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ITERATED TUBULAR ALGEBRAS

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1. Introduction

Let k be a fixed algebraically closed field. Let Λ be a finite dimensional, basic and connected k-algebra.

In this work we are concerned with the study of certain classes of tame algebras. We say that Λ has *acceptable projectives* if the Auslander–Reiten quiver Γ_Λ of Λ has components $\mathscr{P}, \mathscr{C}_1, \ldots, \mathscr{C}_l$ with the following properties:

(i) Any indecomposable projective Λ-module lies on $\mathscr{P}$ or on some $\mathscr{C}_i$.

(ii) $\mathscr{P}$ is a preprojective component of Γ_Λ without injective modules.

(iii) Each $\mathscr{C}_i$ is an inserted-coinserted standard tube.

(iv) If $\mathrm{Hom}_\Lambda(\mathscr{C}_i, \mathscr{C}_j) \neq 0$, then $i \leq j$.

The main result of this work is the following: Let Λ be a directed algebra with acceptable projectives, then Λ is tame iff the Tits form q_Λ of Λ is weakly semi-positive. Moreover, we give an inductive construction of this class of tame algebras and of their module categories. The construction of these algebras is an iteration of the process given by Ringel in [16] for the definition of the domestic tubular and tubular algebras. Hence we call these algebras *iterated tubular algebras*.

Following [8], we write $\Lambda = k[Q_\Lambda]/I$. We assume that Q_Λ has no oriented cycle (i.e. Λ is directed). Our modules are left Λ-modules. By P_x (resp. I_x) we denote the indecomposable projective (resp. injective) Λ-module associated with the vertex $x \in Q_\Lambda$. If $M \in \mathrm{mod}\,\Lambda$ we set $\mathbf{dim}\, M = (\dim_k \mathrm{Hom}_\Lambda(P_x, M))_{x \in Q_\Lambda}$. By Γ_Λ we denote the Auslander–Reiten quiver of Λ and we consider the vertices of Γ_Λ as indecomposable modules. The translation in Γ_Λ is denoted by τ.

For basic notions we refer the reader to [8] and [16].

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1. Some characterizations of domestic tubular algebras

1.1. Following [16, 3.1], a translation quiver $\mathcal{T}$ without double arrows is said to be a *tube* if its associated topological space is homeomorphic to $S^1 \times \mathbb{R}_0^+$ and $\mathcal{T}$ contains a cyclic path. The tube $\mathcal{T}$ is *stable* if it is of the form $\mathbb{Z}\mathbb{A}_\infty/n$, for some natural number $n \geq 1$.

A vertex v in $\mathcal{T}$ is a *ray vertex* if there is an infinite sectional path in $\mathcal{T}$, $v = v[1] \to v[2] \to \cdots \to v[i] \to v[i+1] \to \cdots$ with different vertices $v[i]$, $i \in \mathbb{N}$, such that for each i, there is a unique sectional path of length i starting at v.

Given a branch B and a ray vertex $v \in \mathcal{T}$, the translation quiver $\mathcal{T}[v, B]$, defined in [16, 4.5], is said to be obtained from $\mathcal{T}$ by a *ray insertion*. Given $\mathcal{T}$ a stable tube, $v_1, \ldots, v_t$ different vertices in the mouth of $\mathcal{T}$ and $B_1, \ldots, B_t$ (possibly empty) branches, we say that $\mathcal{T}[v_i, B_i]_{i=1}^t$ is an *inserted tube*.

A vertex v in a tube $\mathcal{T}$ is a *back vertex* if no arrow pointing to infinity [16, 4.6] ends at v. For any vertex ω in $\mathcal{T}$ there is a unique back vertex v and a finite sectional path $v = v[1] \to v[2] \to \cdots \to v[s] = \omega$. In an inserted tube the ray vertices are back vertices.

We have the dual notions: coray vertices, coinserted tubes, front vertices.

Proposition. *Let $\mathcal{T}$ be a connected component of Γ_A and assume that $\mathcal{T}$ is an inserted tube. Let $M_1, \ldots, M_l$ be the back vertices of $\mathcal{T}$. Then for any M in $\mathcal{T}$ there exist $j, j_1, \ldots, j_s \in \{1, \ldots, l\}$ such that $M = M_j[s]$ and* $\mathbf{dim}\, M = \sum_{i=1}^s \mathbf{dim}\, M_{j_i}$.

Proof. We introduce a partial order $\leq$ in $\mathcal{T}$.

If there is no arrow $M \xrightarrow{\alpha} M'$ pointing to the mouth of $\mathcal{T}$ [16, 4.6], we set $W(M) = \{M\}$. Assume $M \xrightarrow{\alpha} M'$ points to the mouth of $\mathcal{T}$ and $M = M_j[s]$ for some $j \in \{1, \ldots, l\}$, then we set $W(M) = W(M') \cup \{M_j[i] : i \leq s\}$. We put $M \leq N$ if $M \in W(N)$.

Let $M \xrightarrow{\alpha} N$ be an arrow pointing to infinity. By induction on $\leq$ we show that α is mono and $N/\operatorname{Im}\alpha \xrightarrow{\sim} M_j$ for some $j \in \{1, \ldots, l\}$. The result then follows.

If $W(M) = \{M\}$, the claim is clear. Otherwise, the Auslander–Reiten sequence starting at M has the form

$$0 \to M \xrightarrow{\begin{pmatrix} \alpha \\ \alpha' \end{pmatrix}} N \oplus N' \xrightarrow{(\beta, \beta')} \tau^{-1} M \to 0.$$

By induction hypothesis, β' is mono and $\tau^{-1}M/\operatorname{Im}\beta' \xrightarrow{\sim} M_i$ for some $i \in \{1, \ldots, l\}$. We get the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
 & & M & \xrightarrow{\alpha} & N & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & N' & \xrightarrow{-\beta'} & \tau^{-1}M & \longrightarrow & M_i & \longrightarrow & 0.
\end{array}$$

It follows that α is mono and γ is iso. $\square$

Corollary. *Let $\mathscr{T}$ be as in the proposition and M in $\mathscr{T}$. Then* $(\mathbf{dim}\,\tau^{-n}M)_{n\geq 0}$ *grows at most linearly with n.* $\square$

1.2. We say that a preinjective component $\mathscr{I}$ of Γ_Λ is *complete* if every indecomposable injective I_x belongs to $\mathscr{I}$ and $\mathscr{I}$ does not have projective modules. Examples of algebras with complete preinjective components are the *domestic tubular algebras* [16, 4.9].

Assume that $\mathscr{I}$ is a complete preinjective component of Γ_Λ. Then $\operatorname{gl\,dim} \Lambda \leq 2$ [16, 2.4(1)]. Moreover, it is well known that Λ is a tilted algebra.

Take ${}_\Lambda T$ a slice module in $\mathscr{I}$, then $A = \operatorname{End}_\Lambda(T)$ is a hereditary algebra. Let $\Sigma = \operatorname{Hom}_\Lambda(T,-)$ and $\Sigma' = \operatorname{Ext}^1_\Lambda(T,-)$ be the functors defining the torsion pair $(\mathscr{F}(T), \mathscr{G}(T))$. Let $\sigma : K_0(\Lambda) \to K_0(A)$ be the isometry defined by $(\mathbf{dim}\,M)\sigma = \mathbf{dim}\,\Sigma M - \mathbf{dim}\,\Sigma' M$. Let ϕ be the Coxeter matrix of Λ and ϕ_A that of A. Then $\phi\sigma = \sigma\phi_A$.

The following is a simple generalization of [2, 1.3]:

Proposition. *Let Λ be as above and assume that the orbit graph $\mathscr{O}(\mathscr{I})$ is wild. Let $M \in \Gamma_\Lambda \setminus \mathscr{I}$. Then* $(\mathbf{dim}\,\tau^{-n}M)_{n\geq 0}$ *grows exponentially.*

Proof. Let the notation be as above. Applying [16, 2.4(3)], we get:

$$\mathbf{dim}\,\tau^{-n}M - (\mathbf{dim}\,M)\phi^{-n} = \sum_{j=0}^{n-1} (\mathbf{dim}\,P_j)\phi^{-j},$$

where P_j is a projective Λ-module. As $\tau^{-n}M \in \mathscr{F}(T)$ and $P_j \in \mathscr{F}(T)$, we have

$$\begin{aligned}\mathbf{dim}\,\Sigma'\tau^{-n}M - (\mathbf{dim}\,\Sigma'M)\phi_A^{-n} &= \sum_{j=0}^{n-1} (\mathbf{dim}\,\Sigma'P_j)\phi_A^{-j} \\ &= \sum_{j=0}^{n-1} (\mathbf{dim}\,\tau_A^{-j}\Sigma'P_j) \geq 0.\end{aligned}$$

As $\Sigma'M$ is not A-preinjective, by [2], $(\mathbf{dim}\,\tau_A^{-n}\Sigma'M)_{n\geq 0}$ grows exponentially.

Therefore, $(\mathbf{dim}\,\Sigma'\tau^{-n}M)_{n\geq 0}$ grows exponentially and so does $(\mathbf{dim}\,\tau^{-n}M)_{n\geq 0}$. $\square$

1.3. Theorem. *Assume that Γ_Λ has a complete preinjective component. Then the following are equivalent:*

(a) *Λ is a domestic tubular algebra.*
(b) *Λ is tilted of a tame hereditary algebra.*
(c) *Λ is tame.*
(d) *The Tits form q_Λ is semipositive.*
(e) *q_Λ is weakly semipositive.*

(f) *The connected components of Γ_Λ are preprojective, preinjective or inserted tubes.*

(g) *Γ_Λ has an inserted tube.*

Proof. (a) $\Leftrightarrow$ (b) is [16, 4.9(1)]. (b) $\Leftrightarrow$ (d) is clear since q_Λ coincides with the Euler characteristic. (b) $\Rightarrow$ (c) is clear. (a) $\Rightarrow$ (f) is [16, 4.9(2)].

(c) $\Rightarrow$ (g). By [5, Corollary F], Γ_Λ has a stable tube. (d) $\Rightarrow$ (e) is clear.

(f) $\Rightarrow$ (g). By [16, 4.5(6)], there is a tame concealed quotient $\bar{\Lambda}$ of Λ. Let M be a regular $\bar{\Lambda}$-module, then M is neither a preprojective nor a preinjective Λ-module. Hence Γ_Λ has an inserted tube.

(e) $\Rightarrow$ (b). Let $\mathcal{J}$ be a slice in the preinjective component $\mathcal{I}$ of Γ_Λ. By [16, 4.2(3)], there is a hereditary algebra A and a tilting module ${}_AT$ such that $\Lambda = \mathrm{End}_A(T)$ and $\mathcal{J} = \{\Sigma I_x : x \in Q_A\}$, where $\Sigma = \mathrm{Hom}_A(T,-)$. Assume that A is wild.

Let $T = \bigoplus_{i=1}^n T_i$ be an indecomposable decomposition of ${}_AT$. As $\mathcal{I}$ has no projectives, none of the T_i are A-preinjective.

Let $M \in \Gamma_A$ be a preinjective module. Then $X = \Sigma M \in \mathcal{I}$ and $\tau^n X = \Sigma \tau_A^n M$ for any $n \geq 0$. Let $n \in \mathbb{N}$ and consider the vector $z_n = \mathbf{dim}\, \tau^n X - \mathbf{dim}\, X$. By [16, 2.4(4) and 4.1(7)],

$$z_n = (\mathbf{dim}\, \tau_A^n M - \mathbf{dim}\, M)\sigma = (\dim \mathrm{Hom}_A(T_i, \tau_A^n M) - \dim \mathrm{Hom}_A(T_i, M))_i.$$

If T_i is preprojective (resp. regular), the ith coordinate of z_n is positive by [7] (resp. [2, 1.3]). On the other hand,

$$q_\Lambda(z_n) = 2 - (\dim \mathrm{Hom}_A(\tau_A^n M, M) - \dim \mathrm{Hom}_A(\tau_A^{n-1} M, M)).$$

By [7], the coordinates of $(\mathbf{dim}\, \tau_A^l M)_{l \geq 0}$ grow exponentially and therefore there exists an $n \in \mathbb{N}$ with $q_\Lambda(z_n) < 0$.

(g) $\Rightarrow$ (b). Assume ${}_AT$ is a slice module with $\Lambda = \mathrm{End}_A(T)$ a hereditary wild algebra. Let M be a module in an inserted tube of Γ_Λ. By 1.1 and 1.2 we obtain a contradiction about the growth of $(\mathbf{dim}\, \tau^{-n} M)_{n \geq 0}$. $\square$

Corollary. *Assume that Γ_Λ has a complete preprojective component and a complete preinjective component. Then the following are equivalent:*

(a) *Λ is tame concealed.*

(b) *q_Λ is semipositive.*

(c) *Γ_Λ has a stable tube.* $\square$

Some parts of the results above also follow from recent work of Kerner [11].

2. Construction of the iterated tubular algebras

2.1. We recall some notions from [16] (we slightly change the notation). Let $\mathcal{T}$ be a standard tubular family in Γ_Λ separating $\mathcal{P}$ from $\mathcal{I}$. Let $E_1, \ldots, E_t$ be a set of pairwise orthogonal ray modules in $\mathcal{T}$ and $K_1, \dots, K_t$ a set of (possibly empty) branches. Then the algebra $\Lambda = A[E_i, K_i]_{i=1}^{t}$ is called a *$\mathcal{T}$-tubular extension* of A.

Let A be tame concealed and $\operatorname{mod} A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{I}$, where $\mathcal{P}$ is the preprojective component, $\mathcal{I}$ the preinjective component and $\mathcal{T}$ is a tubular family separating $\mathcal{P}$ from $\mathcal{I}$.

Let $(n_1, \dots, n_r)$ be the extension type of Λ and $\mathbb{T}_{n_1, \dots, n_r}$ be the associated tree. If $\mathbb{T}_{n_1, \dots, n_r}$ is Dynkin, then Λ is a *domestic tubular algebra* and its module category may be described: $\operatorname{mod} \Lambda = \mathcal{P} \vee \mathcal{T}[E_i, K_i]_{i=1}^{t} \vee \mathcal{I}'$, where $\mathcal{I}'$ is a preinjective component and $\mathcal{T}[E_i, K_i]_{i=1}^{t}$ is a tubular family separating $\mathcal{P}$ from $\mathcal{I}'$.

If $\mathbb{T}_{n_1, \dots, n_r}$ is extended Dynkin, then Λ is a *tubular algebra* and its module category is described: $\operatorname{mod} \Lambda = \mathcal{P} \vee \mathcal{T}_0 \vee \bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma \vee \mathcal{T}_\infty \vee \mathcal{I}'$, where $\mathcal{T}_0 = \mathcal{T}[E_i, K_i]_{i=1}^{t}$ is a tubular family separating $\mathcal{P}$ from $\bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma \vee \mathcal{T}_\infty \vee \mathcal{I}'$; for each $\gamma \in Q^+$, $\mathcal{T}_\gamma$ is a stable tubular family separating $\mathcal{P} \vee \mathcal{T}_0 \vee \bigvee_{\delta < \gamma} \mathcal{T}_\delta$ from $\bigvee_{\gamma < \delta} \mathcal{T}_\delta \vee \mathcal{T}_\infty \vee \mathcal{I}'$.

There is a tame concealed algebra A' such that $\Lambda = {}_{i=1}^{r}[E_i', K_i']A'$ is a $\mathcal{T}'$-tubular coextension, where $\operatorname{mod} A' = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{I}'$ with $\mathcal{T}'$ a tubular family separating $\mathcal{P}'$ from $\mathcal{I}'$. Then $\mathcal{T}_\infty = {}_{i=1}^{r}[E_i', K_i']\mathcal{T}'$ is a tubular family separating $\mathcal{P} \vee \mathcal{T}_0 \vee \bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma$ from $\mathcal{I}'$.

Domestic cotubular algebras are defined dually. Every tubular algebra is also cotubular.

2.2. Domestic tubular, domestic cotubular and tubular algebras are said to be *0-iterated tubular algebras.*

Let Λ_0 be a domestic cotubular algebra or a tubular algebra. In both cases Λ_0 is a coextension $\Lambda_0 = {}_{i=1}^{t}[E_i, K_i]A_0$ of a tame concealed algebra A_0 and $\operatorname{mod} \Lambda_0 = \mathcal{P}^0 \vee \mathcal{T}^0 \vee \mathcal{I}_0$ where $\mathcal{I}_0$ is the preinjective component of both Γ_{Λ_0} and Γ_{A_0} and $\mathcal{T}^0$ is a tubular family separating $\mathcal{P}^0$ from $\mathcal{I}_0$. Let $E_1^0, \dots, E_t^0$ be a set of pairwise orthogonal ray Λ_0-modules in $\mathcal{T}^0$. Let $K_1^0, \dots, K_{t_0}^0$ be a set of branches and assume that $A_0[E_i^0, K_i^0]_{i=1}^{t_0}$ is a domestic tubular algebra or a tubular algebra. Then we say that the extension $\Lambda_1 = \Lambda_0[E_i^0, K_i^0]_{i=1}^{t_0}$ is a *1-iterated tubular algebra*. By [16, 4.7], $\operatorname{mod} \Lambda_1 = \mathcal{P}^0 \vee \mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0} \vee \mathcal{I}^1$, where $\mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0}$ is a tubular family separating $\mathcal{P}^0$ from $\mathcal{I}^1$. We want to describe $\mathcal{I}^1$. Let $\operatorname{mod} A_0 = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{I}_0$, where $\mathcal{T}_0$ is the stable tubular family separating $\mathcal{P}_0$ from $\mathcal{I}_0$. Then

$$\operatorname{mod} A_0[E_i^0, K_i^0]_{i=1}^{t_0} = \mathcal{P}_0 \vee \mathcal{T}_0[E_i^0, K_i^0]_{i=1}^{t_0} \vee \hat{\mathcal{I}}.$$

Lemma. *With the above notation, $\mathcal{I}^1 = \hat{\mathcal{I}}$.*

Proof. Let $A^1 = A_0[E_i^0, K_i^0]_{i=1}^{t_0}$. Let $X \in \hat{\mathcal{I}}$ and P_y be an indecomposable A^1-projective with $\operatorname{Hom}_{A^1}(P_y, X) \neq 0$. As $P_y \in \mathcal{P}_0 \vee \mathcal{T}_0[E_i^0, K_i^0]_{i=1}^{t_0}$, there is an inserted tube $\mathcal{T}$ in $\mathcal{T}_0[E_i^0, K_i^0]_{i=1}^{t_0}$ with $\operatorname{Hom}_{A^1}(\mathcal{T}, X) \neq 0$. Then there is a tube $\mathcal{T}'$ in $\mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0}$ which is obtained from $\mathcal{T}$ by coray insertion and such that $\operatorname{Hom}_{\Lambda_1}(\mathcal{T}', X) \neq 0$. Hence $X \in \mathcal{I}^1$.

Let $X \in \mathcal{I}^1$. Let y be a vertex in Q_{Λ_0} but not in Q_{A_0}. The Λ_0-injective I_y^0 is also Λ_1-injective. Since $I_y^0 \in \mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0}$, $\mathrm{Hom}_{\Lambda_1}(X, I_y^0) = 0$. Thus $X \in \mathrm{mod}\, A^1$ and $X \in \hat{\mathcal{I}}$ as above. □

2.3. Let $\Lambda_1 = \Lambda_0[E_i^0, K_i^0]_{i=1}^{t_0}$ be a 1-iterated tubular algebra as above. Assume that $A^1 = A_0[E_i^0, K_i^0]_{i=1}^{t_0}$ is a tubular algebra and write $\mathrm{mod}\, A^1 = \mathcal{P}_0 \vee \mathcal{T}_0[E_i^0, K_i^0]_{i=1}^{t_0} \vee \bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{I}_1$ as in 2.1. In $\mathrm{mod}\, \Lambda_1$, define $\mathcal{P}^1 = \mathcal{P}^0 \vee \mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0} \vee \bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma^1$, $\mathcal{T}^1 = \mathcal{T}_\infty^1$. Then, by 2.2, $\mathrm{mod}\, \Lambda_1 = \mathcal{P}^1 \vee \mathcal{T}^1 \vee \mathcal{I}_1$, and $\mathcal{I}_1$ is the preinjective component of Γ_{Λ_1}.

The following result is an easy exercise:

Lemma. (a) *$\mathcal{T}^1$ is a tubular family separating $\mathcal{P}^1$ from $\mathcal{I}_1$.*

(b) *$\mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0}$ is a tubular family separating $\mathcal{P}^0$ from $\bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}^1 \vee \mathcal{I}_1$.*

(c) *For each $\gamma \in Q^+$, $\mathcal{T}_\gamma^1$ is a tubular family separating $\mathcal{P}^0 \vee \mathcal{T}^0[E_i^0, K_i^0]_{i=1}^{t_0} \vee \bigvee_{\delta < \gamma} \mathcal{T}_\delta^1$ from $\bigvee_{\delta > \gamma} \mathcal{T}_\delta^1 \vee \mathcal{T}^1 \vee \mathcal{I}_1$.* □

Let A_1 be the tame concealed algebra such that A^1 is a coextension of A_1. Let $E_1^1, \ldots, E_{t_1}^1$ be a set of pairwise orthogonal ray A_1-modules in $\mathcal{T}^1$. Let $K_1^1, \ldots, K_{t_1}^1$ be a set of branches such that $A^2 = A_1[E_i^1, K_i^1]_{i=1}^{t_1}$ is a domestic tubular algebra or a tubular algebra. The extension $\Lambda_2 = \Lambda_1[E_i^1, K_i^1]_{i=1}^{t_1}$ is called a *2-iterated tubular algebra*.

As before, $\mathrm{mod}\, \Lambda_2 = \mathcal{P}^2 \vee \mathcal{T}^2 \vee \mathcal{I}_2$, where $\mathcal{I}_2$ is the preinjective component of Γ_{Λ_2}, $\mathcal{T}^2$ is a tubular family separating $\mathcal{P}^2$ from $\mathcal{I}_2$. Moreover, if $\mathrm{mod}\, A_1 = \mathcal{P}_1 \vee \mathcal{T}_1 \vee \mathcal{I}_1$ and A^2 is a tubular algebra with $\mathrm{mod}\, A^2 = \mathcal{P}_1 \vee \mathcal{T}_1[E_i^1, K_i^1]_{i=1}^{t_1} \vee \bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma^2 \vee \mathcal{T}_\infty^2 \vee \mathcal{I}_2$, then $\mathcal{P}^2 = \mathcal{P}^1 \vee \mathcal{T}^1[E_i^1, K_i^1]_{i=1}^{t_1} \vee \bigvee_{\gamma \in Q^+} \mathcal{T}_\gamma^2$ and $\mathcal{T}^2 = \mathcal{T}_\infty^2$.

By induction, we define the *n-iterated tubular algebras* (or simply *iterated tubular algebras*).

Iterated tubular algebras have already appeared: in the construction of the derived category of a tubular algebra [9]; in the description of the module category of certain group algebras [17].

2.4. We immediately obtain:

Proposition. *Let Λ be an iterated tubular algebra. Then:*

(a) *Λ is tame.*

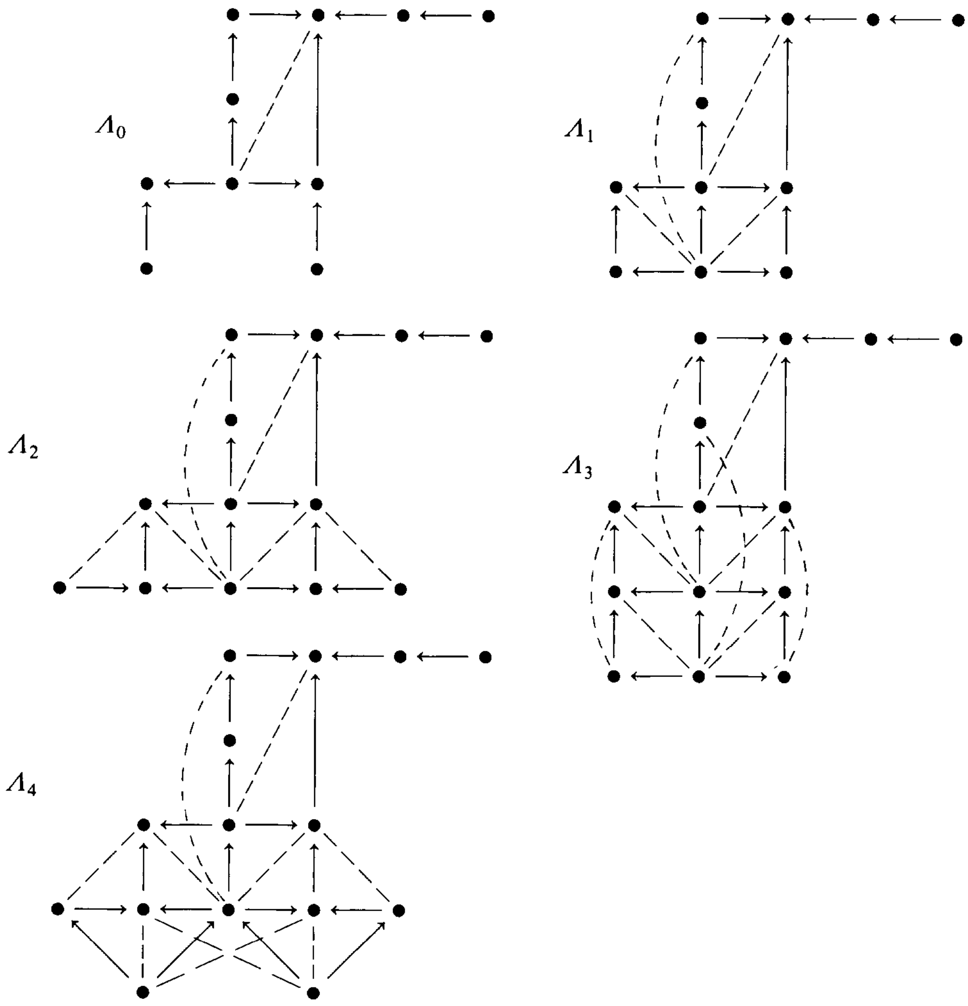
(b) *The Tits form q_Λ is weakly semipositive.*

Proof. (a) follows from the description given above for $\mathrm{mod}\, \Lambda$.

(b) follows from (a) and [14, 1.3]. □

We want to give some *examples* of iterated tubular algebras.

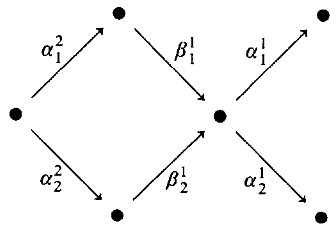
(a)



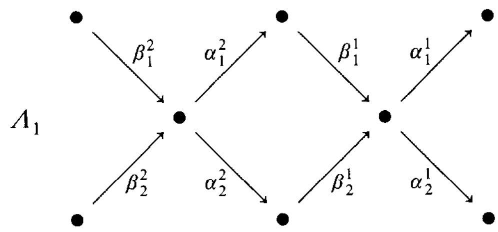
Λ_i is i-iterated tubular. We remark that q_{Λ_1} is not semipositive.

(b) Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in k^*$ pairwise different scalars.

Λ_0



$$\alpha_i^1 \beta_1^1 \alpha_1^2 = \lambda_i \alpha_i^1 \beta_2^1 \alpha_2^2, \quad i = 1, 2,$$



$$\alpha_i^1\beta_1^1\alpha_1^2=\lambda_i\alpha_i^1\beta_2^1\alpha_2^2,$$
$$\beta_1^1\alpha_1^2\beta_i^2=\mu_i\beta_2^1\alpha_2^2\beta_i^2,\quad i=1,2,$$
$$\mathrm{rad}^4\Lambda_1=0.$$

Similarly we may define the i-iterated tubular algebra Λ_i ($i\in\mathbb{N}$).

2.5. Iterated tubular algebras have acceptable projectives.

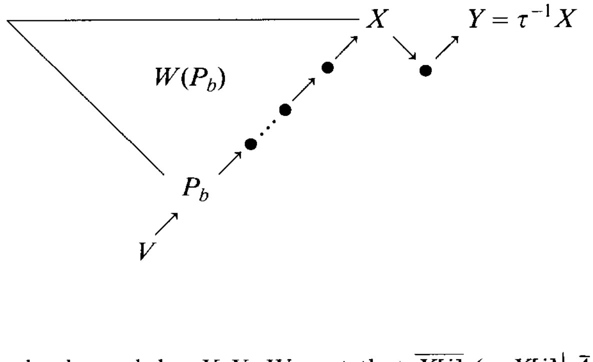
Algebras with acceptable projectives are handy due to the following: Assume that Λ has acceptable projectives and let $\mathscr{P}, \mathscr{C}_1,\ldots,\mathscr{C}_l$ be the components of Γ_Λ where the projectives lie ($\mathscr{P}$ is preprojective, the $\mathscr{C}_i$ are inserted-coinserted tubes). Suppose that $\mathrm{Hom}_\Lambda(\mathscr{C}_i,\mathscr{C}_j)\neq 0$ implies $i\leq j$. Consider $\mathscr{C}_l=\mathscr{C}_l'[V,B]$ where B is a branch and V is a ray module in the inserted tube $\mathscr{C}_l'$. Let b be the root vertex of B, that is, V is a direct summand of $\mathrm{rad}\,P_b$. Consider $W(P_b)$ the wing of P_b in $\mathscr{C}_l$ as defined in 1.1. Let $e=\sum_{P_x\in W(P_b)}e_x$ and $\bar{\Lambda}=\Lambda/\Lambda e\Lambda$.

Proposition. *With the above notation we have*:

(a) $\Lambda=\bar{\Lambda}[V,B]$.

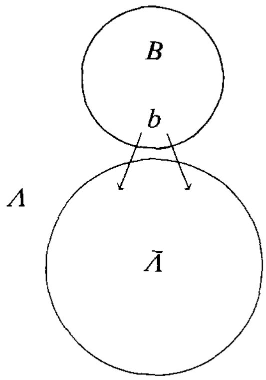
(b) $\bar{\Lambda}$ *has acceptable projectives and* $\mathscr{C}_l'$ *is a standard component of* $\Gamma_{\bar{\Lambda}}$.

Proof. We consider $W(P_b)$ as in the diagram below.



Consider the back modules X, Y. We get that $\overline{X[i]}$ $(=X[i]|\bar{\Lambda})=V[i]$ and $\bar{\tau}Y[i]=V[i]$ where $\bar{\tau}$ is the Auslander–Reiten translation in $\Gamma_{\bar{\Lambda}}$.

(a) Let $s\in Q_{\bar{\Lambda}}$ and $s\to t$ in Q_Λ, then $t\in Q_{\bar{\Lambda}}$. Indeed, if $t\in B$, as $\mathrm{Hom}_\Lambda(P_t,P_s)\neq 0$, then $P_s\in\mathscr{C}_l$. Since $P_t\in W(P_b)$ and $P_s\notin W(P_b)$, then $\mathrm{Hom}_\Lambda(P_t,P_s)=0$, a contradiction. Therefore, $\bar{\Lambda}$ is convex in Λ and Λ has the form



To prove that $\Lambda = \bar{\Lambda}[V, B]$, it is enough to show that there are no zero relations between vertices in $B \setminus \{b\}$ and vertices in $\bar{\Lambda}$. For this purpose, it is enough to show that for a path $b \leftarrow b_1 \leftarrow \cdots \leftarrow b_s$ in B, we have $\bar{P}_{b_i} = V\ (= \overline{\text{rad}\, P_b})$. This is clear since every P_{b_i} lies on the path joining P_b to X in $\mathscr{C}_l$.

(b) The modules on $\mathscr{C}'_l$ are $\bar{\Lambda}$-modules and $\mathscr{C}'_l$ is a component of $\Gamma_{\bar{\Lambda}}$. We show that $\mathscr{C}'_l$ is standard. Assume that $\mathscr{C}_l = \mathscr{C}[V_i, B_i]_{i=1}^{t}$ with $\mathscr{C}$ a coinserted tube and $V_t = V$, $B_t = B$. Let $\mathscr{C} = {}_{i=1}^{t'}[V'_i, B'_i]\mathscr{C}'$ with $\mathscr{C}'$ a stable tube. Then $\mathscr{C}'$ is a component of Γ_{Λ_0} where $\Lambda_0 = \Lambda/\Lambda e_0 \Lambda$ and $e_0 = \sum_{x \in \bigcup B_i} e_x + \sum_{x \in \bigcup B'_i} e_x$. As in (a), we obtain that $\Lambda_1 = {}_{i=1}^{t'}[V'_i, B'_i]\Lambda_0$, $\Lambda = \Lambda_1[V_i, B_i]_{i=1}^{t}$ and $\bar{\Lambda} = \Lambda_1[V_i, B_i]_{i=1}^{t-1}$.

Let $X \in \mathscr{C}'$ and s be a vertex of Q_{Λ_0}. Then $\text{Hom}_{\Lambda_0}(X, P'_s) = \text{Hom}_{\Lambda}(X, P_s) = 0$, where P'_s is the projective Λ_0-module corresponding to s. Hence, $\text{inj dim}_{\Lambda_0} \mathscr{C}' = 1$. By [16, 3.1], $\mathscr{C}'$ is a standard component of Γ_{Λ_0}. By [16, 4.5], $\mathscr{C}$ is a standard component of Γ_{Λ_1} and $\mathscr{C}'_l = \mathscr{C}[V_i, B_i]_{i=1}^{t-1}$ is standard in $\text{mod}\,\bar{\Lambda}$.

Since $\mathscr{P}, \mathscr{C}_1, \ldots, \mathscr{C}_{l-1}, \mathscr{C}'_l$ are the components of $\Gamma_{\bar{\Lambda}}$ where the projectives lie, we get that $\bar{\Lambda}$ has acceptable projectives. □

3. The main theorem

We start with some lemmas.

3.1. The following lemma is well known:

Lemma. *Let A be a tame hereditary algebra and ${}_A T = T_0 \oplus T_1$ a tilting module with T_0 preprojective and T_1 regular. Let $\Lambda = \text{End}_A(T)$ be the corresponding domestic tubular algebra with preinjective component $\mathscr{I}$. Then $\Gamma_\Lambda \setminus \mathscr{I} \subset \text{Im}\,\Sigma$, where $\Sigma = \text{Hom}_A(T, -)$.*

Proof. The set $\{\Sigma I_x : x \in Q_A\}$ is a slice in $\mathscr{I}$. Let $M \in \Gamma_\Lambda \setminus \mathscr{I}$. Then M is a predecessor of some $\Sigma I_x \in \text{Im}\,\Sigma$. By [16, 4.2(1)], $\text{Im}\,\Sigma$ is closed under predecessors. □

3.2. Lemma. *Let A, T and Λ be as in 3.1. Let $R \in \Gamma_\Lambda$. Assume that $\Lambda[R]$ is a tubular extension of $\Lambda_0 = \mathrm{End}_A(T_0)$. Then:*

(a) *There is a simple regular A-module R' with $R = \Sigma R'$.*

(b) *If $A[R']$ is wild, then $\Lambda[R]$ is wild.*

Proof. (a) There is a regular A-module R' with $R = \Sigma R'$. Since $\Lambda[R]$ is a tubular extension of Λ_0, there is only one irreducible $R \xrightarrow{\alpha} X$ starting at R. Thus α is mono.

Assume that R' is not simple regular and let $R' \xrightarrow{\beta} Z$ be an irreducible epimorphism. Since $R' \in \mathcal{G}(T)$, $Z \in \mathcal{G}(T)$ and $R = \Sigma R' \xrightarrow{\Sigma\beta} \Sigma Z$ is irreducible. Therefore, $T \otimes_\Lambda \tau^{-1} R \xrightarrow{\sim} T \otimes_\Lambda \operatorname{coker} \Sigma\beta \xrightarrow{\sim} \operatorname{coker} \beta = 0$. This contradicts with $\tau^{-1} R \in \operatorname{Im} \Sigma$.

(b) Let $\mathcal{I}$ be the preinjective component of Γ_A. Since $A[R']$ is wild, the vector space category $\mathcal{U}(\mathrm{Hom}_A(R', \mathcal{I}))$ is wild. But there is a full embedding

$$F: \mathcal{U}(\mathrm{Hom}_A(R', \mathcal{I})) \to \mathcal{U}(\mathrm{Hom}_\Lambda(R, \mathrm{mod}\, \Lambda)),$$

showing that $\Lambda[R]$ is wild. □

3.3. Proposition. *Let Λ be a tubular extension of a tame concealed algebra Λ_0. Let $(m_1, \ldots, m_r)$ be the extension type of Λ and assume that $\mathbb{T}_{m_1,\ldots,m_r}$ is neither Dynkin nor extended Dynkin. Then:*

(a) *q_Λ is not weakly semipositive.*

(b) *Λ is wild.*

Proof. It is enough to show the result in the case that $\mathbb{T}_{m_1,\ldots,m_r}$ is a minimal tree which is neither Dynkin nor extended Dynkin. By [16, 4.4(4)], we may assume that $\Lambda = \Lambda'[R]$ where Λ' is a tubular extension of Λ_0 of Dynkin or extended Dynkin extension type and $R \in \Gamma_{\Lambda'}$. We distinguish these cases.

(1) Assume that the extension type of Λ' is Dynkin. Then Λ' is domestic tubular and $\Lambda' = \mathrm{End}_A(T)$ with A tame hereditary and ${}_AT = T_0 \oplus T_1$ a tilting module with T_0 preprojective and T_1 regular. Let $\Sigma = \mathrm{Hom}_A(T, \text{-})$. By 3.2, there is a simple regular A-module R' with $\Sigma R' = R$. Let t be the vertex in $Q_{A[R']}$ with $\mathrm{rad}\, P_t = R'$. We claim that:

(a′) There exist $V_1, \ldots, V_m$ preinjective A-modules and $a \in \mathbb{N}$ such that

$$q_{A[R']}\left(\sum_{i=1}^m \mathbf{dim}\, V_i + a e_t \right) < 0.$$

(b′) $A[R']$ is wild.

This implies (a) and (b). Indeed, let s be the vertex in Q_Λ such that $\mathrm{rad}\, P_s = R$. Then

$$q_\Lambda\left(\sum_{i=1}^m \mathbf{dim}\, \Sigma V_i + a e_s \right) = q_{\Lambda'}\left(\sum_{i=1}^m \mathbf{dim}\, \Sigma V_i \right) - a \sum_{i=1}^m \dim \mathrm{Hom}_{\Lambda'}(R, \Sigma V_i) + a^2$$

$$= q_A\left(\sum_{i=1}^{m} \mathbf{dim}\, V_i\right) - a\sum_{i=1}^{m} \dim \mathrm{Hom}_A(R', V_i) + a^2$$

$$= q_{A[R']}\left(\sum_{i=1}^{m} \mathbf{dim}\, V_i + ae_t\right) < 0.$$

That Λ is wild follows from 3.2.

The claim may be proved by an easy case by case inspection of the tables in [6] (to show (b′) use [15]).

(2) Assume that the extension type of Λ' is extended Dynkin. Then Λ' is a tubular algebra and $\mathrm{mod}\,\Lambda' = \mathscr{P} \vee \mathscr{T}_0 \vee \bigvee \mathscr{T}_\gamma \vee \mathscr{T}_\infty \vee \mathscr{I}$ as in 2.1. Since Λ is a tubular extension of Λ_0, $R \in \mathscr{T}_0$. There is a module $X \in \mathscr{T}_1$ with $q_{\Lambda'}(\mathbf{dim}\, X) = 0$ and $\mathrm{Hom}_{\Lambda'}(R, X) \neq 0$. Thus $q_\Lambda(2\,\mathbf{dim}\, X + e_s) = 1 - 2\dim \mathrm{Hom}_{\Lambda'}(R, X) < 0$.

Take a family $\{X_n: 1 \leq n \leq 5\}$ of pairwise orthogonal bricks in $\mathscr{T}_1$ [16, 3.1], such that $\mathrm{Hom}_{\Lambda'}(R, X_n) \neq 0$ for $1 \leq n \leq 5$. There is a full embedding of the vector space category $\mathscr{U} = \mathscr{U}(\mathrm{Hom}_{\Lambda'}(R, \{X_n\}_n))$ in $\mathrm{mod}\,\Lambda$. Let S be the poset consisting of five pairwise non comparable points. There is a full embedding of the vector space category $\mathscr{U}(\mathrm{add}\, kS)$ into $\mathscr{U}$. By [13], S is a representation wild poset. Therefore the categories $\mathscr{U}$ and $\mathrm{mod}\,\Lambda$ are wild. □

3.4. Theorem. *Let Λ be an algebra with acceptable projectives. Then the following are equivalent*:

(a) *Λ is iterated tubular.*

(b) *Λ is tame.*

(c) *q_Λ is weakly semipositive.*

Proof. (a) ⇒ (b) and (a) ⇒ (c) is 2.4.

Let $\mathscr{P}, \mathscr{C}_1, \ldots, \mathscr{C}_l$ be the components of Γ_Λ where the projectives lie. Assume that $\mathscr{P}$ is preprojective and that $\mathrm{Hom}_\Lambda(\mathscr{C}_i, \mathscr{C}_j) \neq 0$ implies $i \leq j$. As in 2.5, let $\mathscr{C}_l = \mathscr{C}_l'[V, B]$ and $\Lambda = \bar{\Lambda}[V, B]$ be such that $\bar{\Lambda}$ has acceptable projectives and $\mathscr{P}, \mathscr{C}_1, \ldots, \mathscr{C}_{l-1}, \mathscr{C}_l'$ are the components of $\Gamma_{\bar{\Lambda}}$ where the projectives lie.

We proceed by induction on the number p of projectives on the inserted tubes $\mathscr{C}_1, \ldots, \mathscr{C}_l$.

(c) ⇒ (a). If $p = 0$, $\mathscr{P}$ is a complete preprojective component. By the dual of 1.3, Λ is a domestic cotubular algebra.

Assume $p > 0$, as $\bar{\Lambda}$ is convex in Λ, $q_{\bar{\Lambda}}$ is also weakly semipositive. By induction hypothesis $\bar{\Lambda}$ is n-iterated tubular. Let $\mathrm{mod}\,\bar{\Lambda} = \bar{\mathscr{P}} \vee \bar{\mathscr{T}} \vee \bar{\mathscr{I}}$, where $\bar{\mathscr{I}}$ is the preinjective component of $\Gamma_{\bar{\Lambda}}$ and $\bar{\mathscr{T}}$ is the tubular family in $\Gamma_{\bar{\Lambda}}$ separating $\bar{\mathscr{P}}$ from $\bar{\mathscr{I}}$. With the notation of Section 2, $\bar{\Lambda} = \Lambda_n = \Lambda_{n-1}[E_i, K_i]_{i=1}^t$, where Λ_{n-1} is an $(n-1)$-iterated algebra, A_{n-1} is tame concealed and $A^n = A_{n-1}[E_i, K_i]_{i=1}^t$ is a domestic tubular or a tubular algebra (w.l.g. $n \geq 1$). Then $\mathrm{mod}\, A^n = \mathscr{P}' \vee \mathscr{T}' \vee \bar{\mathscr{I}}$, where $\mathscr{T}'$ is a tubular family separating $\mathscr{P}'$ from $\bar{\mathscr{I}}$.

As in part 2 of the proof of 3.3, we can show that $\mathscr{C}_l'$ is obtained from a tube in $\mathscr{T}'$ by coray insertion. If A^n is domestic tubular, then, as $q_{A^n[V,B]}$ is weakly semipositive, $A^n[V, B]$ is a domestic tubular or a tubular algebra (3.3). Then $\Lambda = \bar{\Lambda}[V, B]$ is again n-iterated tubular. If A^n is tubular, then A^n is a tubular coextension of a tame concealed algebra A_n. By 3.3, $A_n[V, B]$ is domestic tubular or tubular and therefore Λ is an $(n+1)$-iterated tubular algebra.

(b) ⇒ (a). It follows, as in (c) ⇒ (a), from 3.3. □

Corollary. *Let Λ be a sincere algebra with acceptable projectives. Assume that Γ_Λ has no tubes which are both inserted and coinserted. Then the following are equivalent:*

(a) *Λ is a domestic tubular, a domestic cotubular or a tubular algebra.*
(b) *Λ is tame.*
(c) *q_Λ is weakly semipositive.* □

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