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UV–IR mixing in non-commutative plane

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Abstract

Poincaré-invariant quantum field theories can be formulated on non-commutative planes if the coproduct on the Poincaré group is suitably deformed. As shown in our previous work, this important result implies modification of free field commutation and anti-commutation relations and striking phenomenological consequences such as violations of Pauli principle. In this Letter we prove that with these modifications, UV–IR mixing disappears to all orders in perturbation theory from the S-matrix. This result is in agreement with the previous results of Oeckl.

1. Introduction

The non-commutative Groenwold–Moyal plane is the algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ of functions on $\mathbb{R}^{d+1}$ with the $*$-product as the multiplication law. The latter is defined as follows.

If $\alpha, \beta \in \mathcal{A}_\theta(\mathbb{R}^{d+1})$, then

$$\alpha *_\theta \beta(x) = \left(\alpha e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} \beta\right)(x), \tag{1}$$

$$\theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R}, \qquad x = \left(x^0, x^1, \ldots, x^d\right). \tag{2}$$

Here x^0 is the time coordinate, and the rest are spatial coordinates.

Henceforth, we will write $\alpha *_\theta \beta$ as $\alpha * \beta$.

The appearance of constants $\theta^{\mu\nu}$ would at first sight suggest that the diffeomorphism group $\mathrm{Diff}(\mathbb{R}^{d+1})$ of $\mathbb{R}^{d+1}$, and in particular its Poincaré subgroup is not an automorphism of $\mathcal{A}_\theta(\mathbb{R}^{d+1})$. But the work of [1] and [2] (and the earlier work of [3,4] and [5]) have shown that this appearance is false. Thus there exists a deformed coproduct on $\mathrm{Diff}(\mathbb{R}^{d+1})$ which depends on $\theta^{\mu\nu}$. With this deformation, $\mathrm{Diff}(\mathbb{R}^{d+1})$ does act as the automorphism group of $\mathcal{A}_\theta(\mathbb{R}^{d+1})$.

In [6] (and the earlier work of [4] and [5]), it was shown that the standard commutation relations are not compatible with the deformed action of Poincaré group. Rather they too have to be deformed. If $a(p)$ is the annihilation operator of a free field for momentum p, then for example,

$$a(p)a(q) = \eta e^{i p_\mu \theta^{\mu\nu} q_\nu} a(q)a(p), \tag{3}$$

where η is a Lorentz-invariant function of p and q. The choices $\eta = \pm 1$ correspond, for $\theta = 0$, to bosons and fermions.

There are similar relations involving $a(p)^\dagger$'s as well. All of them follow from the relations

$$a(p) = c(p) e^{+\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \tag{4}$$

$$a(p)^\dagger = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c(p)^\dagger, \tag{5}$$

where $c(p)$ and $c(p)^\dagger$ are the standard oscillators $a(p)|_{\theta=0}$, $a(p)^\dagger|_{\theta=0}$ for $\theta = 0$, and P_μ is the translation generator:

$$P_\mu = \int d\mu(p)\, p_\mu c(p)^\dagger c(p) = \int d\mu(p)\, p_\mu a(p)^\dagger a(p), \tag{6}$$

$d\mu(p)$ here is the Poincaré-invariant measure. For a spin 0 field of mass m,

$$d\mu(p) = \frac{d^3 p}{2 p_0}, \qquad p_0 = \left|\sqrt{\vec{p}^2 + m^2}\right|. \tag{7}$$

There are striking consequences of the deformed commutation relation [6] such as the existence of Pauli-forbidden levels and attendant phenomenology [7]. In this note, we show another striking result: Non-planar graphs and UV–IR mixing completely disappear from the S-matrix S_θ because of the deformed

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statistics. S_θ is in fact independent of $\theta^{\mu\nu}$ so that $S_\theta = S_0$. This does not mean that scattering amplitudes are independent of θ, as the in- and out-state vectors are different, being subject to deformed statistics.

Our treatment here covers both time–space and space–space non-commutativity. In the former case, although there were initial claims of loss of unitarity, the work of Doplicher et al. [8] showed how to construct unitary theories. These ideas were subsequently applied to construct unitary quantum mechanics as well [9,10]. So there is no good theoretical reason to set $\theta^{0i} = 0$. The work we present here is quite general as regards the choice of $\theta^{\mu\nu}$, allowing also the choice $\theta^{0i} \neq 0$.

We present the calculations for a real scalar field with the interaction

$$\phi_*^n := \phi * \phi * \cdots * \phi \quad (n \geqslant 2).$$

The generality of the results will be evident from this example.

There is considerable overlap of the results of this work with those of Oeckl [4]. He too uses non-trivial twisted statistics, but does not use Poincaré symmetry implemented with a twisted coproduct [1,2]. In contrast, our previous work [6] deduced twisted statistics from Poincaré invariance. Oeckl then deduces an expression for the n-point function in agreement with ours. His derivation is based on braided quantum field theory developed by him [3]. Its relation to our approach awaits clarification. But we point out that once the appropriately twisted spacetime algebra and statistics are accepted as axioms, both Oeckl and us get the same final answer without ever invoking Poincaré invariance or any other spacetime symmetry except translations.

2. The model

The free scalar field ϕ of mass m in the Moyal plane has the Fourier expansion

$$\phi(x) = \int d\mu(p) \left[a(p)e^{ip\cdot x} + a(p)^\dagger e^{-ip\cdot x}\right],$$
$$p_0 = \sqrt{\vec{p}^2 + m^2}. \tag{8}$$

The interaction Hamiltonian, in the interaction representation, is taken to be

$$H_I(x_0) = \lambda \int d^d x \,{:}\phi_*^n{:}, \tag{9}$$

where : : denote normal ordering of $a(p)$'s and $a(p)^\dagger$'s.

The operator $H_I(x_0)$ is self-adjoint for any choice of $\theta^{\mu\nu}$, even with time–space non-commutativity. Hence the S-matrix

$$S_\theta = T\exp\left(-i\int dx_0\, H_I(x_0)\right) = T\exp\left(-i\int d^{d+1}x \,{:}\phi_*^n(x){:}\right)$$

is unitary. We will now show that S_θ is independent of θ. That means in particular that there is no UV–IR mixing.

Let e_p be the plane wave of momentum p: $e_p = e^{ip\cdot x}$. The $*$-product of plane waves is simple:

$$e_p * e_q = e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu} e_{p+q}. \tag{10}$$

Let us introduce the notation

$$a(p)^\dagger = a(-p),$$

where p_0 is also reversed by the dagger. Then

$$\phi = \int d\mu(p) \left[a(p)e_p + a(-p)e_{-p}\right]. \tag{11}$$

3. The proof

(i) $n = 2$

First consider $n = 2$, just as an example. Then the $O(\lambda)$ term of S_θ is

$$S_\theta^{(1)} = -i\lambda \int d^{d+1}x \,{:}\phi * \phi{:}(x). \tag{12}$$

A typical term in $\phi * \phi$ is[1]

$$a(p)a(q)e_p * e_q = a(p)a(q)e^{\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu} e_{p+q}. \tag{13}$$

Substituting from (4), we get

$$\begin{aligned}
&\text{R.H.S. of (13)}\\
&\quad = c(p)e^{\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu} c(q)e^{\frac{i}{2}q_\nu\theta^{\mu\nu}P_\nu} e^{\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu} e_{p+q}\\
&\quad = c(p)c(q)e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu} e^{\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu} e^{\frac{i}{2}(p+q)_\mu\theta^{\mu\nu}P_\nu} e_{p+q}\\
&\qquad \big(\text{since } [P_\nu, c(q)] = -q_\nu c(q)\big)\\
&\quad = c(p)c(q)e_{p+q} e^{\frac{i}{2}(p+q)_\mu\theta^{\mu\nu}P_\nu}.
\end{aligned} \tag{14}$$

Note how the phases $e^{\mp\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu}$ cancel.

Using

$$\partial_\mu e_{p+q} = i(p+q)_\mu e_{p+q},$$

we can write this as

$$c(p)c(q)e_{p+q} e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}P_\nu}.$$

Hence

$$-i\lambda \int d^{d+1}x \,{:}\phi * \phi{:}(x) = -i\lambda \int d^{d+1}x \,{:}\phi^2{:}(x) e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}P_\nu}. \tag{15}$$

Expanding the exponential, integrating and discarding the surface terms, we find that

$$-i\lambda \int d^{d+1}x \,{:}\phi * \phi{:}(x) = -i\lambda \int d^{d+1}x \,{:}\phi^2{:}(x)$$

is independent of $\theta^{\mu\nu}$.

The only delicate issue here concerns the surface term. Here and in what follows, we will assume that such surface terms

[1] Here we have used $e_p * e_q = e^{\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu} e_{p+q}$, which requires replacing $\theta^{\mu\nu}$ by $-\theta^{\mu\nu}$ in (1). The reason for this change is explained in [6] after Eq. (2.33).

vanish. In the absence of long range forces, the assumption should be correct.

Next consider the $O(\lambda^2)$ term

$$S_\theta^{(2)} = \frac{(-i\lambda)^2}{2!} \int d^{d+1}x_1\, d^{d+1}x_2 \times \left\{\theta(x_{10}-x_{20}) :\phi * \phi:(x_1) :\phi * \phi:(x_2) + (x_1 \leftrightarrow x_2)\right\}. \tag{16}$$

A typical term in $\theta(x_{10}-x_{20}):\phi*\phi:(x_1):\phi*\phi:(x_2)$ is

$$\begin{aligned}
&\theta(x_{10}-x_{20}):a(p_1)a(q_1):e_{p_1}*e_{q_1}(x_1):a(p_2)a(q_2):e_{p_2}*e_{q_2}(x_2)\\
&\quad= \theta(x_{10}-x_{20}):c(p_1)c(q_1):e_{p_1+q_1}(x_1)e^{+\frac{i}{2}(p_1+q_1)_\mu\theta^{\mu\nu}P_\nu}\\
&\qquad\times :c(p_2)c(q_2):e_{p_2+q_2}(x_2)e^{+\frac{i}{2}(p_2+q_2)_\mu\theta^{\mu\nu}P_\nu}\\
&\quad= \theta(x_{10}-x_{20})\\
&\qquad\times\Big\{:c(p_1)c(p_2)::c(q_1)c(q_2):e^{-\frac{i}{2}(p_1+q_1)_\mu\theta^{\mu\nu}(p_2+q_2)_\nu}\\
&\qquad\times\Big[e_{p_1+q_1}(x_1)e_{p_2+q_2}(x_2)e^{+\frac{1}{2}(\frac{\overleftarrow{\partial}}{\partial x_{1\mu}}+\frac{\overleftarrow{\partial}}{\partial x_{2\mu}})\theta^{\mu\nu}P_\nu}\Big]\Big\},
\end{aligned} \tag{17}$$

where the differentials act only on $e_{p_1+q_1}$, $e_{p_2+q_2}$ and phases involving just p_μ and q_μ cancelling out as before.

Note first that by energy–momentum conservation [enforced by integration over x_1+x_2 and the resultant $\delta^{d+1}(\sum p_i)$], we can set $p_2+q_2=-p_1-q_1$. Hence we can set

$$e^{-\frac{i}{2}(p_1+q_1)_\mu\theta^{\mu\nu}(p_2+q_2)_\nu} = 1.$$

Next note that since

$$\left(\frac{\partial}{\partial x_{10}}+\frac{\partial}{\partial x_{20}}\right)\theta(x_{10}-x_{20}) = 0,$$

we can in fact allow $\frac{\overleftarrow{\partial}}{\partial x_{10}}+\frac{\overleftarrow{\partial}}{\partial x_{20}}$ to act on the θ-function as well. But then all terms involving $\theta^{\mu\nu}$ in the power series expansion of the exponential are total differentials and vanish upon integrating over $d^{d+1}x_1\, d^{d+1}x_2$. Thus

$$S_\theta^{(2)} = S_0^{(2)}.$$

Similar calculations show that S_θ is independent of $\theta^{\mu\nu}$ exactly, to all orders in $\theta^{\mu\nu}$.

$$S_\theta = S_0 \quad \text{for } n=2.$$

(ii) Generic n

The typical term in

$$:\underbrace{\phi*\phi*\cdots*\phi}_{n\text{-terms}}:(x)$$

is

$$:a(p)a(q)\cdots a(s):e_p*e_q*\cdots*e_s(x)$$

which too simplifies to

$$:c(p)c(q)\cdots c(s):e_{p+q+\cdots+s}(x)e^{+\frac{i}{2}(p+q+\cdots+s)_\mu\theta^{\mu\nu}P_\nu}$$

for any n. Hence, we find to $O(\lambda)$, for any n, as before that

$$S_\theta^{(1)} = S_0.$$

The proof to higher orders is similar. Thus to $O(\lambda^2)$, (17) is replaced by

$$\begin{aligned}
&\theta(x_{10}-x_{20})\\
&\quad\times\Big\{:c\big(p_1^{(1)}\big)\cdots c\big(p_1^{(n)}\big)::c\big(p_2^{(1)}\big)\cdots c\big(p_2^{(n)}\big):\\
&\quad\times e^{-\frac{i}{2}(\sum_{j=1}^n (p_1^{(j)})_\mu)\theta^{\mu\nu}(\sum_{k=1}^n (p_2^{(k)})_\nu)}\\
&\quad\times\Big[e_{\sum_j p_1^{(j)}}(x_1)e_{\sum_k p_2^{(k)}}(x_2)e^{+\frac{1}{2}(\frac{\overleftarrow{\partial}}{\partial x_{1\mu}}+\frac{\overleftarrow{\partial}}{\partial x_{2\mu}})\theta^{\mu\nu}P_\nu}\Big]\Big\},
\end{aligned} \tag{18}$$

which can again be shown to be independent of $\theta^{\mu\nu}$ using energy–momentum conservation and partial integration. Therefore

$$S_\theta^{(2)} = S_0^{(2)}.$$

This proof extends to all orders so that

$$S_\theta = S_0.$$

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