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Duality between magnetic field and rotation

V. Dzhunushaliev

Department of Physics and Microelectric Engineering, KRSU, Kievskaya Str. 44, Bishkek 720021, Kyrgyz Republic

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Abstract

It is shown that in 5D Kaluza–Klein theory there are everywhere regular wormhole-like solutions in which the magnetic field at the center is the origin of a rotation on the peripheral part of these solutions. The time on the peripheral part is topologically non-trivial and magnetic field is suppressed in comparison with the electric one.

1. Introduction

In this Letter we would like to show that in 5D gravitational flux tube solutions [1] the magnetic field at the central part of the spacetime (throat) can be an origin for a rotation at the peripheral parts (tails) of the spacetime. Mathematically this duality is based on the fact that for the gravitational flux tube solutions there is an interchange of the signs of metric signature on the throat $\{+,-,-,-,-\}$ and on the tails $\{-,-,-,-,+\}$ (the solution is regular everywhere). It is like to Schwarzschild solution in Schwarzschild's coordinates: where the time and radial coordinates are interchanged under the event horizon. But there is the essential difference: the gravitational flux tube metric is regular everywhere and even on the hypersurface where $ds^2 = 0$ the coordinate singularities are missing in the contrast with Schwarzschild coordinates which have the coordinate singularities on the event horizon. Once this interchanging takes place the time becomes as the space-like coordinate and the 5th coordinate as the time-like coordinate. Then we have to change the metric form in such a way that to isolate the term $(dx^5_{\text{new}} + A_\mu\, dx^\mu)$. Such algebraical manipulations cause to the appearance of a new off-diagonal metric term (at the peripheral parts of the gravitational flux tube spacetime) $dt_{\text{new}}\, d\varphi$ that is an indication of a rotation. Another exhibition of this rotation is that the angular momentum density appears for the electromagnetic field.

E-mail address: dzhun@hotmail.kg (V. Dzhunushaliev).

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2. Recalculation of electromagnetic fields

Let us write the 5D metric in the following form

$$ds^2 = -\delta e^{2\phi}\left(d\chi + A_\mu\, dx^\mu\right)^2 + \frac{1}{\delta}dt^2 + 2g_{0i}\, dt\, dx^i + g_{ij}\, dx^i\, dx^j, \tag{1}$$

where A_μ is the electromagnetic potential, $x^\mu = t, x^i$ is the 4D coordinates, χ is the 5th coordinate, $\mu = 0, 1, 2, 3$; $i = 1, 2, 3$. We suppose that exists such everywhere regular solution of 5D Einstein's equations that $\delta > 0$ for one part of a spacetime (on the throat) and $\delta < 0$ for another one (on the tails). The metric (1) is written for $\delta > 0$ part. Now we would like to rewrite this metric for the $\delta < 0$ part of a spacetime (on the tails). After some algebraical manipulations

$$ds^2 = -\tilde{\delta} e^{2\phi}\left(dt + \tilde{A}_\mu\, d\tilde{x}^\mu\right)^2 + \frac{1}{\tilde{\delta}}\left(d\chi + A_i\, dx^i\right)^2 + g_{ij}\, dx^i\, dx^j, \tag{2}$$

here for the simplicity we consider the case with $g_{0i} = 0$ and

$$\tilde{x}^\mu = \left\{\chi, x^i\right\}, \qquad \tilde{x}^0 = \chi, \qquad \tilde{x}^5 = t, \tag{3}$$

$$\tilde{\delta} = A_0^2\delta - \frac{e^{-2\phi}}{\delta}, \tag{4}$$

$$\tilde{A}_0 = \frac{\delta}{\tilde{\delta}} A_0, \tag{5}$$

$$\tilde{A}_i = \frac{A_0\delta}{A_0^2\delta - e^{-2\phi}/\delta} A_i = \frac{A_0\delta}{\tilde{\delta}} A_i, \tag{6}$$

where $\tilde{A}_\mu$ is the new electromagnetic potential (on the tails); $\tilde{x}^\mu$ and $\tilde{x}^5$ are the new coordinates. We suppose that $\tilde{\delta} > 0$ (below we will present solutions with these properties). On the peripheral part t is the 5th space-like coordinate and χ is the time-like coordinate.

3. Application for the gravitational flux tube solutions

The 5D metric for the gravitational flux tube metric has the following form [1]

$$ds^2 = \frac{a(r)}{\Delta(r)}dt^2 - dr^2 - a(r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) - \frac{\Delta(r)}{a(r)}e^{2\phi(r)}\left(d\chi + \omega(r)\, dt + Q\cos\theta\, d\varphi\right)^2, \tag{7}$$

the functions $a(r), \delta(r) = \Delta(r)/a(r)$ and $\phi(r)$ are the even functions; $r \in \{-\infty, +\infty\}$ and consequently the metric has a wormhole-like form; Q is the magnetic charge. The off-diagonal metric components are $G_{5\mu}/G_{55} = (\omega(r), 0, 0, Q\cos\theta)$ and consequently we have the radial electric and magnetic fields. The 5D vacuum Einstein's equations are

$$R_{AB} - \frac{1}{2}\eta_{AB} R = 0, \tag{8}$$

here A, B are 5-bein indices; R_{AB} and R are 5D Ricci tensor and the scalar curvature, respectively; $\eta_{AB} = \mathrm{diag}\{1, -1, -1, -1, -1\}$. The 5D substitution of the metric (7) in Eq. (8) gives us

$$\omega'' + \omega'\left(-\frac{a'}{a} + 2\frac{\Delta'}{\Delta} + 3\phi'\right) = 0, \tag{9}$$

$$\frac{a''}{a} + \frac{a'\phi'}{a} - \frac{2}{a} + \frac{Q^2 \Delta e^{2\phi}}{a^3} = 0, \tag{10}$$

$$\phi'' + \phi'^2 + \frac{a'\phi'}{a} - \frac{Q^2 \Delta e^{2\phi}}{2a^3} = 0, \tag{11}$$

$$\frac{\Delta''}{\Delta} - \frac{\Delta' a'}{\Delta a} + 3\frac{\Delta'\phi'}{\Delta} + \frac{2}{a} - 6\frac{a'\phi'}{a} = 0, \tag{12}$$

$$\frac{\Delta'^2}{\Delta^2} + \frac{4}{a} - \frac{q^2 e^{-4\phi}}{\Delta^2} - \frac{Q^2 \Delta e^{2\phi}}{a^3} - 6\frac{a'\phi'}{a} - 2\frac{\Delta' a'}{\Delta a} + 2\frac{\Delta'\phi'}{\Delta} = 0. \tag{13}$$

From the Maxwell's equation (9) we have

$$\omega' = \frac{qae^{-3\phi}}{\Delta^2}, \tag{14}$$

here q is the electric charge. The solutions of Eqs. (9)–(13) are parametrized by electric q and magnetic Q charges [1]:

(1) $0 < Q < q$. The solution is a wormhole-like object. The throat between the surfaces at $\pm r_H$ [1] is a finite flux tube filled with both electric and magnetic fields. The longitudinal distance between the $\pm r_H$ surfaces increases by $q \to Q$. We will call these solutions as gravitational flux tube solutions.
(2) $q = Q$. In this case the solution is an infinite flux tube filled with constant electrical and magnetic fields. The cross-sectional size of this solution is constant ($a = \text{const}$).
(3) $0 < q < Q$. In this case we have a singular finite flux tube located between two $(+)$ and $(-)$ electrical and magnetic charges located at $\pm r_0$.

In this Letter we consider the first case. The detailed numerical and approximate analytical investigations of the properties of the gravitational flux tube solutions is given in Refs. [1,2]. This spacetime can be divided on three parts: the first one (throat) is the central part of the solution located between $r = \pm r_H$ and $\Delta > 0$; the second and third parts are located accordingly by $r < -r_H$ and $r > r_H$, here $\Delta < 0$. On the throat the electromagnetic potential is

$$A_0 = \omega(r), \tag{15}$$

$$A_\varphi = Q\cos\theta \tag{16}$$

with the radial electric and magnetic fields

$$E_r = \frac{d\omega}{dr} = \frac{qae^{-3\varphi}}{\Delta^2}, \tag{17}$$

$$H_r = \frac{Q}{a}. \tag{18}$$

But for the peripheral parts $\Delta(r) < 0$ that means that the time t becomes as 5th coordinate and 5th coordinate χ becomes as time coordinate.

At the tails according to Eq. (6)

$$\tilde{\delta}(r) = \frac{\Delta(r)}{a(r)}\omega^2(r) - \frac{a(r)}{\Delta(r)e^{2\phi(r)}}, \tag{19}$$

[1] r_H is defined as follows: $\Delta(\pm r_H) = 0$.

$$\tilde{A}_0(r) = \frac{\Delta(r)}{a(r)\tilde{\delta}(r)}\omega(r), \tag{20}$$

$$\tilde{A}_\varphi(r,\theta) = \frac{\Delta(r)\omega(r)}{a(r)\tilde{\delta}(r)} Q\cos\theta. \tag{21}$$

4. Non-singularity of the gravitational flux tube metric

The most important for the idea presented here is the regularity of the solution everywhere. Evidently that the metric (7) can have a singularity only at the points where $\Delta = 0$. In this section we would like to show that the gravitational flux tube solutions is non-singular at the points $\pm r_H$ where $\Delta(\pm r_H) = 0$. Let us investigate the solution near to the point $|r| \approx r_H$, where the metric functions have the following view

$$a(r) = a_0 + a_1(r - r_H) + a_2(r - r_H)^2 + \cdots, \tag{22}$$

$$\phi(r) = \phi_H + \phi_1(r - r_H) + \phi_2(r - r_H)^2 + \cdots, \tag{23}$$

$$\Delta(r) = \Delta_1(r - r_H) + \Delta_1\Delta_2(r - r_H)^2 + \cdots. \tag{24}$$

The substitution these functions in Einstein's equations (9)–(13) gives us the following coefficients

$$\Delta_1 = \pm q e^{-2\phi_H}, \quad (+) \quad \text{for } r \to -r_H \quad \text{and} \quad (-) \quad \text{for } r \to +r_H, \tag{25}$$

$$\phi_2 = -\phi_1 \frac{a_1 + a_0\phi_1}{2a_0}, \tag{26}$$

$$\Delta_2 = \frac{-3a_0\phi_1 + a_1}{2a_0}, \tag{27}$$

$$a_2 = \frac{2 - a_1\phi_1}{2}. \tag{28}$$

In this case the electric field (14) has the following behaviour near to the points $r = \pm r_H$

$$\omega'(r) = \frac{a_0 e^{\phi_H}}{q} \frac{1}{(r - r_H)^2} + \omega_1 + \mathcal{O}(r - r_H), \tag{29}$$

where ω_1 is some constant depending on $a_{0,1}, \phi_{1,2}, \Delta_{1,2}$. Then $\omega(r)$ is

$$\omega(r) = -\frac{a_0 e^{\phi_H}}{q} \frac{1}{(r - r_H)} + \omega_0 + \mathcal{O}(r - r_H), \tag{30}$$

where ω_0 is some integration constant. The G_{tt} metric component is

$$G_{tt} = \frac{a(r)}{\Delta(r)} - \frac{\Delta(r)e^{2\phi(r)}}{a(r)}\omega^2(r) = -e^{2\phi_H}\frac{2qe^{-\phi_H}\omega_0 - a_1 - a_0\phi_1}{q} + \mathcal{O}(r - r_H). \tag{31}$$

Finally the metric (7) has the following approximate behaviour near to $r = \pm r_H$ points

$$\begin{aligned} ds^2 = {} & \left[g_H + \mathcal{O}(r - r_H)\right]dt^2 - \mathcal{O}(r - r_H)(d\chi + Q\cos\theta\, d\varphi)^2 \\ & - \left[e^{\phi_H} + \mathcal{O}(r - r_H)\right]dt(d\chi + Q\cos\theta\, d\varphi) - dr^2 - \left[a(r_H) + \mathcal{O}(r - r_H)\right]\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\ \approx {} & e^{\phi_H}\, dt(d\chi + Q\cos\theta\, d\varphi) - dr^2 - a(r_H)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \end{aligned} \tag{32}$$

where $g_H = -e^{2\phi_H}(2qe^{-\phi_H}\omega_0 - a_1 - a_0\phi_1)/q$. It means that at the points $r = \pm r_H$ the metric (7) is non-singular one.

5. Asymptotical behaviour of the gravitational flux tube metric

Let us consider $|r| \gg r_H$ parts of the solution. We search the asymptotical behaviour of the gravitational flux tube metric as

$$a(r) = r^2 + m_1 r + q_1 + \cdots, \tag{33}$$

$$\Delta(r) = -\Delta_\infty r^2 + \Delta_\infty m_2 r + \Delta_\infty q_2 + \cdots, \tag{34}$$

$$\phi(r) = \phi_\infty + \frac{\phi_1}{r^2} + \cdots. \tag{35}$$

The Einstein's equations give

$$\phi_1 = -\frac{Q^2 \Delta_\infty e^{2\phi_\infty}}{4}, \tag{36}$$

$$q_1 = \frac{q^2 e^{-4\phi_\infty} + 3Q^2 \Delta_\infty^3 e^{2\phi_\infty} - \Delta_\infty^2 m_2^2 - 2\Delta_\infty^2 m_1 m_2}{4\Delta_\infty^2}, \tag{37}$$

$$q_2 = \frac{q^2 e^{-4\phi_\infty} - 3Q^2 \Delta_\infty^3 e^{2\phi_\infty} - \Delta_\infty^2 m_2^2}{4\Delta_\infty^2}. \tag{38}$$

It shows us that at the infinity

$$\frac{\Delta(r) e^{2\phi(r)}}{a(r)} \approx -\Delta_\infty e^{2\phi_\infty} \left(1 - \frac{m_1 + m_2}{r} - \frac{m_1^2 - q_1 - q_2 + 2\phi_1}{r^2}\right). \tag{39}$$

The numerical investigations [2] give us arguments that $\Delta(r) e^{2\phi(r)}/a(r) \to -1$ and consequently

$$\Delta_\infty = e^{-2\phi_\infty}. \tag{40}$$

After substitution in Eqs. (36)–(38) we have

$$\phi_1 = -\frac{Q^2}{4}, \tag{41}$$

$$q_1 = \frac{q^2 + 3Q^2 - m_2^2 - 2m_1 m_2}{4}, \tag{42}$$

$$q_2 = \frac{q^2 - 3Q^2 - m_2^2}{4}. \tag{43}$$

Thus asymptotically we have

$$\frac{d\omega}{dr} = \frac{qa}{\Delta e^{2\phi}} \frac{e^{-\phi}}{\Delta} \to \frac{q e^{\phi_\infty}}{r^2}. \tag{44}$$

Therefore,

$$\omega = -\frac{q e^{\phi_\infty}}{r}. \tag{45}$$

According to Eq. (6)

$$\tilde{\delta} = \frac{\Delta}{a}\omega^2 - \frac{a}{\Delta e^{2\phi}} \to 1, \tag{46}$$

$$\tilde{A}_0 \to \frac{q e^{-\phi_\infty}}{r}, \tag{47}$$

$$\tilde{A}_\varphi \to \frac{qQ}{r} e^{-\phi_\infty} \cos\theta, \tag{48}$$

and asymptotically the metric is

$$ds^2 \approx -\left(d\tilde{t} + \frac{q}{r}d\chi + \frac{qQ}{r}\cos\theta\, d\varphi\right)^2 + (d\chi + Q\cos\theta\, d\varphi)^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{49}$$

where $\tilde{t} = e^{\phi_\infty} t$. We see that asymptotically there are the radial electric and magnetic fields

$$|E_r| \approx \frac{q}{r^2}, \qquad |H_r| \approx \frac{qQ}{r^3}. \tag{50}$$

Remarkably that the magnetic field is suppressed in comparison with the electric field.

The 4D part of the metric (49) is

$$ds^2_{(4)} \approx Q^2\left(\frac{d\chi}{Q} + \cos\theta\, d\varphi\right)^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) - dr^2. \tag{51}$$

The first two terms give the Lorentzian metric on the deformed Hopf bundle $S^3/S^1 \to S^2$, where the total space S^3 is spanned on coordinates χ, θ, φ; the fiber S^1 on χ coordinate and the base S^2 on θ, φ coordinates. On the principal Hopf bundle the angle $0 \leqslant \chi \leqslant 4\pi$ consequently the time χ is closed one and topologically non-trivial in the sense that the spacetime is not the direct product: spacetime $\neq$ time $\times$ space. Physically it means that we cannot introduce a global time but only a local one. The situation is similar to Gödel's and Taub–NUTs metrics. In order to avoid closed time one can consider the associated Hopf bundle with the fibers $R^1 = u(1) = \mathrm{Lie}\{U(1)\}$. In this case the time χ becomes infinite but nevertheless the bundle remains non-trivial and as before we cannot introduce a global time.

Finally we would like to note that the off-diagonality of the metrics (1) and (2) is very important that was emphasized in Ref. [3].

6. A duality between magnetic field and an angular momentum density

Another interesting peculiarity of the gravitational flux tube spacetime is that by $|r| > r_H$ the metric has the 4D off-diagonal metric component

$$g_{\chi\varphi} = \frac{1}{\tilde{\Delta}} Q\cos\theta\, d\chi\, d\varphi, \tag{52}$$

where χ is the time-like coordinate. It shows us that like to Kerr and Gödel metrics in this spacetime there is a rotation.

In order to understand what kind of the rotation there is here let us consider the electromagnetic field. From Eq. (6) we see that by $|r| > r_H$ we have the following electromagnetic potential

$$\tilde{A}_0 = -\frac{\Delta(r)}{a(r)\tilde{\Delta}(r)}\omega(r), \tag{53}$$

$$\tilde{A}_\varphi = -\frac{\Delta(r)\omega(r)}{a(r)\tilde{\Delta}(r)} Q\cos\theta. \tag{54}$$

Consequently we have the following non-zero components of the tensor of the electromagnetic field

$$\tilde{F}_{r\chi} = -\frac{d}{dr}\left(\frac{\Delta(r)}{a(r)\tilde{\Delta}(r)}\omega(r)\right) \neq 0, \tag{55}$$

$$\tilde{F}_{\theta\phi} = \frac{\Delta(r)\omega(r)}{a(r)\tilde{\Delta}(r)} Q\sin\theta \neq 0, \tag{56}$$

$$\tilde{F}_{r\phi} = Q\cos\theta \frac{d}{dr}\left(\frac{-\Delta(r)\omega(r)}{a(r)\tilde{\Delta}(r)}\right) \neq 0. \tag{57}$$

The definition of the tensor of the angular momentum density for the electromagnetic field is

$$M^{\mu\nu} = x^\mu T^{\nu 0} - x^\nu T^{\mu 0}, \tag{58}$$

where

$$T^{\mu\nu} = F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \tag{59}$$

is the energy–momentum tensor for the electromagnetic field. The substitution electromagnetic field (55)–(57) into definitions (58) and (59) gives us that

$$M^{r\theta},\ M^{r\varphi} \neq 0. \tag{60}$$

In vector notation the angular momentum is defined as

$$M_i = \epsilon_{ijk} M^{jk}. \tag{61}$$

According to (60)

$$M_\varphi,\ M_\theta \neq 0. \tag{62}$$

Thus the gravitational flux tube solutions have the rotation term (52) in the metric connected with the angular momentum density of the electromagnetic field (62).

It is necessary to deduce why the metric (7) has not the form of the Kerr metric for the rotating black hole. The reason is that the χ coordinate is topologically non-trivial. At the center $(r = 0)$ the metric is

$$ds^2 = \frac{a(0)}{\Delta(0)} dt^2 - dr^2 - \left[Q^2 \frac{\Delta(0)}{a(0)} e^{2\phi(0)} \left(\frac{d\chi}{Q} + \cos\theta\, d\phi \right)^2 + a(0) \left(d\theta^2 + \sin\theta\, d\varphi^2 \right) \right], \tag{63}$$

as $\omega(0) = 0$. We can choose the scale of time in such a way that $a(0)/\Delta(0) = 1$. One can see that the last term

$$dl^2 = Q^2 e^{2\phi_0} \left(\frac{d\chi}{Q} + \cos\theta\, d\varphi \right)^2 + a_0 \left(d\theta^2 + \sin\theta\, d\phi^2 \right) \tag{64}$$

again is the metric on the deformed S^3-sphere presented as the Hopf bundle: $S^3/S^1 \to S^2$. The coordinate χ is directed along the fibers. Thus one can say that the space-like coordinate χ on the throat and the time-like coordinate χ on the tails of the gravitational flux tube spacetime are non-trivial. This non-triviality in combination with the non-zero angular momentum density of the electromagnetic field show us that on the tails of gravitational flux tube solutions there is a rotation connected with the magnetic field on the throat.

7. Conclusions and discussion

In this Letter it is shown that:

- The regular gravitational flux tube metric has two different kind of metric signature: the first type is $\{+, -, -, -, -,\}$ on the throat by $|r| < r_H$, the second one is $\{-, -, -, -, +,\}$ on the tails by $|r| > r_H$. On the throat there are the electric and magnetic fields.
- On the tails the magnetic field decreases faster the electric field, although on the throat the fields are almost equal.
- On the tails there is a rotation connected with the magnetic field on the throat, i.e., a rotation on an external universe and the magnetic field on the throat are dual each other.
- It is very important that the gravitational flux tube solutions are the vacuum solutions of the 5D Einstein's equations and consequently do not depend on the properties of any matter sort.

- The time direction on the tails is topologically non-trivial. Probably it is like to the time in the Gödel solution.
- Probably the most interesting case is when the length of χ coordinate is in the Planck region and $\delta_q = 1 - Q/q \ll 1$. In this case this dimension is invisible and effectively we have two Euclidean spacetime connected with super-thin and super-long flux tube (namely, Δ-string [2]).

The most interesting question arising here is: one can extend these results to the topologically trivial time when spacetime = space × time? In this case the gravitational flux tube solutions can be a geometrical model of electric charge with the explanation why the magnetic charge is unobservable in the nature.

It is necessary to note that although the metric (7) at the tails is spherically symmetric one but this solution is not listed in review [4] as the presented solution has a rotation connected with the magnetic field and consequently the off-diagonal 4D metric component.

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