Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Differential Equations 229 (2006) 257–269
Journal of Differential Equations
www.elsevier.com/locate/jde

Asymptotic regularity conditions for the strong convergence towards weak limit sets and weak attractors of the 3D Navier–Stokes equations

Ricardo M.S. Rosa *

Departamento de Matemática Aplicada, Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68530 Ilha do Fundão, Rio de Janeiro RJ 21945-970, Brazil

Received 3 October 2005

Abstract

The asymptotic behavior of solutions of the three-dimensional Navier–Stokes equations is considered on bounded smooth domains with no-slip boundary conditions and on periodic domains. Asymptotic regularity conditions are presented to ensure that the convergence of a Leray–Hopf weak solution to its weak ω-limit set (weak in the sense of the weak topology of the space H of square-integrable divergence-free velocity fields with the appropriate boundary conditions) are achieved also in the strong topology. It is proved that the weak ω-limit set is strongly compact and strongly attracts the corresponding solution if and only if all the solutions in the weak ω-limit set are continuous in the strong topology of H. Corresponding results for the strong convergence towards the weak global attractor of Foias and Temam are also presented. In this case, it is proved that the weak global attractor is strongly compact and strongly attracts the weak solutions, uniformly with respect to uniformly bounded sets of weak solutions, if and only if all the global weak solutions in the weak global attractor are strongly continuous in H.

MSC: primary 35Q30; secondary 35B40, 35B41, 37L30

Keywords: Navier–Stokes equations; Weak global attractor; Global attractor; Limit sets

* Fax: +55 21 2562 7374.
E-mail address: rrosa@ufrj.br.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.03.004

1. Introduction

The notions of limit sets and attractors (whether local or global) permeate the theory of dynamical systems both in finite and in infinite dimensions. In the case of infinite dimensions, the existence of such sets, in particular that of the global attractor, is a major issue. We recall here that the global attractor is a compact set which is minimal for the inclusion relation among the sets that uniformly attract, as time goes to infinity, all bounded sets of initial conditions. The global attractor contains, for instance, all locally attracting sets and ω-limit sets. Existence results of such limit sets and attractors have been obtained for a number of nonlinear partial differential equations modeling various phenomena.

In this note we address the celebrated system associated with the Navier–Stokes equations for an incompressible fluid filling a region in a three-dimensional space. Due in particular to the lack of a result on the global well-posedness for this system the notion of attractor in this case is not settled, and the study of the asymptotic behavior of this system is a major challenge. The classical notion of global attractor should be modified since there may not be a unique solution associated with a given initial condition and defined for all positive times. And even if we start with solutions defined for all positive times, it is not known whether there is a "global attractor" attracting all such solutions in the strong topology of any suitable phase space.

In [8] Foias and Temam introduced the notion of weak global attractor (see the definition in (3.1)), which is loosely speaking a global attractor for the weak topology of the natural phase space of square-integrable divergence-free velocity fields with the appropriate boundary conditions. They have proved that such a weak global attractor exists, having the properties that it is weakly compact, attracts the weak solutions in the weak topology, and is positively invariant (see Section 3 for more details).

Another important concept is that of a trajectory attractor, which has been considered in [2,4,17]. Such trajectory attractor is a classical global attractor but in the space of weak solutions defined on $[0, \infty)$, with the corresponding semigroup being simply the translation in time of such solutions. Although seemingly physically unrealistic, since the "initial condition" contains the information of the state of the system at all positive times, this is an important notion since it allows for the use of classical results in dynamical system and ergodic theories, which can be used in the study of the evolution of the system back in the phase space. In particular, the weak global attractor of [8] can, in fact, be recovered from the trajectory attractor by taking a projection, into the phase space, of the solutions in the trajectory attractor at an arbitrary instant of time.

Similarly to the notion of weak global attractor, the concept of weak limit set (limit set for the weak topology, see the definition in (3.4)) can also be considered. Our aim in this note is to consider weak limit sets and the weak global attractor of Foias and Temam and present necessary and sufficient conditions for the attraction to hold in the strong topology of the phase space. The main condition is an asymptotic regularity condition. More precisely, we prove that the weak ω-limit set of a given weak solution is strongly compact and strongly attracts the corresponding weak solution if and only if all the global weak solutions in the weak ω-limit set are strongly continuous. This result is given in Theorem 4.1. It implies, in particular, that weakly attracting fixed points are necessarily strongly attracting since they are constant and, hence, strongly continuous.

Similarly, it is proved that the weak global attractor is strongly compact and strongly attracts the weak solutions, uniformly with respect to uniformly bounded weak solutions, if and only if all the global weak solutions in the weak global attractor are strongly continuous. This result is presented in Theorem 5.1.

The proof of these results is based on techniques devised by Ball (see [1–3,9], and also [11,12,16] for some applications of these techniques to the two-dimensional Navier–Stokes equations), originating from the use of energy-type equations to prove asymptotic compactness of the trajectories. Let us consider limit sets for simplicity. We start with a trajectory $\mathbf{u} = \mathbf{u}(t)$. This trajectory is said to be asymptotically compact (in a given space) if given any time sequence $t_n \to \infty$, there exists a convergent subsequence for $\{\mathbf{u}(t_n)\}_n$. This asymptotic compactness implies the existence of the corresponding ω-limit set. A similar result holds for global attractors [10,14,18,20].

The idea of the energy-equation method to obtain the asymptotic compactness can be divided in two steps:

(i) weak compactness of the sequence $\{\mathbf{u}(t_n)\}_n$, and
(ii) norm convergence $|\mathbf{u}(t_{n_j})| \to |\mathbf{v}_0|$ of a weakly convergent subsequence $\mathbf{u}(t_{n_j}) \rightharpoonup \mathbf{v}_0$, as $j \to \infty$.

In uniformly convex spaces (such as Hilbert spaces), weak plus norm convergences implies strong convergence, hence asymptotic compactness in the strong topology. In practice, the first step follows from classical a priori estimates obtained from energy-type *inequalities*, while the second one, as developed in [1,3], follows from energy-type *equations*.

In the three-dimensional Navier–Stokes equations, however, the (Leray–Hopf) weak solutions are known to satisfy only an energy inequality. This problem can be partly overcome by noting that the equality in the argument cited above is used only for the limit solution passing through $\mathbf{v}_0$. Hence, the strong convergence can be obtained under the assumption that the limit solution be more regular in the sense of satisfying the energy equation. This condition can in fact be further relaxed by using another technique of Ball presented in [2], related to the notion of generalized semiflows. This technique allows one to prove the strong convergence under the sole assumption that the limit solution be strongly continuous in H. This result is given in Lemma 4.2, which is a fundamental step towards Theorem 4.1.

In a concurrent work, Cheskidov and Foias [5] address similar issues. They also obtain that the strong continuity of the solutions in the weak global attractor is a sufficient condition for the weak global attractor to be strongly compact and strongly attracting. Their technique is different and yields a number of other results. Necessary conditions, however, are not given. The weak ω-limit sets are not mentioned explicitly in [5], either, but it is clear that their technique is directly applicable to these objects as well.

As a final remark we mention that this idea may be adapted to yield similar results for other differential equations in which uniqueness and lack of regularity are troublesome, such as wave equations with critical nonlinearities. It can also be adapted for weak α-limit sets.

2. Preliminaries

We recall now some classical results which can be found, for instance, in [6,13,15,19]. We consider the three-dimensional Navier–Stokes equations with either periodic or no-slip boundary conditions. In the periodic case, we consider the whole space $\mathbb{R}^3$, and the flow is assumed periodic with period L_i in each direction x_i. We define $\Omega = \prod_{i=1}^{3}(0, L_i)$ and assume that the average flow on Ω vanishes, i.e.

$$\int_{\Omega} \mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

Here, $\mathbf{u} = (u_1, u_2, u_3)$ denotes the velocity vector, and $\mathbf{x} = (x_1, x_2, x_3)$, the space variable.

In the no-slip case, the flow is considered in a bounded domain Ω of $\mathbb{R}^3$ with the no-slip boundary condition $\mathbf{u} = 0$ on $\partial\Omega$.

In either the periodic or the no-slip case, we obtain a functional equation formulation for the time-dependent velocity field $\mathbf{u} = \mathbf{u}(t)$ in a suitable space H:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}. \tag{2.1}$$

We consider the test spaces

$$\mathcal{V} = \biggl\{\mathbf{u} = \mathbf{w}|_{\Omega}\colon\ \mathbf{w} \in \mathcal{C}^{\infty}\bigl(\mathbb{R}^3\bigr),\ \nabla \cdot \mathbf{w} = 0,\ \int_{\Omega} \mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,\ \text{and } \mathbf{w}(\mathbf{x}) \text{ is}$$
$$\text{periodic with period } L_i \text{ in each direction } x_i\biggr\},$$

in the periodic case, and

$$\mathcal{V} = \bigl\{\mathbf{u} \in \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)^3\colon\ \nabla \cdot \mathbf{u} = 0\bigr\},$$

in the no-slip case, where $\mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$ denotes the space of infinitely-differentiable real-valued functions with compact support on Ω. In either case, we define H as the completion of $\mathcal{V}$ under the $L^2(\Omega)^3$ norm. We also consider the space V defined as the completion of $\mathcal{V}$ under the $H^1(\Omega)^3$ norm. We identify H with its dual and consider the dual space V', so that $V \subset H \subset V'$. We denote by H_{w} the space H endowed with its weak topology.

We consider the inner products in H and V given respectively by

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad ((\mathbf{u}, \mathbf{v})) = \int_{\Omega} \sum_{i=1,3} \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{v}}{\partial x_i}\,\mathrm{d}\mathbf{x},$$

and the associated norms $|\mathbf{u}| = (\mathbf{u}, \mathbf{u})^{1/2}$, $\|\mathbf{u}\| = ((\mathbf{u}, \mathbf{u}))^{1/2}$. The norm in the dual space V' is denoted by $\|\mathbf{u}\|_{V'}$.

We denote by P_{LH} the (Leray–Helmholtz) orthogonal projector in $L^2(\Omega)^3$ onto the subspace H. In (2.1), A is the Stokes operator $A\mathbf{u} = -P_{\mathrm{LH}}\Delta\mathbf{u}$; $B(\mathbf{u}, \mathbf{v}) = P_{\mathrm{LH}}((\mathbf{u} \cdot \nabla)\mathbf{v})$ is a bilinear term corresponding to the inertial term; $\mathbf{f}$ represents the mass density of volume forces applied to the fluid, and we assume that $\mathbf{f} \in V'$; and $\nu > 0$ is the kinematic viscosity. The Stokes operator is a positive self-adjoint operator on H, and we denote by $\lambda_1 > 0$ its first eigenvalue.

A *Leray–Hopf weak solution* on an open time interval $I = (t_0, t_1)$, $-\infty \leqslant t_0 < t_1 \leqslant \infty$, is defined as a function $\mathbf{u} = \mathbf{u}(t)$ on (t_0, t_1) with values in H and satisfying the following properties:

(i) $\mathbf{u} \in L^{\infty}(t_0, t_1; H) \cap L^2_{\mathrm{loc}}(t_0, t_1; V)$;
(ii) $\partial\mathbf{u}/\partial t \in L^{4/3}_{\mathrm{loc}}(t_0, t_1; V')$;

(iii) $\mathbf{u} \in \mathcal{C}(I; H_{\mathrm{w}})$;
(iv) $\mathbf{u}$ satisfies the functional equation (2.1) almost everywhere on $I = (t_0, t_1)$;
(v) $\mathbf{u}$ satisfies the following energy inequality in the distribution sense on $I = (t_0, t_1)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathbf{u}(t)\right|^2 + \nu\left\|\mathbf{u}(t)\right\|^2 \leqslant \left(\mathbf{f}, \mathbf{u}(t)\right). \tag{2.2}$$

The set $\mathcal{C}(I, H_{\mathrm{w}})$ is simply the set of functions $\mathbf{u}: I \to H$ which are weakly continuous in H, i.e. for every $\mathbf{w}$ in H, $t \mapsto (\mathbf{u}(t), \mathbf{w})$ is a continuous real-valued function. A related space is obtained when considering subsets of functions bounded in H. More precisely, given $R > 0$, we consider the closed ball

$$B_H(R) = \left\{\mathbf{u} \in H\colon\ |\mathbf{u}| \leqslant R\right\}$$

and denote by $B_H(R)_{\mathrm{w}}$ this ball endowed with the weak topology of H. Since H is separable, the weak topology in $B_H(R)$ is metrizable, and we denote its metric by $d_{B_H(R)_{\mathbf{w}}}(\cdot,\cdot)$. Then, we consider the space $\mathcal{C}(I; B_H(R)_{\mathbf{w}})$ endowed with the uniform metric

$$d_{\mathcal{C}(I; B_H(R)_{\mathbf{w}})}(\mathbf{u}, \mathbf{v}) = \sup_{t \in I} d_{B_H(R)_{\mathbf{w}}}\left(\mathbf{u}(t), \mathbf{v}(t)\right).$$

This is a complete metric space.

A *Leray–Hopf weak solution* on an interval of the form $[t_0, t_1)$ is defined as a Leray–Hopf weak solution on (t_0, t_1) which is strongly continuous at $t = t_0$, i.e.

(vi) $\mathbf{u}(t) \to \mathbf{u}(t_0)$ in H, as $t \to t_0^+$.

A *global Leray–Hopf weak solution* for us means a Leray–Hopf weak solution on $\mathbb{R}$.

From now on, for notational simplicity, a weak solution will always mean a Leray–Hopf weak solution.

For a weak solution on an arbitrary interval I, it follows that

$$\left|\mathbf{u}(t)\right|^2 \leqslant \left|\mathbf{u}(t')\right|^2 e^{-\nu\lambda_1(t-t')} + \frac{1}{\nu^2\lambda_1}\|\mathbf{f}\|_{V'}^2\left(1 - e^{-\nu\lambda_1(t-t')}\right), \tag{2.3}$$

for all t in I and almost all t' in I with $t' < t$. The allowed times t' are the Lebesgue points of the function $t \mapsto |\mathbf{u}(t)|^2$ in the sense that

$$\frac{1}{h}\int_{t'}^{t'+h} \left|\mathbf{u}(t)\right|^2 \mathrm{d}t \to \left|\mathbf{u}(t')\right|^2, \quad \text{as } h \to 0^+. \tag{2.4}$$

In the case of a weak solution on an interval of the form $[t_0, t_1)$, the point t_0 is a point of continuity of $t \mapsto |\mathbf{u}(t)|^2$, hence a Lebesgue point, so that the estimate above is also valid for the initial time $t' = t_0$.

Another classical estimate obtained from the energy inequality is

$$\frac{1}{2}\left|\mathbf{u}(t)\right|^2 + \nu \int_{t'}^{t} \left\|\mathbf{u}(s)\right\|^2 \mathrm{d}s \leqslant \frac{1}{2}\left|\mathbf{u}(t')\right|^2 + \int_{t'}^{t} \big(\mathbf{f}, \mathbf{u}(s)\big)\,\mathrm{d}s, \tag{2.5}$$

for all t in I and almost all t' in I with $t' < t$, with the set of allowed times t' consisting again of the Lebesgue points of the function $t \mapsto |\mathbf{u}(t)|^2$.

From (2.5) one obtains

$$\left|\mathbf{u}(t)\right|^2 + \nu \int_{t'}^{t} \left\|\mathbf{u}(s)\right\|^2 \mathrm{d}s \leqslant \left|\mathbf{u}(t')\right|^2 + \frac{1}{\nu}\|\mathbf{f}\|_{V'}^2 (t - t'), \tag{2.6}$$

for all t in I and almost all t' in I with $t' < t$, with the set of allowed times t' consisting again of the Lebesgue points of the function $t \mapsto |\mathbf{u}(t)|^2$.

It is well established that given any initial time t_0 and any initial condition $\mathbf{u}_0$ in H, there exists at least one weak solution on $[t_0, \infty)$ satisfying $\mathbf{u}(t_0) = \mathbf{u}_0$.

From the energy inequalities above one deduces the following classical result.

Let $\{\mathbf{u}_n\}_n$ be a sequence of weak solutions on a certain interval of the form $I = (a, b)$, with $-\infty \leqslant a < b \leqslant \infty$. Suppose $\{\mathbf{u}_n\}_n$ is uniformly bounded in H, i.e. $\sup_{t \in I,\, n \in \mathbb{N}} \left|\mathbf{u}_n(t)\right| \leqslant R$, for some $R > 0$. Then there exists a subsequence $\{\mathbf{u}_{n_k}\}_k$ which converges to a weak solution $\mathbf{v}$ on I in $\mathcal{C}\big(J, B_H(R)_{\mathrm{w}}\big)$, strongly in $L^2(J; H)$, and weakly in $L^2(J; V)$, for every compact subinterval $J \subset I$, and almost everywhere in H on I. (2.7)

From the energy inequality (2.2) and the weak continuity of the weak solutions one deduces that any weak solution is strongly continuous from the right at its Lebesgue points. To obtain the strong convergence towards the weak limit sets and the weak global attractor, however, we will need strong continuity from the left also and at all points. The main idea is expressed in the following lemma, which is based on the ideas used in [2, Proposition 7.4] and is the fundamental convergence result for the subsequent results.

Lemma 2.1. *Let $\{\mathbf{u}_n\}_n$ be a sequence of weak solutions defined on some interval of the form $I = (a, b)$, with $-\infty \leqslant a < b \leqslant \infty$, and which is uniformly (in n and t) bounded in H. Suppose that $\{\mathbf{u}_n\}_n$ converges weakly in H to a weak solution $\mathbf{v}$ on I, pointwise in this interval, and that $\mathbf{v}$ is strongly continuous in H at some $t \in I$. Then, $\mathbf{u}_n(t)$ converges strongly in H to $\mathbf{v}(t)$.*

Proof. The weak solutions satisfy the energy inequality (2.6), which yields

$$\left|\mathbf{u}_n(t)\right|^2 \leqslant \left|\mathbf{u}_n(t')\right|^2 + \frac{1}{\nu}\|\mathbf{f}\|_{V'}^2 (t - t'), \tag{2.8}$$

for any $t' \in I$, $t' < t$, such that t' is a Lebesgue point of $|\mathbf{u}_n(\cdot)|^2$.

Since $\{\mathbf{u}_n\}_n$ converges weakly to $\mathbf{v}$ it follows from (2.7) and the uniqueness of the limit that $\{\mathbf{u}_n\}_n$ converges strongly in H to $\mathbf{v}$ almost everywhere on I. (For the a.e. convergence for the whole sequence instead of only a subsequence we may use that the a.e. convergence is equivalent to the set $\bigcap_{k\geqslant 1}\bigcup_{n\geqslant k}\{t\in I\colon\ |\mathbf{u}_n(t)-\mathbf{v}(t)|>\eta\}$ being of null Lebesgue measure for every $\eta>0$.)

Since the functions $\mathbf{u}_n$ together with $\mathbf{v}$ form a countable set, since the set of Lebesgue points associated with each of these functions is of full measure, and since $\mathbf{u}_n$ converges strongly in H to $\mathbf{v}$ almost everywhere on I, we can choose a sequence $\{t'_l\}_l$ in I with $t'_l<t$ and $t'_l\to t$, such that t'_l are Lebesgue points associated with all these functions and that $\mathbf{u}_n(t'_l)$ converges strongly in H to $\mathbf{v}(t'_l)$ for each l.

Then we consider (2.8) with t' replaced by t'_l, and pass to the limit as n goes to infinity to find

$$\limsup_{n\to\infty}\left|\mathbf{u}_n(t)\right|^2\leqslant\left|\mathbf{v}(t'_l)\right|^2+\frac{1}{\nu}\|\mathbf{f}\|_{V'}^2(t-t'_l),\tag{2.9}$$

for each l. We let l go to infinity and use the strong continuity of $\mathbf{v}$ in H at t to find

$$\limsup_{n\to\infty}\left|\mathbf{u}_n(t)\right|^2\leqslant\left|\mathbf{v}(t)\right|^2.\tag{2.10}$$

On the other hand, since $\mathbf{u}_n(t)$ converges weakly to $\mathbf{v}(t)$ in H, as $n\to\infty$, we have that $|\mathbf{v}(t)|\leqslant\liminf_{n\to\infty}|\mathbf{u}_n(t)|$. Thus $\lim_{n\to\infty}|\mathbf{u}_n(t)|=|\mathbf{v}(t)|$, which together with the weak convergence implies the strong convergence $\mathbf{u}_n(t)\to\mathbf{v}(t)$ in H. This concludes the proof. □

Remark 2.1. A perusal of the proof of Lemma 2.1 reveals that the assumption of the strong continuity in H of the limit solution $\mathbf{v}$ at t can be relaxed to the strong continuity from the left along sequences $t_n\to t^-$ belonging to a set of positive measure in every neighborhood of t. However, this condition turns out to be equivalent to that of strong continuity. This fact was brought to my attention by Alexey Cheskidov and the idea in his argument is as follows: Take any sequence $t_n\to t$ in the interval of definition of $\mathbf{v}$. Under the relaxed assumption above, we can find another sequence $t'_n\to t^-$ with $t'_n<t_n$ and with t'_n in both the set of Lebesgue points of $|\mathbf{v}(\cdot)|^2$ and the set along which the strong continuity of $\mathbf{v}$ at t from the left holds. Then, we have from the energy inequality that $|\mathbf{v}(t_n)|^2\leqslant|\mathbf{v}(t'_n)|^2+\nu^{-1}\|\mathbf{f}\|_{V'}^2(t_n-t'_n)$. Letting $n\to\infty$ and using the strong continuity of $\mathbf{v}$ at t from the left along $\{t'_n\}_n$ we find that $\limsup_n|\mathbf{v}(t_n)|^2\leqslant|\mathbf{v}(t)|^2$. This together with the weak continuity of the weak solutions implies that $\mathbf{v}(t_n)\to\mathbf{v}(t)$ in H, which shows that $\mathbf{v}$ is strongly continuous in H at t.

We will also use the following lemma.

Lemma 2.2. *Let* $\mathbf{u}$ *be a weak solution on some interval* $I\subset\mathbb{R}$. *If the set* $\{\mathbf{u}(t)\in H;\ t\in I\}$ *is strongly precompact in* H, *then* $\mathbf{u}$ *is strongly continuous in* H *on* I.

Proof. Let $t_0\in I$ and let $t_n\to t_0$ in I. Since $\mathbf{u}$ is weakly continuous, we have that $\mathbf{u}(t_n)$ converges weakly to $\mathbf{u}(t_0)$. On the other hand, since $\{\mathbf{u}(t)\in H;\ t\in I\}$ is strongly precompact, then any subsequence $\{n_j\}_j$ has a further subsequence $\{n_{j_k}\}$ such that $\mathbf{u}(t_{n_{j_k}})$ converges strongly to some point in H. Since strong convergence implies weak convergence and the weak limit is unique, it follows that this limit point must coincide with $\mathbf{u}(t_0)$. Since this happens with any subsequence, the whole sequence $\mathbf{u}(t_n)$ must converge strongly to $\mathbf{u}(t_0)$, which proves that $\mathbf{u}$ is strongly continuous at t_0. Since $t_0\in\mathbb{R}$ is arbitrary, it follows that $\mathbf{v}$ is strongly continuous in H on all I. □

3. Weak limit sets and the weak global attractor

The weak global attractor, as defined in [8], is the set

$$\mathcal{A}_\mathrm{w} = \Big\{ \mathbf{v}_0 \in H\colon \text{ there exists at least one global weak solution } \mathbf{v} = \mathbf{v}(t), \\ \text{defined for all } t \in \mathbb{R}, \text{ which is uniformly bounded in } H, \\ \text{i.e., } \sup_{t\in\mathbb{R}} \big|\mathbf{v}(t)\big| < \infty, \text{ and such that } \mathbf{v}(0) = \mathbf{v}_0 \Big\}. \tag{3.1}$$

Due to the energy estimate (2.3) and the uniform bound on the global solutions in the definition of $\mathcal{A}_\mathrm{w}$ it follows that $\mathcal{A}_\mathrm{w}$ is a bounded set in H:

$$|\mathbf{v}_0| \leqslant \frac{1}{\nu\lambda_1^{1/2}} \|\mathbf{f}\|_{V'}, \quad \forall \mathbf{v}_0 \in \mathcal{A}_\mathrm{w}.$$

It is proved in [8] that $\mathcal{A}_\mathrm{w}$ is weakly compact in H and that it attracts all weak solutions in the following sense: If $\mathbf{u} = \mathbf{u}(t)$ is a weak solution on $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$, then for any neighborhood $\mathcal{O}$ of $\mathcal{A}_\mathrm{w}$ in the weak topology of H, there exists a time $T \geqslant t_0$ such that $\mathbf{u}(t) \in \mathcal{O}$ for all $t \geqslant T$. Since H is separable, the weak topology is metrizable on bounded sets, and the convergence above can be rewritten in terms of this metric. The weak global attractor is also positively invariant in the sense that if $\mathbf{v}_0$ belongs to $\mathcal{A}_\mathrm{w}$ and $\mathbf{v}$ is a weak solution on $[t_0, \infty)$, $t_0 \in \mathbb{R}$, with $\mathbf{v}(t_0) = \mathbf{v}_0$, then $\mathbf{v}(t) \in \mathcal{A}_\mathrm{w}$ for all $t \geqslant t_0$. However, as far as we know there is no result preventing the existence of a weak solution through some point $\mathbf{v}_0$ in $\mathcal{A}_\mathrm{w}$ which blows up backwards in time in either finite or infinite time, so that backward invariance is not assured in general.

Besides the pointwise attraction (attraction of individual weak solutions) of the weak global attractor, one can show that the attraction is, in fact, uniform with respect to uniformly bounded sets of initial condition (see [7]). More precisely, given $t_0 \in \mathbb{R}$ and $R > 0$, then for every neighborhood $\mathcal{O}$ of $\mathcal{A}_\mathrm{w}$ in the weak topology of H, there exists a time $T \geqslant t_0$ such that $\mathbf{u}(t) \in \mathcal{O}$ for all $t \geqslant T$ and for every weak solution $\mathbf{u}$ on $[t_0, \infty)$ with $\sup_{t\geqslant t_0} |\mathbf{u}(t)| \leqslant R$. Since $\mathcal{A}_\mathrm{w}$ is bounded in H and the weak topology of H is metrizable on bounded subsets this uniform attraction in the weak topology can be rewritten in terms of an associated metric.

These properties define $\mathcal{A}_\mathrm{w}$ and justify its definition as the weak global attractor. They can also be used to characterize $\mathcal{A}_\mathrm{w}$ in a more classical way:

$$\mathcal{A}_\mathrm{w} = \Big\{ \mathbf{v}_0 \in H\colon \text{ there exist } t_0 \in \mathbb{R}, \text{ a sequence of weak solutions } \{\mathbf{u}_n\}_n \text{ defined} \\ \text{on } [t_0, \infty) \text{ with } \sup_{n\in\mathbb{N},\, t>t_0} \big|\mathbf{u}_n(t)\big| < \infty, \text{ and a time sequence} \\ \{t_n\}_n,\ t_n \geqslant t_0,\ t_n \to \infty, \text{ such that } \mathbf{u}_n(t_n) \rightharpoonup \mathbf{v}_0 \text{ weakly in } H \Big\}. \tag{3.2}$$

One of the aims of this paper is to obtain necessary and sufficient conditions for the weak global attractor to be a strongly compact and strongly attracting. The strong attraction that we are interested in is in fact a uniform strong attraction in the following sense:

$\mathcal{A}_w$ is said to strongly attract the weak solutions uniformly with respect to uniformly bounded sets of weak solutions if for $t_0 \in \mathbb{R},\ R > 0,$ and every $\varepsilon > 0,$ there exists a time $T \geqslant t_0$ such that $\operatorname{dist}_H(\mathbf{u}(t), \mathcal{A}_w) \stackrel{\text{def}}{=} \sup_{\mathbf{v}_0 \in \mathcal{A}_w} |\mathbf{u}(t) - \mathbf{v}_0| < \varepsilon,$ for all $t \geqslant T,$ and for every weak solution $\mathbf{u}$ on $[t_0, \infty)$ with $\sup_{t \geqslant t_0} |\mathbf{u}(t)| \leqslant R.$ (3.3)

Now, given an arbitrary weak solution $\mathbf{u} = \mathbf{u}(t)$ on an interval of the form $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$, we define its weak ω-limit set by

$$\omega_w(\mathbf{u}) = \{\mathbf{v}_0 \in H;\ \exists \{t_n\}_n,\ t_n \geqslant t_0,\ t_n \to \infty,\ \mathbf{u}(t_n) \rightharpoonup \mathbf{v}_0 \text{ weakly in } H\}. \tag{3.4}$$

This set is always nonempty since $\{\mathbf{u}(t)\}_{t \geqslant t_0}$ is bounded in H (thanks to (2.3)), hence weakly precompact. Since the weak topology is metrizable on bounded subsets of H, the classical characterization $\omega_w(\mathbf{u}) = \bigcap_{t \geqslant 0} \overline{\bigcup_{t \geqslant s} \{\mathbf{u}(t)\}}^w$ holds, where $\overline{\cdot}^w$ denotes the closure in the weak topology. Hence, $\omega_w(\mathbf{u})$ is weakly compact. By classical dynamical system arguments one can also show that $\omega_w(\mathbf{u})$ attracts $\mathbf{u}$ in the sense that for any weakly open set $\mathcal{O}$ containing $\omega_w(\mathbf{u})$, there exists a time $T \geqslant t_0$ such that $\mathbf{u}(t) \in \mathcal{O}$ for all $t \geqslant T$.

As for the invariance property, it is possible to show that for every $\mathbf{v}_0$ in $\omega_w(\mathbf{u})$, there exists a global weak solution $\mathbf{v} = \mathbf{v}(t)$, $t \in \mathbb{R}$, with $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{v}(t) \in \omega_w(\mathbf{u})$ for all $t \in \mathbb{R}$. This is achieved by passing to the limit in the solutions $\mathbf{u}(t_n + \cdot)$ over time intervals $[-T, T]$, for arbitrarily large times T. A diagonalization argument using (2.7) guarantees the existence of a subsequence converging weakly to a global weak solution $\mathbf{v}$, so that $\mathbf{v}(t)$ belongs to $\omega_w(\mathbf{u})$ for all $t \in \mathbb{R}$. However, due to the possible lack of uniqueness one cannot guarantee the invariance for every bounded global weak solution passing through $\mathbf{v}_0$, neither backward nor forward in time.

A similar argument for the weak global attractor yields that for every $\mathbf{v}_0$ in $\mathcal{A}_w$ and every pair of sequences $\{\mathbf{u}_n\}_n$ and $\{t_n\}_n$ as in the characterization (3.2), with $\mathbf{u}_n(t_n) \rightharpoonup \mathbf{v}_0$ weakly in H, there exist subsequences $\{\mathbf{u}_{n_j}\}_j$ and $\{t_{n_j}\}_j$ such that $\mathbf{u}_{n_j}(t_{n_j} + \cdot)$ converges weakly to a global weak solution $\mathbf{v} = \mathbf{v}(t)$, with $\mathbf{v}(t) \in \mathcal{A}_w$, for all $t \in \mathbb{R}$, and with $\mathbf{v}(0) = \mathbf{v}_0$.

4. Asymptotic regularity conditions for the strong convergence towards weak limit sets

As mentioned in the Introduction the required asymptotic regularity condition is that the limit solutions be strongly continuous in H. More precisely, we have the following result.

Lemma 4.1. *Let $\mathbf{u}$ be a weak solution defined on some interval of the form $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$. Let $\mathbf{v}_0 \in \omega_w(\mathbf{u})$ and let $\{t_n\}_n$ be such that $t_n \geqslant t_0$, $t_n \to \infty$, and $\mathbf{u}(t_n) \rightharpoonup \mathbf{v}_0$ weakly in H. If there exists a weak solution $\mathbf{v} = \mathbf{v}(t)$ on an interval $(-\delta, \delta)$, for some $\delta > 0$, such that $\mathbf{u}(t_n + t)$ converges weakly to $\mathbf{v}(t)$ for all $t \in (-\delta, \delta)$ and such that $\mathbf{v}$ is strongly continuous in H at $t = 0$, then $\mathbf{u}(t_n)$ converges strongly in H to $\mathbf{v}_0$.*

Proof. Just apply Lemma 2.1 to the sequence $\mathbf{u}_n(t) = \mathbf{u}(t_n + t)$. □

Lemma 4.2. *Let* $\mathbf{u}$ *be a weak solution defined on some interval of the form* $[t_0, \infty)$*, for some* $t_0 \in \mathbb{R}$*. If all the global weak solutions in* $\omega_{\mathrm{w}}(\mathbf{u})$ *are strongly continuous in* H *on* $\mathbb{R}$*, then* $\omega_{\mathrm{w}}(\mathbf{u})$ *attracts* $\mathbf{u}$ *in the strong topology of* H*.*

Proof. If this were not true we would find a time sequence $\{t_n\}_n$, with $t_n \geqslant t_0$, $t_n \to \infty$, and such that $\{\mathbf{u}(t_n)\}_n$ does not have any subsequence converging strongly in H. Now, from (2.7), and using a diagonalization argument, we know that $\{\mathbf{u}(t_n + \cdot)\}_n$ has a subsequence $\{\mathbf{u}(t_{n_j} + \cdot)\}_j$ which converges weakly in H to some global weak solution $\mathbf{v}$. By the definition of the weak ω-limit set, we have $\mathbf{v}(t) \in \omega_{\mathrm{w}}(\mathbf{u})$ for each $t \in \mathbb{R}$. By hypothesis, we have that $\mathbf{v}$ is strongly continuous in H at any time $t \in \mathbb{R}$. Then, applying Lemma 4.1 at $t = 0$, we deduce that $\{\mathbf{u}(t_{n_j})\}_j$ converges strongly in H to $\mathbf{v}(0)$, which is a contradiction. Thus, $\omega_{\mathrm{w}}(\mathbf{u})$ attracts $\mathbf{u}$ in the strong topology of H. □

Lemma 4.3. *Let* $\mathbf{u}$ *be a weak solution defined on some interval of the form* $[t_0, \infty)$*, for some* $t_0 \in \mathbb{R}$*. If all the global weak solutions in* $\omega_{\mathrm{w}}(\mathbf{u})$ *are strongly continuous in* H *on* $\mathbb{R}$*, then* $\omega_{\mathrm{w}}(\mathbf{u})$ *is strongly compact in* H*.*

Proof. Let $\{\mathbf{v}_{0n}\}_n$ be a sequence of elements in $\omega_{\mathrm{w}}(\mathbf{u})$. By definition, for each n, there exists a sequence of times $\{t_j^n\}_j$, with $t_j^n \geqslant t_0$, $t_j^n \to \infty$, and $\mathbf{u}(t_j^n) \rightharpoonup \mathbf{v}_{0n}$ weakly in H, as $j \to \infty$. By (2.7) and a diagonalization argument we can assume, passing to a further subsequence (in j) if necessary, that the solution $\mathbf{u}(t_j^n + \cdot)$ converges weakly to some global weak solution $\mathbf{v}_n$, uniformly on bounded intervals in $\mathbb{R}$, with $\mathbf{v}_n(0) = \mathbf{v}_{0n}$.

Since the weak solution $\mathbf{u}$ is bounded in H we may consider a ball $B_H(R)$ in H, with $R > 0$ large enough, containing the orbit of $\mathbf{u}$. Then, from the convergences above, there exists, for each n, an integer $j_n \geqslant n$ such that

$$\mathrm{d}_{B_H(R)_{\mathrm{w}}}\big(\mathbf{u}\big(t_{j_n}^n + t\big), \mathbf{v}_n(t)\big) \leqslant \frac{1}{n}, \quad \forall t \in [-n, n]. \tag{4.1}$$

From (2.7) and a diagonalization argument we deduce that $\mathbf{u}(t_{j_{n_k}}^{n_k} + \cdot)$ converges weakly to a global weak solution $\mathbf{v}$, uniformly on every bounded interval in $\mathbb{R}$, for some subsequence $\{n_k\}_k$. By the definition of the weak ω-limit set, we have that $\mathbf{v}(t) \in \omega_{\mathrm{w}}(\mathbf{u})$ for all $t \in \mathbb{R}$. From (4.1) we have that $\mathbf{v}_{n_k}(\cdot)$ also converges weakly to $\mathbf{v}$, uniformly on every bounded interval. Since by hypothesis all solutions in $\omega_{\mathrm{w}}(\mathbf{u})$ are strongly continuous in H on $\mathbb{R}$, we have that $\mathbf{v}$ is strongly continuous in H at $t = 0$. Then we apply Lemma 2.1 to deduce that $\mathbf{v}_{0n_k} = \mathbf{v}_{n_k}(0)$ converges strongly in H to $\mathbf{v}(0)$, which belongs to $\omega_{\mathrm{w}}(\mathbf{u})$. This proves that $\omega_{\mathrm{w}}(\mathbf{u})$ is strongly compact. □

So far we have worked with asymptotic regularity conditions for the strong convergence and the strong compactness of the weak ω-limit sets. Let us now prove a converse statement.

Lemma 4.4. *Let* $\mathbf{u}$ *be a weak solution defined on some interval of the form* $[t_0, \infty)$*, for some* $t_0 \in \mathbb{R}$*. If* $\omega_{\mathrm{w}}(\mathbf{u})$ *is strongly compact in* H*, then all the global weak solutions in* $\omega_{\mathrm{w}}(\mathbf{u})$ *are strongly continuous in* H*.*

Proof. Let $\mathbf{v}$ be a global weak solution in $\omega_{\mathrm{w}}(\mathbf{u})$. Since $\omega_{\mathrm{w}}(\mathbf{u})$ is strongly compact, it follows that the orbit $\{\mathbf{v}(t);\ t \in \mathbb{R}\}$ is precompact in H. Hence, by Lemma 2.2, we conclude that $\mathbf{v}$ is strongly continuous in H on $\mathbb{R}$. □

Combining the previous results, we obtain:

Theorem 4.1. *Let* $\mathbf{u}$ *be a weak solution defined on some interval of the form* $[t_0, \infty)$*, for some* $t_0 \in \mathbb{R}$*. Then the following statements are equivalent*:

(i) *The set* $\omega_w(\mathbf{u})$ *is strongly compact.*
(ii) *All the global weak solutions in* $\omega_w(\mathbf{u})$ *are strongly continuous in* H.
(iii) *The set* $\omega_w(\mathbf{u})$ *is strongly compact and strongly attracts* $\mathbf{u}$ *in* H.

Proof. That (i) implies (ii) follows from Lemma 4.4. That (ii) implies (iii) follows from Lemmas 4.3 and 4.2. That (iii) implies (i) is trivial. □

5. Asymptotic regularity conditions for the strong convergence towards the weak global attractor

We now study the weak global attractor. We recall that the weak global attractor is not simply the union of weak ω-limit sets since it may also contain, for instance, non-wandering connecting orbits. Therefore, the results for the weak global attractor do not follow from those for the weak ω-limit sets. However, the extension of the previous results to the weak global attractor is straightforward. The proof of the strong compactness in Lemma 5.3 is in fact easier than the corresponding one for weak ω-limit sets, in Lemma 4.3, due to the fact that the global weak attractor attracts every weak solution while the weak ω-limit set attracts only the corresponding orbit. The extension of the previous results to the weak global attractor is based on the following simple modification of Lemma 4.1, concerning individual trajectories in $\mathcal{A}_w$.

Lemma 5.1. *Let* $\{\mathbf{u}_n\}_n$ *and* $\{t_n\}_n$ *be as in the characterization* (3.2) *of* $\mathcal{A}_w$*. If there exists a weak solution* $\mathbf{v} = \mathbf{v}(t)$ *on an interval* $(-\delta, \delta)$*, for some* $\delta > 0$*, such that* $\mathbf{u}_n(t_n + t)$ *converges weakly in* H *to* $\mathbf{v}(t)$ *for all* $t \in (-\delta, \delta)$ *and such that* $\mathbf{v}$ *is strongly continuous in* H *at* $t = 0$*, then* $\mathbf{u}_n(t_n)$ *converges strongly in* H *to* $\mathbf{v}(0)$.

Proof. Just apply Lemma 2.1 to the functions $\mathbf{u}_n(t_n + \cdot)$. □

This result can be extended to all the weak global attractor as stated in the following way.

Lemma 5.2. *If all the global weak solutions in* $\mathcal{A}_w$ *are strongly continuous in* H *on* $\mathbb{R}$ *then* $\mathcal{A}_w$ *attracts every weak solution in the strong topology of* H*, and this attraction is uniform with respect to uniformly bounded sets of weak solutions in the sense of* (3.3).

Proof. Suppose the result is not true. Then there exists $t_0 \in \mathbb{R}$, $R > 0$, $\varepsilon > 0$, a sequence $\mathbf{u}_n$ of weak solutions on $[t_0, \infty)$ with $\sup_{t \geqslant t_0} |\mathbf{u}_n(t)| \leqslant R$, and a time sequence $\{t_n\}_n$, $t_n \geqslant t_0$, $t_n \to \infty$, such that

$$\left|\mathbf{u}_n(t_n) - \mathbf{v}_0\right| \geqslant \varepsilon, \quad \text{for all } n \text{ and all } \mathbf{v}_0 \in \mathcal{A}_w. \tag{5.1}$$

Consider the sequence $\mathbf{v}_n(t) = \mathbf{u}_n(t_n + t)$, defined for $t \geqslant t_0 - t_n$. Using the assumption of the uniform estimate on $\{\mathbf{u}_n\}_n$ we apply (2.7) and a diagonalization argument to obtain the existence of a subsequence $\{\mathbf{v}_{n_j}\}_j$ converging weakly in H to a global weak solution $\mathbf{v} = \mathbf{v}(t)$ on $\mathbb{R}$. In particular, $\mathbf{v}_{n_j}(0) = \mathbf{u}_{n_j}(t_{n_j})$ converges weakly to $\mathbf{v}_0 = \mathbf{v}(0)$ in H.

At the limit, we retain a uniform bound for $\mathbf{v}$, namely $\sup_{t\in\mathbb{R}} |\mathbf{v}(t)| \leqslant R$, so that $\mathbf{v}(t)$ belongs to $\mathcal{A}_\mathrm{w}$ for all $t \in \mathbb{R}$. In particular, $\mathbf{v}_0 \in \mathcal{A}_\mathrm{w}$. By hypothesis, we have that $\mathbf{v}$ is strongly continuous in H on $\mathbb{R}$. Then, it follows from Lemma 5.1 that $\mathbf{u}_{n_j}(t_{n_j})$ converges strongly in H to $\mathbf{v}_0 \in \mathcal{A}_\mathrm{w}$. But this contradicts (5.1). Hence, the stated result must be true. □

Lemma 5.3. *If all the global weak solutions in $\mathcal{A}_\mathrm{w}$ are strongly continuous in H on $\mathbb{R}$ then $\mathcal{A}_\mathrm{w}$ is strongly compact in H.*

Proof. Let $\{\mathbf{v}_{0n}\}_n$ be a sequence of elements in $\mathcal{A}_\mathrm{w}$. Then, there exist global weak solutions $\mathbf{v}_n$ on $\mathbb{R}$ with $\mathbf{v}_n(0) = \mathbf{v}_{0n}$ and $\mathbf{v}_n(t) \in \mathcal{A}_\mathrm{w}$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. Let $\{t_n\}_n$ be an arbitrary time sequence with $t_n \to \infty$. Consider $\mathbf{u}_n(t) = \mathbf{v}(t - t_n)$, so that $\mathbf{u}_n(t_n) = \mathbf{v}_n(0) = \mathbf{v}_{0n}$. Since $\mathbf{u}_n(t) \in \mathcal{A}_\mathrm{w}$, for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$, and since $\mathcal{A}_\mathrm{w}$ is bounded in H we apply (2.7) and a diagonalization argument to deduce that there exists a subsequence $\{n_j\}_j$ and a global weak solution $\mathbf{v}$ such that $\mathbf{u}_{n_j}(t_{n_j} + \cdot)$ converges weakly in H to $\mathbf{v}$, as $j \to \infty$. Moreover, $\mathbf{v}(t)$ belongs to $\mathcal{A}_\mathrm{w}$ for all $t \in \mathbb{R}$. Thus, by hypothesis, $\mathbf{v}$ is strongly continuous in H on $\mathbb{R}$. Hence, by Lemma 4.1, it follows that $\mathbf{u}_{n_j}(t_{n_j})$ converges strongly to $\mathbf{v}(0)$. Since $\mathbf{u}_{n_j}(t_{n_j}) = \mathbf{v}_{0n_j}$ and $\mathbf{v}(0)$ belongs to $\mathcal{A}_\mathrm{w}$, this means that $\mathbf{v}_{0n_j}$ converges strongly to an element of $\mathcal{A}_\mathrm{w}$, which proves that $\mathcal{A}_\mathrm{w}$ is strongly compact. □

So far we have worked with asymptotic regularity conditions for the strong convergence and the strong compactness of the weak global attractor. Let us now prove a converse statement.

Lemma 5.4. *Suppose that $\mathcal{A}_\mathrm{w}$ is strongly compact. Then any global weak solution in $\mathcal{A}_\mathrm{w}$ is strongly continuous in H.*

Proof. Let $\mathbf{v}$ be a global weak solution in $\mathcal{A}_\mathrm{w}$. Since $\mathcal{A}_\mathrm{w}$ is strongly compact it follows that the orbit $\{\mathbf{v}(t)\colon t \in \mathbb{R}\}$ is precompact in H. Hence, by Lemma 2.2, we conclude that $\mathbf{v}$ is strongly continuous in H on $\mathbb{R}$. □

As a consequence of these results, we have:

Theorem 5.1. *The following statements are equivalent*:

(i) *The weak global attractor $\mathcal{A}_\mathrm{w}$ is strongly compact in H.*
(ii) *All the global weak solutions in $\mathcal{A}_\mathrm{w}$ are strongly continuous in H.*
(iii) *$\mathcal{A}_\mathrm{w}$ is strongly compact and strongly attracts every weak solution in the strong topology of H, uniformly with respect to uniformly bounded sets of weak solutions in the sense of* (3.3).

Proof. That (i) implies (ii) follows from Lemma 5.4. That (ii) implies (iii) follows from Lemmas 5.3 and 5.2. That (iii) implies (i) is trivial. □

Acknowledgments

I thank an anonymous referee for bringing to my attention the result in [2], which enabled the improvement of the convergence result to the form stated in Lemma 2.1. I am also greatly indebted to Alexey Cheskidov for many pertinent remarks that led to various improvements to this paper. The author was partly supported by a fellowship from CNPq, Brasília, Brazil, grant number 30.0185/97, and by the Pronex in Turbulence, CNPq/FAPERJ, Brazil, grant number E-26/171.198/2003.

References

[1] J.M. Ball, A proof of the existence of global attractors for damped semilinear wave equations, unpublished (cited in [9]).
[2] J.M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier–Stokes equations, J. Nonlinear Sci. 7 (5) (1997) 475–502, Erratum in: J. Nonlinear Sci. 8 (2) (1998) 233.
[3] J.M. Ball, Global attractors for damped semilinear wave equations, Discrete Contin. Dyn. Syst. 10 (1/2) (2004) 31–52.
[4] V.V. Chepyzhov, M.I. Vishik, Attractors for Equations of Mathematical Physics, Amer. Math. Soc. Colloq. Publ., vol. 49, Amer. Math. Soc., Providence, RI, 2002.
[5] A. Cheskidov, C. Foias, On global attractors of the 3D Navier–Stokes equations, in preparation.
[6] P. Constantin, C. Foias, Navier–Stokes Equation, Univ. of Chicago Press, Chicago, 1989.
[7] C. Foias, O.P. Manley, R. Rosa, R. Temam, Navier–Stokes Equation and Turbulence, Encyclopedia Math. Appl., vol. 83, Cambridge Univ. Press, Cambridge, 2001.
[8] C. Foias, R. Temam, The connection between the Navier–Stokes equations, dynamical systems, and turbulence theory, in: Directions in Partial Differential Equations, Madison, WI, 1985, in: Publ. Math. Res. Center Univ. Wisconsin, vol. 54, Academic Press, Boston, MA, 1987, pp. 55–73.
[9] J.M. Ghidaglia, A note on the strong convergence towards attractors of damped forced KdV equations, J. Differential Equations 110 (1994) 356–359.
[10] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., Providence, RI, 1988.
[11] I. Moise, R. Rosa, X. Wang, Attractors for non-compact semigroups via energy equations, Nonlinearity 11 (5) (1998) 1369–1393.
[12] I. Moise, R. Rosa, X. Wang, Existence of uniform attractors for noncompact nonautonomous dissipative systems via energy equations, Discrete Contin. Dyn. Syst. A 10 (1/2) (2004) 473–496.
[13] O. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Revised English edition, translated from Russian by R.A. Silverman, Gordon and Breach, New York/London, 1963.
[14] O. Ladyzhenskaya, Attractors for Semigroups and Evolution Equations, Lezioni Lincei/Cambridge Univ. Press, 1991.
[15] J.-L. Lions, Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires, Dunod/Gauthier–Villars, Paris, 1969.
[16] R. Rosa, The global attractor for the 2D Navier–Stokes flow on some unbounded domains, Nonlinear Anal. 32 (1) (1998) 71–85.
[17] G.R. Sell, Global attractors for the three-dimensional Navier–Stokes equations, J. Dynam. Differential Equations 8 (1996) 1–33.
[18] G.R. Sell, Y. You, Dynamics of Evolutionary Equations, Appl. Math. Sci., vol. 143, Springer, New York, 2002.
[19] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, third ed., Stud. Math. Appl., vol. 2, North-Holland, Amsterdam/New York, 1984; reedition: AMS Chelsea Series, Amer. Math. Soc., Providence, RI, 2001.
[20] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, second ed., Appl. Math. Sci., vol. 68, Springer, New York, 1988.