dQI			iQI		
	$\widetilde{v}_{4,m}$	$\|\widetilde{Q}_{4,m}\|_\infty$	$\widetilde{C}_m$	$\widetilde{v}_{4,m}$	$\|\widetilde{Q}_{4,m}\|_\infty$	$\widetilde{C}_m$
1	1.6667	1.2222	0.0214	2.3333	1.5278	0.0468
2	1.1667	1.1389	0.0398	1.3333	1.2778	0.0781
3	1.0741	1.0741	0.0633	1.1481	1.1481	0.1206
4	1.0417	1.0417	0.0887	1.0833	1.0833	0.1685

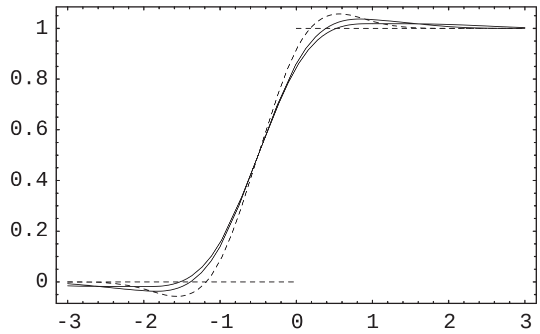

Fig. 1. The Heaviside function and the QIs $\widetilde{Q}_{4,1}H$, $\widetilde{Q}_{4,2}H$, and $\widetilde{Q}_{4,3}H$.

I. An approximant *Af* of f on I is usually accepted when $\|f - Af\|_{\infty,I} \leqslant \varepsilon$ for a prescribed tolerance $\varepsilon > 0$. For h_* small enough, there exists an open subset $J \subset I$ such that $\|f - \widetilde{Q}^{h_*}_{4,1} f\|_{\infty,I\setminus J} \leqslant \varepsilon$ and $\|f - \widetilde{Q}^{h_*}_{4,1} f\|_{\infty,J} > \varepsilon$. The choice of $h \leqslant h_*$ would permit that $\|f - \widetilde{Q}^{h}_{4,1} f\|_{\infty,I} \leqslant \varepsilon$ holds, but in this case $\widetilde{Q}^{h}_{4,1} f$ must be computed on I without taking advantage of the good performance of $\widetilde{Q}^{h_*}_{4,1} f$ on $I\setminus J$.

In [15], two methods are proposed in order to define *Af*. Both methods produce approximants which coincide with $\widetilde{Q}^{h}_{4,1} f$ on $I\setminus J$, except a small subset. They give smaller errors on J than $\widetilde{Q}^{h_*}_{4,1} f$. One of them is based on the classical dQI $\widetilde{Q}_{4,1}$, and the other one uses the NMN dQI $\widetilde{Q}_{4,2}$. It this proved that this second method permits us a better control of the error near the boundary of $I\setminus J$. We refer to [15] for more details.

8. Conclusion

In this paper, we have constructed discrete and integral spline QIs on uniform partitions of the real line with optimal approximation orders and small norms by minimizing a simple upper bound of the true norm. They can be used to approximate functions with isolated discontinuities. Although this method gives good results w.r.t. the infinity norm, the constant appearing in the standard estimation of the quasi-interpolation error is too much crude, because the bound is independent of the class of the function to be approximated. It will be interesting to construct QIs in order to obtain a better bound for smooth enough functions.

Future works would include a similar treatment for nonuniform partitions in the univariate case (cf. [2]), as well as the construction of dQIs and iQIs based on B-splines on the uniform three or four-directional meshes of the plane, in the bivariate case (cf. [3]), and the numerical solution of integral equations with discontinuous kernels.

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