On a Theorem of Schreier and Ulam for Countable Permutations

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Communicated by Marshall Hall, Jr.

Received September 2, 1971

In 1933 J. Schreier and S. Ulam [3] gave the Jordan–Hölder composition series for the group Sym(S) of all permutations of a countable set S. In particular, they showed that any normal subgroup containing a permutation P with infinite support must in fact be all of Sym(S). This is equivalent to the statement that the conjugacy class containing such a P must generate Sym(S), i.e., that every permutation \( T \in \text{Sym}(S) \) can be represented as a finite product of permutations conjugate to P.

More recently, in [2], it has been shown that if the disjoint-cycle decomposition of \( P \) consists of infinitely many infinite cycles (and no other cycles) then every permutation in Sym(S) can be represented as a product of two permutations, both conjugate to \( P \). On the other hand the present author proved, in [1], that if the disjoint-cycle decomposition of \( P \) consists of precisely one infinite cycle (and no others) then every permutation in Sym(S) can be represented as a product of three permutations each conjugate to \( P \); however, no finite odd permutation is a product of two such infinite cycles.

In this paper we exhibit a new conjugacy class which has the property that every permutation of S is a product of two permutations from that class, and make use of this and a result from [1] in considering a more general question which now arises naturally: Given a permutation \( P \) of S with infinite support, is there an integer \( m(P) \) such that every permutation in Sym(S) can be represented as a product of \( m \) permutations, each conjugate to \( P \)? Can such an \( m \) be found which is independent of \( P \)? We answer both questions affirmatively in proving the following theorem which is nearly best possible:

**Theorem.** Let \( P \) be any permutation of the countable set \( S \) such that \( P \) has infinite support. Then every permutation of \( S \) is a product of four permutations, each conjugate to \( P \).
Notation. \( M(T) \) denotes the set of symbols of \( S \) which are moved by \( T \in \text{Sym}(S) \); the d.c.d. of \( T \) means the collection of all cycles in the disjoint-cycle decomposition of \( T \); \( (T)_k \) denotes the cardinality of the set of cycles of length \( k \) in the d.c.d. of \( T \), \( 1 \leq k \leq \infty \). \( \mathcal{E}_S \) denotes the conjugacy class of permutations of \( S \) consisting of infinitely many 1-cycles, infinitely many 2-cycles, and infinitely many 3-cycles, with no infinite cycles and no finite cycles of length \( \geq 4 \). \( \mathcal{A} \setminus \mathcal{B} \) means the set-theoretic difference of \( \mathcal{A} \) and \( \mathcal{B} \). All products of permutations are executed from right to left, and the \( \circ \) notation will sometimes be used for emphasis to separate two permutations, and means composition (product).

Lemma 1. Let \( c = (\cdots x_{-3} x_{-2} x_{-1} x_0 x_1 x_2 x_3 \cdots) \) be the permutation of \( \{x_i\}_{-\infty}^{\infty} \) given by \( c(x_i) = x_{i+1} \). Then there exist permutations \( R, S \in \mathcal{E}_{\{x_i\}_{-\infty}^{\infty}} \) such that \( c = R \circ S \).

Proof. We first define \( R \) and \( S \) on the subset
\[ \Omega = \{x_{-5}, x_{-4}, x_{-3}, \ldots, x_0, x_1, x_2, x_3\} \]
as follows:

\[
\begin{align*}
S(x_0) &= x_0, & S(x_{-1}) &= x_1, & R(x_0) &= x_1, & R(x_{-1}) &= x_2, \\
S(x_{-1}) &= x_{-1}, & S(x_{-2}) &= x_2, & R(x_{-1}) &= x_0, & R(x_{-2}) &= x_{-2}, \\
S(x_{-2}) &= x_{-3}, & S(x_{-3}) &= x_{-2}, & R(x_{-2}) &= x_{-1}, & R(x_{-3}) &= x_3, \\
S(x_{-3}) &= x_{-5}, & S(x_{-4}) &= x_{-4}, & R(x_{-3}) &= x_{-4}, & R(x_{-4}) &= x_{-3}, \\
S(x_{-4}) &= x_3, & S(x_{-5}) &= x_3, & R(x_{-4}) &= x_{-5}, & R(x_{-5}) &= x_4.
\end{align*}
\]

It is easy to check that both \( R \) and \( S \), restricted to \( \Omega \setminus \{x_{-5}\} \) consist only of 1-cycles, 2-cycles, and 3-cycles. Furthermore, \( RS(x_i) = x_{i+1} \) for each \( x_i \in \Omega \).

In order to define \( R \) and \( S \) on the remainder of \( \{x_i\}_{-\infty}^{\infty} \), we partition the \( \{x_i\}_{-\infty}^{\infty} \setminus \Omega \) according to the residues \( \text{mod} 3 \) of their subscripts \( i \).

If \( i \geq 4 \) and \( i \equiv 1 \pmod{3} \), let \( S(x_i) = x_{-(i+2)} \) and \( R(x_i) = x_{-(i+1)} \), 
if \( i \equiv 2 \pmod{3} \), let \( S(x_i) = x_{i+1} \) and \( R(x_i) = x_{-(i+2)} \), 
if \( i \equiv 0 \pmod{3} \), let \( S(x_i) = x_{-(i+2)} \) and \( R(x_i) = x_i \).

If \( i \leq -6 \) and \( i \equiv 0 \pmod{3} \), let \( S(x_i) = x_{-(i+2)} \) and \( R(x_i) = x_{-(i+1)} \), 
if \( i \equiv 2 \pmod{3} \), let \( S(x_i) = x_i \) and \( R(x_i) = x_{i+1} \), 
if \( i \equiv 1 \pmod{3} \), let \( S(x_i) = x_{-(i+3)} \) and \( R(x_i) = x_{-(i+1)} \).

We may now verify directly that for each \( x_i \in \{x_i\}_{-\infty}^{\infty} \setminus \Omega \) we have \( RS(x_i) = x_{i+1} \). Furthermore, whenever \( i \leq -6 \) and \( i \equiv 0 \pmod{3} \), \( R \) contains the 3-cycle \( (x_i x_{-(i+1)} x_{i-1}) \); if \( i \leq -6 \) and \( i \equiv 1 \pmod{3} \), \( R \) contains the transposition
(x_{i} x_{i+1}); for i \geq -4, i \equiv 0 \pmod{3} R contains the 1-cycle (x_i). Thus \( R \in \mathcal{E}_{(x_i)_{i=\infty}^{\infty}} \). Similarly, for i \leq -6 and i \equiv 0 \pmod{3}, S contains the 2-cycle (x_{i} x_{i+3}); if i \leq -6 and i \equiv 1 \pmod{3}, S contains the 3-cycle (x_{i} x_{i+3} x_{i+6}); whenever i \leq -6, i \equiv 2 \pmod{3}, S contains the 1-cycle (x_i). Thus the d.c.d. of S also consists of infinitely many 1-cycles, infinitely many 2-cycles, and infinitely many 3-cycles, and no other cycles, so \( S \in \mathcal{E}_{(x_i)_{i=-\infty}^{\infty}} \).

Now suppose \( \{x_i\}_{i=1}^{k} \) is any collection of k symbols from S. For even \( k \geq 4 \), let \( \mathcal{H}_k \) denote the conjugacy class of permutations of \( \{x_i\}_{i=1}^{k} \) whose d.c.d. consists of one 1-cycle, one 3-cycle, and \( (k-4)/2 \) 2-cycles; let \( \mathcal{I}_k \) denote the class of permutations of \( \{x_i\}_{i=1}^{k} \) whose d.c.d. consists of two 1-cycles and \( (k-2)/2 \) 2-cycles. For odd \( k \geq 5 \), let \( \mathcal{L}_k \) denote the class of permutations of \( \{x_i\}_{i=1}^{k} \) whose d.c.d. consists of two 1-cycles, one 3-cycle, and \( (k-5)/2 \) 2-cycles. We then have the following

**Lemma 2.** Let \( (x_1 x_2 x_3 \cdots x_k) \) be a finite cycle of length \( k \geq 4 \), and \( \mathcal{C}_k \) the conjugacy class of permutations of \( \{x_i\}_{i=1}^{k} \) which are \( k \)-cycles. Then, with \( \mathcal{H}_k \), \( \mathcal{I}_k \) and \( \mathcal{L}_k \) defined as above we have \( \mathcal{C}_k \subseteq \mathcal{I}_k \mathcal{H}_k = \mathcal{H}_k \mathcal{I}_k \) for \( k \) even, and \( \mathcal{C}_k \subseteq \mathcal{L}_k \mathcal{L}_k \) for \( k \) odd.

**Proof.** Since

\[
(x_1 x_2 x_3 x_4) = (x_3)(x_4)(x_1 x_2) \circ (x_1)(x_2 x_3 x_4) = (x_2)(x_1 x_3 x_4) \circ (x_3)(x_4)(x_1 x_2),
\]

the lemma is true for \( k = 4 \). When \( k \geq 6 \) is even, we have

\[
(x_1 x_2 \cdots x_k) = (x_{(k+2)}/2)(x_{(k-4)}/2)(x_{(k+4)}/2)(x_{(k-6)}/2)(x_{(k+6)}/2)
\]

\[
\cdots (x_{4} x_{2} x_{1})(x_{3} x_{1})(x_{1})(x_{2} x_{3} x_{4})(x_{3} x_{5} x_{1})
\]

\[
\cdots (x_{(k-2)}/2 x_{(k+4)}/2)(x_{k}/2 x_{(k+2)}/2 x_{(k+4)}/2) \in \mathcal{I}_k \circ \mathcal{H}_k.
\]

Note that \( \mathcal{I}_k \circ \mathcal{H}_k = \mathcal{H}_k \circ \mathcal{I}_k \) since both are conjugacy classes. Since

\[
(x_1 x_2 x_3 x_4 x_5) = (x_3)(x_4)(x_1 x_2 x_3) \circ (x_1)(x_2)(x_3 x_4 x_5),
\]

the lemma is true when \( k = 5 \). When \( k \geq 7 \) is odd, we have

\[
(x_1 x_2 \cdots x_k) = (x_{(k+3)}/2)(x_{(k+1)}/2)(x_{(k+5)}/2)(x_{(k+7)}/2)
\]

\[
\cdots (x_{k-1} x_{k})(x_{k})(x_1 x_2 x_3)(x_2 x_4 x_5)(x_5 x_{k-1})(x_6 x_{k-2})
\]

\[
\cdots (x_{(k+1)}/2 x_{(k+3)}/2)(x_{(k+5)}/2) \in \mathcal{L}_k \circ \mathcal{L}_k.
\]

**Theorem 1.** Let \( T \) be any permutation of a countably infinite set \( S \). Then \( T \) may be expressed as a product \( R_1 \circ R_2 \) of two conjugate permutations belonging to \( \mathcal{E}_S \).
Proof. (i) \( M(T) \) is infinite. Let \( \{f_i\}_{i \in I} \) denote the collection of symbols which appear among the 1-, 2-, or 3-cycles of \( T \), and assume first that \( \sum_{n \geq 4} (T)_n < \infty \) and \( (T)_\infty = 0 \). Since \( M(T) \) is infinite, either \( (T)_2 = \infty \) or \( (T)_3 = \infty \). If \( (T)_2 = \infty \) [and \( (T)_3 \) is arbitrary], we put \( R_1(f_i) = T(f_i) \) and \( R_2(f_i) = f_i \) if either \( f_i \) is fixed by \( T \) or appears in a 3-cycle of \( T \). Since

\[
(f_i, f_i, f_i)(f_i, f_i, f_i)(f_i, f_i)(f_i, f_i, f_i) = (f_i, f_i, f_i)(f_i, f_i, f_i)(f_i, f_i)(f_i, f_i, f_i),
\]

we can express \( T \) restricted to all of \( \{f_i\}_{i \in I} \) as a product of two permutations in \( \mathcal{E}_{\{f_i\}_{i \in I}} \). Using Lemma 2 on the finite cycles of \( T \) of length \( \geq 4 \), we may now define \( R_1 \) and \( R_2 \) on all of \( S \) so that \( R_1 \circ R_2 = T \) and \( R_1 \), \( R_2 \in \mathcal{E}_S \).

If \( (T)_2 < \infty \) and \( (T)_3 = \infty \), we put \( R_1(f_i) = T(f_i) \) and \( R_2(f_i) = f_i \) if \( f_i \) is either fixed by \( T \) or appears in a 2-cycle of \( T \). Since

\[
(f_i, f_i, f_i)(f_i, f_i, f_i)(f_i, f_i, f_i)(f_i, f_i, f_i) = (f_i, f_i, f_i)(f_i, f_i, f_i)(f_i, f_i, f_i)(f_i, f_i, f_i),
\]

we can again express \( T \) restricted to all of \( \{f_i\}_{i \in I} \) as a product of two permutations belonging to \( \mathcal{E}_{\{f_i\}} \). Just as before, one can use Lemma 2 on the finite cycles of \( T \) of length \( \geq 4 \) to get \( T = R_1 \circ R_2 \), \( R_i \in \mathcal{E}_S \).

Now suppose \( (T)_\infty > 0 \) [and \( \sum_{n \geq 1} (T)_n \) arbitrary]. By Lemma 1 we can express each infinite cycle \( c \) as a product of two permutations in \( \mathcal{E}_{M(c)} \). Thus, if \( C \) denotes the product of the (disjoint) infinite cycles of \( T \), we can express \( T \), restricted to \( M(C) \), as a product of two permutations in \( \mathcal{E}_{M(C)} \).

If \( C' \) denotes the product of the (disjoint) finite cycles of \( T \) of length \( \geq 4 \), we can use Lemma 2 to express \( T \), restricted to \( M(C') \), as a product of two permutations of \( M(C') \), each containing at most 1-, 2-, or 3-cycles. Furthermore, the restriction of \( T \) to \( \{f_i\}_{i \in I} \) can easily be expressed as a product of two permutations of \( \{f_i\}_{i \in I} \), each with at most 1-, 2-, or 3-cycles. Again we can find \( R_1 \) and \( R_2 \in \mathcal{E}_S \) with \( R_1 \circ R_2 = T \).

A slight subtlety arises when \( (T)_\infty = 0 \) and \( \sum_{n \geq 4} (T)_n = \infty \). Let \( C_0 \) denote the product of the (disjoint) odd-length (\( \geq 5 \)) cycles in the d.c.d. of \( T \), and \( C_e \) the product of the even-length (\( \geq 4 \)) cycles of \( T \). If \( \sum_{n \geq 5 \text{ even}} (T)_n = \infty \), we know using Lemma 2 that \( T \) restricted to \( M(C_0) \) can be expressed as a product of two permutations belonging to \( \mathcal{E}_{M(C_0)} \). Also by Lemma 2, \( T \) restricted to \( M(C_e) \) can be expressed as a product of two permutations of \( M(C_e) \) each with at most 1-, 2-, or 3-cycles. As before the restriction of \( T \) to \( \{f_i\}_{i \in I} \) is easily decomposed, and thus \( T \) may be correctly decomposed on all of \( S \). On the other hand, if \( \sum_{n \geq 5 \text{ odd}} (T)_n < \infty \) and \( \sum_{n \geq 4 \text{ even}} (T)_n = \infty \), let \( e_1, e_2, e_3, \ldots \) be the cycles of \( T \) of finite even length \( \geq 4 \). By Lemma 2 we can express each \( e_i \in \{e_1, e_3, e_5, e_7, \ldots \} \).
as a product $e_i^{(1)} \circ e_i^{(2)}$ of permutations of $M(e_i)$, $e_i^{(1)}$ containing only 1- and 2-cycles, $e_i^{(2)}$ containing 1-, 2-, and 3-cycles only. Also, each $e_j \in \{e_2, e_4, e_6, e_8, \ldots\}$ can be expressed as a product $e_j^{(1)} \circ e_j^{(2)}$ of permutations of $M(e_j)$, where this time $e_j^{(1)}$ contains just 1-, 2-, and 3-cycles and $e_j^{(2)}$ contains only 1- and 2-cycles. As before, $T$ restricted to $M(C_0)$ can be expressed as a product of two permutations of $M(C_0)$, each with only 1-, 2-, and 3-cycles. The restriction of $T$ to $\{f_i\}_{i \in I}$ is handled exactly as before, and now the theorem is proved for all $T \in \text{Sym}(S)$, with $M(T)$ infinite.

(ii) $M(T)$ is finite. Here, let $\{f_i\}_{i=1}^{\infty}$ denote just the infinitely many symbols which are fixed by $T$. Define $R_1$ on $\{f_i\}$ by

$$R_1 = (f_1)(f_2f_3)(f_6f_5f_4)(f_7)(f_8f_9)(f_{12}f_{11}f_{10}) \cdots$$

and define $R_2$ on $\{f_i\}$ by

$$R_2 = (f_1)(f_2f_3)(f_4f_5f_6)(f_7)(f_8f_9)(f_{10}f_{11}f_{12}) \cdots.$$ 

Then $R_1 \circ R_2 = T$ restricted to $\{f_i\}$, and $R_1, R_2 \in \mathcal{S}(f_i)$. As before, by Lemma 2 we can define $R_1$ and $R_2$ on $M(T) = S \setminus \{f_i\}$ in such a way that $R_1$ and $R_2$ each have at most 1-, 2-, or 3-cycles and $R_1 \circ R_2 = T$ restricted to $M(T)$. Again we conclude that $R_1$ and $R_2$ can be defined on all of $S$ so that $T = R_1 \circ R_2$ and $R_1, R_2 \in \mathcal{S}$.

**Theorem 2.** Let $P$ be any permutation of the countable set $S$, with $M(P)$ infinite. Suppose $R$ is any permutation in $\mathcal{S}$. Then there exist permutations $U, V$ of $S$ such that $U$ and $V$ are conjugate to $P$ in $\text{Sym}(S)$ and $R = U \circ V$.

**Proof.** We partition the 1-, 2-, and 3-cycles of the d.c.d. of $R$ according to the d.c.d. of $P$. Corresponding to each infinite cycle $C_n$ of $P$, we choose a different infinite subcollection of 1-cycles, infinite subcollection of 2-cycles, and infinite subcollection of 3-cycles from the d.c.d. of $R$. For each $n$, let $\{x_{j}^{(m)}\}_{j=-\infty}^{\infty}$ denote the set of symbols of $S$ in these 1-, 2-, and 3-cycles; we have chosen cycles so that $\{x_{j}^{(m)}\}_{j=-\infty}^{\infty} \cap \{x_{j}^{(n)}\}_{j=-\infty}^{\infty} = \emptyset$ if $m \neq n$. According to Theorem 4.2 of [1], we know that for each $n$ there exist two infinite cycles $a_n, b_n$, with $M(a_n) = M(b_n) = \{x_{j}^{(n)}\}_{j=-\infty}^{\infty}$ and $a_n \circ b_n = R$ restricted to $\{x_{j}^{(n)}\}_{j=-\infty}^{\infty}$. We define the permutations

$$U, V \text{ on } \bigcup_n \{x_{j}^{(n)}\}_{j=-\infty}^{\infty} \text{ by } U = a_1a_2 \cdots \text{ and } V = b_1b_2 \cdots.$$ 

Since $M(a_k) \cap M(b_l) = \emptyset$ if $k \neq l$, we find that

$$U \circ V = a_1a_2 \cdots b_1b_2 \cdots = a_1b_1a_2b_2a_3b_3 \cdots = R,$$

restricted to $\bigcup_n \{x_{j}^{(n)}\}_{j=-\infty}^{\infty}$. 

\
Corresponding to each 1-cycle in the d.c.d. of \( P \), we choose a 1-cycle \((v_n)\) from the remaining cycles of \( R \), and now we include this 1-cycle in the definitions of \( U \) and \( V \). Thus \( U \circ V = R \) restricted also to \( \bigcup_n \{v_n\} \). Corresponding to each 2-cycle in the d.c.d. of \( P \), we choose a pair of 1-cycles from the remaining cycles of \( R \), say \((w_{n,1}, w_{n,2})\). If we define both \( U \) and \( V \) on \( \bigcup_n \{w_{n,1}, w_{n,2}\} \) by \( U = V = (w_{1,1}w_{1,2})(w_{2,1}w_{2,2})(w_{3,1}w_{3,2}) \cdots \), then it is obvious that now \( U \circ V = R \) restricted also to \( \bigcup_n \{w_{n,1}, w_{n,2}\} \). Corresponding to each 3-cycle in the d.c.d. of \( P \), we choose a 3-cycle \((u_{n,1}u_{n,2}u_{n,3})\) from the remaining cycles of \( R \). If we define \( U \) and \( V \) on \( \bigcup_n \{u_{n,1}, u_{n,2}, u_{n,3}\} \) by

\[
U - V = (u_{1,1}u_{1,3}u_{1,2})(u_{2,1}u_{2,3}u_{2,2}) \cdots (u_{n,1}u_{n,3}u_{n,2}) \cdots ,
\]

we have \( U \circ V = R \), restricted now also to \( \bigcup_n \{u_{n,1}, u_{n,2}, u_{n,3}\} \).

Corresponding to each \( k \)-cycle \((k \geq 4)\) in the d.c.d. of \( P \), we assign a collection of the remaining cycles of \( R \) and define \( U \) and \( V \) on the corresponding subsets of symbols of \( S \) as follows:

(i) When \( k \equiv 0 \pmod{4} \) choose, corresponding to each cycle \( k \) of length \( k \), \( k/2 \) new 2-cycles from the d.c.d. of \( R \), say

\[
(t_{n,1}t_{n,2}), (t_{n,3}t_{n,4}), \ldots, (t_{n,k-3}t_{n,k-2}), (t_{n,k-1}t_{n,k}).
\]

Define \( U \) and \( V \) on \( \bigcup_n \{t_{n,i}\}_{i=1}^k \), each as a product of \( (P)_k \) disjoint \( k \)-cycles, by

\[
U = \cdots (t_{n,k}t_{n,k-3}t_{n,k-1}t_{n,k-2}t_{n,k-4}t_{n,k-7}t_{n,k-3}t_{n,k-6} \cdots \cdot t_{n,3}t_{n,5}t_{n,7}t_{n,9}t_{n,11}t_{n,3} \cdots ) \cdots ,
\]

\[
V = \cdots (t_{n,2}t_{n,4}t_{n,1}t_{n,3}t_{n,6}t_{n,8}t_{n,3}t_{n,5}t_{n,7} \cdots \cdot t_{n,k-6}t_{n,k-4}t_{n,k-7}t_{n,k-3}t_{n,k-2}t_{n,k-4}t_{n,k-3}t_{n,k-1} \cdots ) \cdots .
\]

Then we may easily check that \( U \circ V = R \) restricted now also to \( \bigcup_n \{t_{n,i}\}_{i=1}^k \).

(ii) When \( k \equiv 2 \pmod{4} \) choose, corresponding to each cycle \( k \) of length \( k \), \((k - 2)/2 \) new 2-cycles and 2 new 1-cycles, say

\[
(s_{n,1}s_{n,2}), \ldots, (s_{n,k-3}s_{n,k-2}), (s_{n,k-1}), (s_{n,k}).
\]

Define \( U \) and \( V \) on \( \bigcup_n \{s_{n,i}\}_{i=1}^k \), each as a product of \( (P)_k \) disjoint \( k \)-cycles, as follows:

\[
U = \cdots (s_{n,k-2}s_{n,k-5}s_{n,k-3}s_{n,k-4}s_{n,k-6}s_{n,k-9}s_{n,k-7}s_{n,k-8} \cdots \cdot s_{n,5}s_{n,7}s_{n,9}s_{n,4}s_{n,1}s_{n,3}s_{n,2}s_{n,1} \cdots ) \cdots ,
\]

\[
V = \cdots (s_{n,2}s_{n,4}s_{n,1}s_{n,3}s_{n,6}s_{n,8}s_{n,5}s_{n,7} \cdots \cdot s_{n,k-6}s_{n,k-4}s_{n,k-9}s_{n,k-7}s_{n,k-4}s_{n,k-5}s_{n,k-3}s_{n,k-1} \cdots ) \cdots .
\]
Again, it is easily checked that $U \circ V = R$ restricted now also to $\bigcup_n \{ s_n,i \}_{i=1}^k$. In case $k \equiv 1$ or $3 \pmod{4}$ the method is nearly the same. In the first instance we choose $(k - 1)/2$ 2-cycles and one 1-cycle from the remaining cycles of the d.c.d. of $R$; in the latter case we choose $(k - 3)/2$ 2-cycles and three 1-cycles. The construction of $U$ and $V$ so that $U \circ V = R$ proceeds almost exactly as in the case $k \equiv 2 \pmod{4}$, the only difference being that for these odd $k$ we introduce a different number of fixed symbols of $R$ at the end of both $U$ and $V$.

Since $M(P)$ is infinite, the preceding constructions may be carried out in such a way that each 1-, 2-, or 3-cycle of $R$ has been chosen to correspond to some cycle of $P$ (though in general the same cycle of $P$ may correspond to many cycles of $R$). Furthermore, we have insured at each stage that $U \circ V = R$, that $(U)_k = (V)_k = (P)_k$ for each $k \geq 1$, and that $(U)_\infty = (V)_\infty = (P)_\infty$; the theorem is completely proved.

**Theorem 3.** Let $P$ be any permutation of the countable set $S$, with $M(P)$ infinite. Then every permutation $T$ in $\text{Sym}(S)$ is a product of four permutations, each conjugate to $P$.

**Proof.** By Theorem 1, $T = R_1 \circ R_2$, where $R_1, R_2 \in \mathcal{O}_S$. By Theorem 2, $R_1 = U_1 \circ V_1$ and $R_2 = U_2 \circ V_2$, where $U_1, U_2, V_1, V_2$ are each conjugate to $P$ in $\text{Sym}(S)$. Thus $T = R_1 \circ R_2 = U_1 \circ V_1 \circ U_2 \circ V_2$.

As was mentioned earlier in the introduction, we have shown elsewhere that if the d.c.d. of $P$ consists of precisely one infinite cycle (and no others) then every permutation of $S$ is a product of three permutations conjugate to $P$. Since we have not been able to construct a counterexample, we make the following:

**Conjecture.** Let $P$ be any permutation of the countable set $S$, with $M(P)$ infinite. Then every $T \in \text{Sym}(S)$ is a product of three permutations, each conjugate to $P$.

**References**