



Shelling Coxeter-like complexes and sorting on trees

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Abstract

In their work on ‘Coxeter-like complexes’, Babson and Reiner introduced a simplicial complex Δ_T associated to each tree T on n nodes, generalizing chessboard complexes and type A Coxeter complexes. They conjectured that Δ_T is $(n - b - 1)$ -connected when the tree has b leaves. We provide a shelling for the $(n - b)$ -skeleton of Δ_T , thereby proving this conjecture. In the process, we introduce notions of weak order and inversion functions on the labellings of a tree T which imply shellability of Δ_T , and we construct such inversion functions for a large enough class of trees to deduce the aforementioned conjecture and also recover the shellability of chessboard complexes $M_{m,n}$ with $n \geq 2m - 1$. We also prove that the existence or nonexistence of an inversion function for a fixed tree governs which networks with a tree structure admit greedy sorting algorithms by inversion elimination and provide an inversion function for trees where each vertex has capacity at least its degree minus one.

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1. Introduction

Coxeter-like complexes were introduced in [2] as a common generalization of Coxeter complexes and chessboard complexes. The point was to associate a cell complex analogous to a Coxeter complex to any minimal generating set for any finite group – for instance, to any set of $n - 1$ transpositions in the symmetric group S_n which generate S_n . While traditional Coxeter

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complexes are endowed with a wealth of beautiful and remarkable properties (see e.g. [7,10]), including shellability (as proven in [4]), the topological structure of more general Coxeter-like complexes is often much more subtle. The chessboard complexes already demonstrate this.

Chessboard complexes have been studied extensively, motivated both by applications to computational geometry (see e.g. [17]) and also because of a direct relation established in [12] between their homology groups and the Tor groups of Segre modules. The homology groups for chessboard complexes are not easy to determine, for example involving 3-torsion; results in [6] and [13] together show that they are exactly $(v_{m,n} - 1)$ -connected, for $v_{m,n} = \min\{m, n, \lfloor \frac{m+n+1}{3} \rfloor\} - 1$. Our main focus will be on topological properties of more general Coxeter-like complexes. Let us now briefly recall some terminology and establish some notation, before describing our main results.

Babson and Reiner associate to any finite group G with chosen minimal generating set S a cell complex which they call a *Coxeter-like complex*, denoted $\Delta(G, S)$, as follows. The cells of $\Delta(G, S)$ are the cosets of parabolic subgroups of G , where a *parabolic subgroup* is defined to be any subgroup generated by a subset of S . A cell gT is said to be in the closure of a cell $g'T'$ whenever $g'T' \subseteq gT$. $\Delta(G, S)$ is the unique such regular cell complex with the property that each cell has the combinatorial type of a simplex. In other words, all lower intervals in its face poset are Boolean algebras, implying that its face poset is a simplicial poset (as in [14]), and $\Delta(G, S)$ is what is known as a cell complex of Boolean type (cf. [3]).

Our focus will be on the case where the group G is the symmetric group S_n and the minimal generating set S is a set of $n - 1$ transpositions that generate S_n . In this case, $\Delta(G, S)$ is a simplicial complex which is specified by the tree T on n vertices having the edge set $\{e_{i,j} \mid (i, j) \in S\}$, where (i, j) denotes the transposition swapping i and j . Here we are using the fact that a set S of transpositions is a minimal generating set for S_n if and only if $\{e_{i,j} \mid (i, j) \in S\}$ is a tree T on n vertices. Denote $\Delta(G, S)$ by Δ_T in this case, where T is this associated tree. Observe that the i -dimensional faces of Δ_T may be interpreted as the labelled forests obtained by deleting $i + 1$ edges from T and then assigning $|C|$ labels to each of the resulting components C in such a way that each of the labels $1, \dots, n$ is assigned to exactly one component. When defining the “inversions” of a face later, we will think of such a labelled forest as a tree whose vertices are the forest components and whose edges are the (deleted) edges which connect the components, with the “capacity” of a component being the number of labels to be assigned to it.

To allow for capacities larger than one, we study the more general type-selected Coxeter-like complexes of [2], denoted $\Delta_{(T, \mathbf{m})}$, where T is a tree and \mathbf{m} is a vector consisting of $|V(T)|$ non-negative integers which specify the capacities of the various vertices of T . Denote the capacity of vertex v by $\text{cap}(v)$. One way to define $\Delta_{(T, \mathbf{m})}$ is as a Coxeter-like complex for the quotient group $S_{\sum_{v \in T} \text{cap}(v)} / S_{\text{cap}(v_1)} \times \dots \times S_{\text{cap}(v_n)}$ with generating set S consisting again of transpositions corresponding to the edges of T . Notice that one can also then make a combinatorial definition, analogous to the one given above for Δ_T , again obtaining a face F by deleting a set of edges $E^C(F)$ and then assigning the appropriate number of labels to each of the resulting graph components, i.e. assigning exactly $\sum_{v \in C} \text{cap}(v)$ labels to component C , where the labels are taken from the set $\{1, \dots, \sum_{v \in T} \text{cap}(v)\}$. Just as above, this is a simplicial complex with face containments of the form $\sigma \subseteq \tau$ if and only if σ is obtained from τ by merging neighboring components. Working in the generality of these type-selected complexes $\Delta_{(T, \mathbf{m})}$ will be quite helpful to our analysis of the topological structure of skeleta of the complexes Δ_T .

Recall that the *chessboard complex* $M_{m,n}$ is the simplicial complex whose i -dimensional faces are the collections of $i + 1$ mutually non-attacking rooks on an m by n chessboard. It was observed in [2] that $M_{m,n}$ is isomorphic to the Coxeter-like complex resulting from a tree with m

leaves, each having capacity one and exactly one nonleaf vertex v with $\text{cap}(v) = n - m$. The edges from v to the leaves specify the m rows of the chessboard while the n labels specify the columns. A collection of non-attacking rooks then corresponds to an assignment of labels to the subset of the leaves whose incident edges have been chosen.

It was shown in [6] that $M_{m,n}$ is at least $(v_{m,n} - 1)$ -connected. Shareshian and Wachs later proved that $\tilde{H}_{v_{m,n}}(M_{m,n}; \mathbb{Z}) \neq 0$ in [13] and exhibited 3-torsion in many cases. Ziegler proved vertex decomposibility (and thus shellability) of the $v_{m,n}$ -skeleta of chessboard complexes in [16] (see also [1]), while Friedman and Hanlon determined S_n -module structure in [8] and also showed in many cases that there is no rational homology in degree $v_{m,n}$; specifically they showed in [8] that $M_{m,n}$ has rational homology in degree $v_{m,n}$ for $m \leq n$ if and only if $n \geq 2m - 4$ or $(m, n) \in \{(6, 6), (7, 7), (8, 9)\}$. See [15] for a clear and quite comprehensive survey article regarding chessboard complexes and related complexes.

Our starting point was the following conjecture from [2], which we will prove in Theorem 3.8.

Conjecture 1.1 (*Babson–Reiner*). *If a tree T has n nodes and b leaves, then the Coxeter-like complex Δ_T is at least $(n - 1 - b)$ -connected.*

In fact, we prove something stronger, namely that if T has a collection E of edges such that the tree (T', \mathbf{m}) associated to (T, E) via Definition 2.1 satisfies $\text{cap}(v) \geq \text{deg}(v) - 1$ for each vertex $v \in T$, and a similar condition holds for all subtrees, then the $(|E| - 1)$ -skeleton of Δ_T is shellable, hence homotopy equivalent to a wedge of $(|E| - 1)$ -spheres. The conjecture follows from the case in which E is the set of edges from parents to first children in a depth-first-search of a planar embedding of T with a leaf serving as the tree root. In the course of proving this result, we also do several other things:

- (1) axiomatize the notions of inversions and weak order for permutations in a way that also makes sense for the labellings of any fixed tree (regarding permutations as the labellings of a path);
- (2) provide such an inversion function and weak order for any tree (T, \mathbf{m}) in which each vertex $v \in T$ satisfies $\text{cap}(v) \geq \text{deg}(v) - 1$, i.e. any so-called “distributable capacity tree;”
- (3) prove that if (T, \mathbf{m}) admits an inversion function, then $\Delta_{(T, \mathbf{m})}$ is shellable;
- (4) extend a shelling constructed using (2) and (3) to one for the entire $(n - b)$ -skeleton of Δ_T for any tree T with n nodes and b leaves.

Our approach to shelling via an inversion function also turns out to be related to the question of how to route packets of data efficiently on a network of computers which has a tree structure, viewing vertices as processors and edges as connections between them. Specifically, in Theorem 4.1 we prove that the existence of an inversion function on the labellings of a tree is equivalent to that tree admitting greedy sorting based on this same inversion function; the requirements of an inversion function ensure that any series of local moves which swap labels at neighboring nodes that are out of order will terminate, and that all such series of local moves terminate at the same fully-sorted tree.

Our setting is an idealized one which does not even begin to capture the complexity of the networks of widest current interest such as the internet graph, particularly since the graph of the internet is very far from being a tree. However, one still might hope that our topological viewpoint on sorting and routing could offer some useful new insight. For instance, the connection we establish between greedy sorting and shellability gives a topological obstruction to greedy sorting

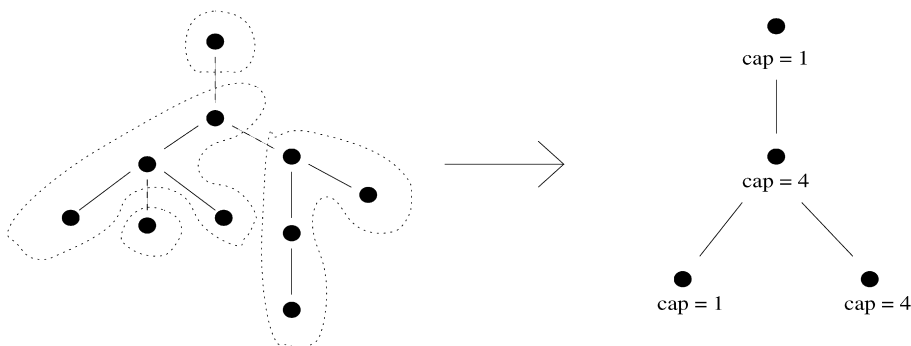


Fig. 1. Tree derived from (T, E) .

as follows: showing that $\Delta_{(T, \mathbf{m})}$ has nonvanishing homology in low degrees directly implies that greedy sorting based on an inversion set is not possible on the tree T with vertex capacities \mathbf{m} . Chessboard complexes already provide a class of such trees, since they are known to have nonvanishing homology in low enough degrees to imply that their associated trees, namely stars, do not admit such greedy sorting algorithms unless the central node of the star has capacity at least its degree minus one. We refer readers to [11] for results in theoretical computer science regarding sorting and routing on various networks, though there are also many more recent results in this direction.

2. Further terminology, notation and a key lemma

The main focus of this paper is the Coxeter-like complex $\Delta_T = \Delta(S_n, S)$ in which G is the symmetric group S_n and S is a set of $n - 1$ transpositions which generate S_n . The faces and incidences in this complex are as described in Section 1. Given a face F , denote by $E^C(F)$ the set of edges in $E(T) \setminus E(F)$, i.e. the edges deleted from T to obtain the labelled forest representing F . Thus, $\dim(F) = |E^C(F)| - 1$.

Definition 2.1. Associate to a tree (T, \mathbf{m}) and a choice of subset E of the set of edges of T a new tree (T', \mathbf{m}') as follows. The vertices of T' are the connected components C in the forest obtained by deleting from T the edges in E , and the edges of T' are the edges in E . The capacity of a component C is the sum of its vertex capacities.

Fig. 1 gives one example of this process, with edges in E depicted by dashed lines. For another example, suppose that T is a tree of four nodes, three of which are leaves, and suppose each leaf has capacity one while the central node has capacity three. Letting E consist of any one edge of this tree, then the new tree (T', \mathbf{m}') will be a path on two nodes, one having capacity one and the other having capacity five.

Definition 2.2. A tree (T, \mathbf{m}) has *distributable capacity* if each of its vertices v satisfies $\text{cap}(v) \geq \text{deg}(v) - 1$.

For example, a tree with four nodes, three of which are leaves, has distributable capacity iff the non-leaf has capacity at least two.

Definition 2.3. Given a tree T , a face F in Δ_T is said to have *distributable capacity* if the tree (T', \mathbf{m}') obtained by applying Definition 2.1 to tree T and edge set $E^C(F)$ has distributable capacity.

Lemma 2.4. *The faces of Δ_T with distributable capacity comprise a subcomplex. Moreover, any component C obtained by merging two neighboring components of a tree T' with distributable capacity satisfies $\text{cap}(C) \geq \text{deg}(C)$.*

Proof. Proper faces are obtained by successively merging neighboring nodes. If neighboring nodes C_1, C_2 are merged to form $C_1 \cup C_2$, and if $\text{deg}(C_i) \leq \text{cap}(C_i) + 1$, for $i \in \{1, 2\}$, then $C_1 \cup C_2$ satisfies

$$\begin{aligned} \text{deg}(C_1 \cup C_2) &= \text{deg}(C_1) + \text{deg}(C_2) - 2 \\ &\leq \text{cap}(C_1) + \text{cap}(C_2) \\ &< \text{cap}(C_1 \cup C_2) + 1. \quad \square \end{aligned}$$

Denote by \overline{F} the simplicial complex comprised of a face F and all the proper faces of F . A *shelling* of a simplicial complex Δ is a total order F_1, \dots, F_k on the facets of Δ , i.e. on its maximal faces, such that for each $k > 1$ the simplicial complex $\overline{F}_k \cap (\bigcup_{j < k} \overline{F}_j)$ is a pure, codimension one subcomplex of \overline{F}_k ; recall that a simplicial complex is *pure* if all maximal faces are equidimensional. It is well known (and easy to show) that any shellable simplicial complex which is pure of dimension d for some $d \geq 2$ has homology concentrated in top degree and is simply connected, hence is $(d - 1)$ -connected. Recall that the link of a face G in a simplicial complex Δ , denoted $lk_\Delta(G)$, is the subcomplex $lk_\Delta(G) = \{F \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}$, and that a shelling for Δ induces a shelling on the link of any face of Δ .

Denote the k -skeleton of a simplicial complex Δ by $\Delta^{(k)}$. For T a tree with n nodes and b leaves, we will prove that $\Delta_T^{(n-b)}$ has a pure, full-dimensional subcomplex isomorphic to $\Delta_{(T', \mathbf{m})}$ for some (T', \mathbf{m}) with distributable capacity, and use the upcoming inversion function for (T', \mathbf{m}) to construct a shelling for $\Delta_{(T', \mathbf{m})}$ which extends to a shelling of $\Delta_T^{(n-b)}$. By convention, we will always work in terms of a choice of rooted, planar embedding for T which has a leaf as its root. The *destination* of a label $i \in [n]$ in a tree (T, \mathbf{m}) , denoted $\text{dest}(i)$, is the a_i th vertex visited in a depth-first-search of T , where $\{a_1, \dots, a_n\}$ is the integer-valued sequence with $a_1 = 1$, $a_i \leq a_{i+1} \leq a_i + 1$ for all i which assigns $\text{cap}(v)$ consecutive labels to each v as it is encountered in a depth-first-search. Denote by $\lambda(T, \mathbf{m})$ the set of labellings of a tree (T, \mathbf{m}) with labels $\{1, \dots, \sum_{v \in T} \text{cap}(v)\}$ which assign $\text{cap}(v)$ labels to vertex v for each $v \in (T, \mathbf{m})$. For further background in topological combinatorics and in Coxeter group theory, see [5] and [10], respectively.

3. Inversion function implies shelling

In this section, we prove the following implications, after first defining what we mean by inversion function:

- (1) If (T, \mathbf{m}) admits an inversion function, then $\Delta_{(T, \mathbf{m})}$ is shellable.

- (2) If $\Delta_{(T, \mathbf{m})}$ is shellable via an inversion function for each (T, \mathbf{m}) having distributable capacity, then $\Delta_T^{(n-b)}$ is shellable for T any tree with n nodes, b of which are leaves.

Theorem 3.2 proves (1). Theorem 3.5 implies (2), but is more general. In Section 5, we will construct an inversion function for all distributable capacity trees, enabling us to use (1) and (2) to give a shelling for $\Delta_T^{(n-b)}$ and thereby deduce the connectivity lower bound conjectured in [2].

Generalizing Definition 2.1, associate to any face F in $\Delta_{(T, \mathbf{m})}$ a tree $T(F)$ by making a vertex for each connected component in $E(T) \setminus E^C(F)$ and an edge for each element of $E^C(F)$, letting the capacity of any component be the sum of the capacities of the vertices comprising that component. Label the vertices of this tree with the collections of labels in the components of $E(T) \setminus E^C(F)$. Denote by $\lambda(T, \mathbf{m})$ the set of labellings of the tree (T, \mathbf{m}) . Now let I be a function which assigns to each face F in $\Delta_{(T, \mathbf{m})}$ a collection of pairs of neighboring components in $T(F)$, called the *inversion pairs* of F . Further require of I that for each pair of neighboring components C_1, C_2 , there is a unique way to redistribute the labels collectively assigned to C_1 and C_2 so that (C_1, C_2) is not an inversion pair of the resulting labelled tree τ , i.e. so that $(C_1, C_2) \notin I(\tau)$. This condition will in fact be subsumed by part 1 of Definition 3.1.

For neighboring components i, j with $(i, j) \in I(\tau)$, define $(i, j)\tau$ to be the labelling obtained from τ by redistributing the labels among i and j in the unique way so that $(i, j) \notin I((i, j)\tau)$. Make a covering relation $(i, j)\tau \prec_{\text{weak}} \tau$ for each $(i, j) \in I(\tau)$ where (i, j) are tree neighbors.

Definition 3.1. A function I as above is an *inversion function* for (T, \mathbf{m}) if I has the following properties:

- (1) Each face $F \in \Delta_{(T, \mathbf{m})}$ is contained in a unique facet which is inversion-free on its restriction to each component of F .
- (2) The transitive closure of \prec_{weak} is a partial order, which we then call the *weak order* with respect to I , denoted \leq_{weak} .
- (3) These properties also hold on the restriction to any subforest of T .

For example, suppose T is a path with one end of the path chosen as tree root. Say that a labelling of T has an inversion pair (v_i, v_j) exactly when v_i is closer than v_j to the root but some label assigned to v_i is larger than some label assigned to v_j . It is easy to check that this meets the above requirements for an inversion function, and in fact is equivalent to the usual notion of inversions between adjacent letters in permutations. However, it is much more difficult, and in many cases is impossible, to define an inversion function for other trees besides paths. One of our main goals is to determine for which trees it is possible to construct an inversion function, and one of the main results in this paper will be a sufficient condition which will enable us to prove the conjecture of Babson and Reiner.

Theorem 3.2. *If (T, \mathbf{m}) admits an inversion function, then $\Delta_{(T, \mathbf{m})}$ is shellable.*

Proof. The facets in $\Delta_{(T, \mathbf{m})}$ may be viewed as labellings of (T, \mathbf{m}) with $\text{cap}(v_i)$ labels assigned to the vertex v_i for each i . Denote by F_w the facet associated to tree labelling w .

We first prove for each facet F_w with w not the minimal element in \leq_{weak} that $\bar{F}_w \cap (\bigcup_{u \prec_{\text{weak}} w} \bar{F}_u)$ is a pure codimension one subcomplex of \bar{F}_w . Implicitly here we use (2) to guarantee that \leq_{weak} is indeed a partial order. By (1), merging neighboring components $j, j + 1$ in F_w for any $(j, j + 1) \in I(w)$ yields a codimension one face of F_w also contained in the earlier

facet $F_{(j,j+1)w}$. Given any face $G \in \overline{F}_w \cap (\bigcup_{u <_{\text{weak}} w} \overline{F}_u)$, let $F_{u'}$ be a facet containing G such that $F_{u'} <_{\text{weak}} F_w$. Requirement (3) ensures that u' may be obtained from w by a series of steps each eliminating an inversion between two neighboring nodes, with the further restriction that these pairs of neighboring nodes are each connected by edges not belonging to $E^C(G)$. Thus, we obtain a codimension one face of F_w , denoted G' , with $G \subseteq G' \in \overline{F}_w \cap (\bigcup_{u <_{\text{weak}} w} \overline{F}_u)$ as follows: merge two neighboring components of F_w whose inversion may be eliminated as the first step in proceeding from F_w to $F_{u'}$ in weak order.

Now let \leq be any linear extension of \leq_{weak} . Consider G any face in $\overline{F}_n \cap (\bigcup_{m < n} \overline{F}_m)$. The earliest facet containing G is the unique one that is inversion-free on each component of G . This will already come before F_n in weak order, so $\overline{F}_n \cap (\bigcup_{m <_{\text{weak}} n} \overline{F}_m) = \overline{F}_n \cap (\bigcup_{m < n} \overline{F}_m)$ regardless of our choice of linear extension \leq . \square

Remark 3.3. It is easy to check that the case of a path amounts to exactly the shellability of the type A Coxeter complex by using any linear extension of weak order to order the facets.

It seems to be much easier for our upcoming main example to confirm that a particular function I is indeed an inversion function by considering the much more detailed data of inversions between non-neighboring nodes as well as neighbors. In particular, the extra information will make it easier to prove that the transitive closure of the set of covering relations is indeed a partial order. Therefore, we make the following variation on the definition of inversion function, where now we let I be a function assigning to each tree labelling a collection of (not necessarily adjacent) pairs of nodes.

Definition 3.4. A function I assigning to each tree labelling a collection of pairs of nodes is a *full inversion function* for (T, \mathbf{m}) if I has the following properties:

- (1) For each $\sigma \in \lambda(T, \mathbf{m})$ and each $(i, k) \in I(\sigma)$, there is an inversion pair (denoted $(j, j + 1) \in I(\sigma)$ to keep notation simple) between two neighboring components on the unique path from i to k in (T, \mathbf{m}) .
- (2) Each face F is contained in a unique facet F_m which is inversion-free on the restriction of F_m to each component of F .
- (3) The transitive closure of $<_{\text{weak}}$ is a partial order, which we then call the *weak order* with respect to I , denoted \leq_{weak} .
- (4) These properties also hold on the restriction to any subtree.

Note that any full inversion function induces an inversion function by assigning to each labelling its inversions which are between neighbors.

3.1. Facet labelling giving rise to $\Delta_T^{(n-b)}$ shelling

Throughout this section, E will always be a set of edges in a tree T such that (T, E) gives rise to a distributable capacity tree via Definition 2.1. Moreover, if we delete from T any set E' of edges such that $E' \cap E = \emptyset$, and we let U be a connected component of the resulting forest with $E(U)$ denoting the set of edges in U , then we also assume that the tree (T', \mathbf{m}) obtained from $(U, E(U))$ via Definition 2.1 will also have distributable capacity. Theorem 3.5 will show for such (T, E) how to shell $\Delta_T^{(|E|-1)}$. Proposition 3.6 will show that the set of edges from parents to their first children in any depth first search of a tree whose root is a leaf will meet the above

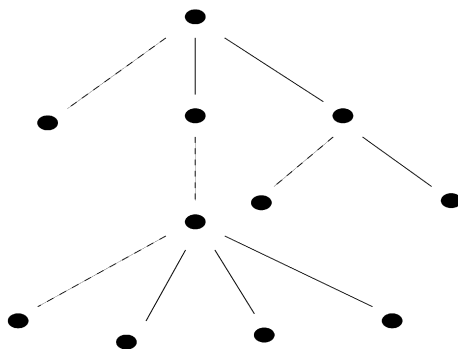


Fig. 2. The set E of edges to first children.

conditions on a set E . See Fig. 2 for what would be an example of such a set E , except that in this example we have not chosen a leaf as the root. We will use this special case to prove that $\Delta_T^{(n-b)}$ is shellable, for T any tree with n nodes and b leaves.

Label each facet F in $\Delta_T^{(|E|-1)}$ with coordinates $(c, S, I(F))$ as described next, and then order the facets by letting earlier coordinates take precedence over later ones, using the orders specified below on individual coordinates. Let $c = |E^C(F) \setminus E|$, and order this first coordinate linearly. Let $S = E^C(F) \cap E$, and order this second coordinate by lexicographic order on the words obtained from the sets S by listing the elements of S in increasing order with respect to depth-first-search order on the edges of the tree T .

It will be convenient to first determine c, S for a given facet F , and then describe the labels $I(F)$ for the various facets having this fixed choice of c and S . In fact, we will need to fix somewhat more, namely the edge set $E^C(F)$, and then use the fact that the facets of $\Delta_T^{(n-b)}$ with fixed edge set are the labellings of a fixed tree (T', \mathbf{m}) whose edges are exactly the edges in $E^C(F)$. Two facets F_i, F_j may have the same coordinates c and S despite having $E^C(F_i) \neq E^C(F_j)$, but we deal with this by first ordering collections of facets using the labels c and S , and then choosing any ordering on the edge sets arising for a fixed c and S to extend to a total order on edge sets. Next we describe how to assign sets $I(F)$ to the facets with a given edge set $E^C(F)$ and how to order these inversion sets.

When $c = 0$, the tree (T', \mathbf{m}) obtained from $(T, E^C(F))$ by Definition 2.1 has distributable capacity, as shown in Proposition 3.6. Let $I(F)$ then be the set of inversions given by the inversion function for this distributable capacity tree (as provided in Section 5). Partially order the facets by the weak order on their inversion sets, i.e., \leq_{weak} , and then choose any total order extension of \leq_{weak} to obtain a total order on the facets having a fixed edge set $E^C(F)$. All total order extensions of \leq_{weak} are equivalent for purpose of shelling, since $\bar{F}_j \cap (\bigcup_{i <_{\text{weak}} j} \bar{F}_i) = \bar{F}_j \cap (\bigcup_{i <_T j} \bar{F}_i)$ for each j regardless of our choice of linear extension $<$, so there is no need to specify a choice.

For $c > 0$, consider each component U in the forest obtained from T by deleting the edges in $E^C(F) \setminus E$. For each such U , consider the tree (T', \mathbf{m}) obtained from $(U, E^C(F)|_U)$ via Definition 2.1. This tree also has distributable capacity, so let $I(F)$ be the union over all these components U of the inversion sets for the various U . Theorem 3.5 will show that these sets $I(F)$ come close enough to being inversion functions to allow us to extend the shelling of the subcomplex generated by the facets having $c = 0$ to a shelling of the entire $(|E| - 1)$ -skeleton.

3.2. Shelling skeleta of Δ_T

Theorem 3.5. *Suppose there exists $E \subseteq E(T)$ such that (T, E) gives rise to a tree (T', \mathbf{m}) via Definition 2.1 which has distributable capacity. Moreover, for U any subtree arising as a connected component in a forest obtained from T by deleting any subset of the set of edges not in E , suppose that the tree (T', \mathbf{m}) obtained from $(U, E \setminus U)$ also has distributable capacity. Then $\Delta_T^{(|E|-1)}$ is shellable.*

Proof. Let F_1, \dots, F_r be the facets of $\Delta_T^{(|E|-1)}$, ordered as described in Section 3.1. We will show this is a shelling in which the minimal face G_m contributed by facet F_m consists of the following three types of edges in $E^C(F_m)$:

- (1) $E^C(F_m) \setminus E$;
- (2) $\{\sigma \in E^C(F_m) \cap E \mid \sigma > \min(E \setminus E^C(F_m))\}$;
- (3) $\{e_{i,j} \in E^C(F_m) \cap E \mid (i, j) \in I(F_m)\}$.

The first thing to show is that deleting an edge of any of these three types from the facet F_m yields a codimension one face that is also contained in an earlier facet. Secondly, we must prove that every face in $\bar{F}_m \cap (\bigcup_{m' < m} \bar{F}_{m'})$ is contained in some such codimension one face. Together, these will imply that $\bar{F}_m \cap (\bigcup_{m' < m} \bar{F}_{m'})$ is a pure, codimension one subcomplex of \bar{F}_m , as is needed for the shelling.

To verify the first claim, we show that omitting from F_m any vertex of G_m will yield a face F that has codimension one in F_m and is shared with a facet $F_{m'}$ in which one of the coordinates in the facet labelling for F_m has been decremented. If we delete from F_m an edge in $E^C(F_m) \setminus E$, then there must be some edge in $E \setminus E^C(F_m)$ that may be added to F to obtain $F_{m'}$ with smaller c coordinate and with $F \subsetneq F_{m'}$. If F is obtained by deleting from F_m an edge from $E^C(F_m) \cap E$ such that $E^C(F_m)$ lacks some earlier edge $e \in E$, then we may insert e into F to obtain a facet with the same c coordinate as F_m but a smaller S coordinate. Deleting an edge where there is an inversion between neighboring nodes means there will be an earlier facet in \leq_{weak} with this inversion eliminated, by Theorem 5.11.

Now we turn to the second claim. Suppose G is a face in $\bar{F}_m \cap (\bigcup_{m' < m} \bar{F}_{m'})$ not contained in any codimension one face of F_m . Then G must contain each vertex in G_m , since otherwise our first claim would imply that G would be contained in some codimension one face of F_m . In particular, G must include $E^C(F_m) \setminus E$, hence may only merge components of F_m that are connected by edges $e \in E$. Moreover, G may only merge components such that the edge e connecting the pair of components satisfies $e < \min(E \setminus E^C(F_m))$. Thus, G is not contained in any $(|E| - 1)$ -faces in which either of the first two coordinates in the facet labelling for F_m have been decremented from their value in F_m . Additionally, G is not contained in any earlier facet $F_{m'}$ satisfying $E^C(F_{m'}) \neq E^C(F_m)$ since G contains all edges in $E^C(F_m) \setminus E$ as well as all edges in $E^C(F_m) \cap E$ that could possibly be replaced by earlier edges. Finally, our first and third requirements for inversion sets completely determine the distribution of labels in F_m and force F_m to come earlier than all other facets containing G , a contradiction to G being shared with an earlier facet. \square

Proposition 3.6. *The tree T together with the set E of edges from parents to first children in a depth first search of T with a leaf as root gives rise by Definition 2.1 to a tree (T', \mathbf{m}) with distributable capacity. Moreover, if we delete from T any subset E' of the edges with $E' \cap E = \emptyset$*

and restrict to a component U of the resulting forest, then the tree associated to $(U, U \cap E)$ by Definition 2.1 also has distributable capacity.

Proof. Each component of the forest obtained by deleting these edges has at least as many vertices as it has edges from its vertices to their first children; the only other possible edge to another component is from the root of the component to its parent, so F has distributable capacity. The same clearly holds on the restriction to any U , by virtue of our choice of edge set E . \square

Corollary 3.7. *If T has n nodes and b leaves, then $\Delta_T^{(n-b)}$ is shellable.*

Proof. There are $n - (b - 1)$ edges from parents to first children, since T has a leaf as root. Letting E be this collection of edges, (T, E) gives rise to a distributable-capacity tree, and this property also restricts to subtrees as needed; this is verified in Proposition 3.6, so we may apply Theorem 3.5. \square

From this, the conjecture of [2] is immediate:

Theorem 3.8. *If T has n nodes and b leaves, then Δ_T is at least $(n - b - 1)$ -connected.*

4. Connection to greedy sorting on trees

Define a *local sorting step* to be the unique redistribution of labels between two neighboring nodes eliminating the inversion pair between the two nodes. A tree labelling is said to be *completely sorted* if it has no inversion pairs.

Theorem 4.1. *Definition 3.1 may be rephrased in terms of greedy sorting algorithms as follows:*

- (1) *Any tree whose labels are not completely sorted admits a local sorting step. This is also true of the restriction to any subtree which is not completely sorted.*
- (2) *All sequences of local sorting steps lead to the same completely sorted labelling. This also holds for restrictions to subtrees, using only local swaps within the subtree.*
- (3) *Any series of local sorting steps will eventually terminate at a completely sorted tree.*
- (4) *The above properties also hold for the restriction to any subtree.*

Proof. It is easy to see that the conditions of Definition 3.1 imply the conditions above, so we focus on proving the other direction. The fact that any series of local sorting steps terminates implies that the transitive closure of $<_{\text{weak}}$ is a partial order on the (finite) set of tree labellings, since otherwise there would be a cycle enabling the same series of sorting steps to be repeated indefinitely. The fact that every series of local swaps terminates with the same outcome implies that \leq_{weak} has a unique minimal element, giving the unique inversion-free labelling of the entire tree. The analogous requirement for restrictions to subtrees implies condition (2) also holds for all other faces. \square

If one instead considers full inversion functions, i.e., Definition 3.4, note then that condition (1) in that definition is essentially equivalent to condition (1) of Theorem 4.1. Combining Theorems 4.1 and 3.2 yields:

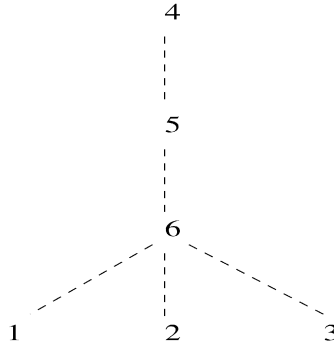


Fig. 3. A queue of inverted labels.

Corollary 4.2. *If $\Delta_{(T, \mathbf{m})}$ is not shellable, then (T, \mathbf{m}) does not admit a greedy sorting algorithm in the sense of Theorem 4.1.*

Thus, we obtain a homological obstruction to many trees admitting greedy sorting algorithms based on inversion functions. For instance, chessboard complexes arise as the special case in which T is a star, namely a tree with a single vertex v that is not a leaf. By Corollary 4.2, known results on the homology of chessboard complexes from [13] tell us that stars with $\text{cap}(v) < \text{deg}(v) - 1$ do not admit greedy sorting.

We should note that in the context of sorting/routing algorithms, operations are typically performed simultaneously at the edges connecting many different pairs of processors. However, an inversion function still indicates which such operations would constitute progress in sorting data. The distributable-capacity hypothesis which will make possible an inversion function is very much in the spirit of results on routing in the sense that allowing queues of bounded size to accumulate at nodes greatly improves efficiency in algorithms.

5. An inversion function for distributable capacity trees

Let us begin with an example demonstrating some challenges to be overcome. Trees are depicted with the root at the top and siblings ordered from left to right. Edges in $E^C(F)$ are represented by dashed lines, while all other edges in $E(T)$ are depicted by solid lines. We say $\lambda \in v_i$ when λ is one of the labels assigned to vertex v_i .

Example 5.1. A natural approach would be to make an inversion (i, j) whenever there are labels at positions i and j traversing the same edge in opposite directions on the most direct routes to their destinations in depth-first-search order. However, this inversion set is too sparse to satisfy condition one of Definition 3.4 in general (and will also fail Definition 3.1). The example in Fig. 3 also illustrates a second issue, namely that of contention. There are inversion pairs between the vertices with label pairs $(1, 4)$, $(2, 5)$, $(3, 6)$, but it is not clear where there should be an inversion between neighboring nodes on the path between the locations of labels 1 and 4, as required by Definition 3.4, condition one. Notice also that this tree labelling would be inversion-free in the sense of Definition 3.1, but would not be the unique such labelling because the labelling which puts labels in exactly depth-first-search order would also be inversion-free.

The distributable capacity requirement will handle this sort of contention by ensuring that each vertex has enough capacity to allow distinct labels at the node to form inversions with its various neighbors, to exactly the extent that will be needed to define an inversion function; Remark 5.2 makes this precise. This simple idea will be crucial to the inversion function which is constructed over the remainder of this section.

For any face $F \in \Delta_{(T,m)}$, define the *components* of F to be the components of the graph $G(F)$ obtained by deleting the edges in $E^C(F)$ from the tree T . The inversions of F will be the inversions of the labelled tree whose vertices are the components of $G(F)$ and whose edges are the edges in $E^C(F)$, letting the labels of a vertex be the set of labels assigned to that component in $G(F)$.

Remark 5.2. Let T be a tree with distributable capacity and let T' be any tree obtained from T by merging some neighboring nodes. Then for any vertex $C \in T'$ comprised of more than one vertex of T and for any vertex $v \in T$ belonging to C , Lemma 2.4 ensures that the number of edges from v to vertices of T not in C is at least as large as the capacity of v .

Call the edges $e_{v,w}$ from a fixed vertex v the *capacity channels* of v . Define the *path* of a label λ in a tree labelling to be the minimal path from λ 's position in the tree labelling to $\text{dest}(\lambda)$. When we speak of an edge $e_{u,v}$ in the path of a label, by convention the directed path proceeds from u to v . A label λ in a tree labelling *fills a capacity channel* $e_{x,y}$ if the path of λ includes the edge $e_{x,y}$.

The following notion will help us generalize ideas from linear arrays to trees, by specifying a hierarchy of linear arrays within a tree:

Definition 5.3. A *coarsening* with respect to a node r (which we will regard as a “local root”) is the decomposition of a rooted tree T with respect to a marked node r into as many as three parts as follows: (1) if r has multiple children, then there is a part C_1 consisting of r 's first child v along with all descendants of v , (2) if r has more than two children, then there is a part C_3 consisting of r 's last child w and all descendants of w , and (3) in any case there is a part C_2 consisting of r and all remaining nodes of T (some of which will not be descendants of r if r is not the global root). Now repeatedly subdivide C_2 in this fashion to obtain successive levels of coarsening on smaller and smaller subtrees, each of which includes r , until reaching a subtree in which r has at most one remaining child.

Definition 5.4. The *coarsening-path-distance* of a label λ with respect to a particular coarsening at a chosen local root r is $|i - j|$, where C_i and C_j are the coarsening parts where the path of λ begins and ends.

While our upcoming inversion function may seem rather complicated, we are not aware of any simpler choice that provably works for more general trees than just paths. It would be interesting if a simpler inversion function exists in our level of generality, i.e. for all distributable capacity trees.

Part (1) of Definition 5.6 and part E1 of Definition 5.7 below are both forced by our upcoming base-labelling algorithm (which constructs the unique inversion-free labelling with prescribed labels). Part (4) of Definition 5.6 and parts V1–V4 of Definition 5.7 are chosen so as to make inversion-free labellings tend towards ordering labels consistently with depth-first-search order. Other parts constitute more arbitrary choices we have made where some choice was needed.

First we prioritize the capacity channels out of a vertex:

Definition 5.5. Given a vertex u , say that the edge $e_{u,w}$ is a *higher priority edge* at u than $e_{u,v}$ if either (1) there is a coarsening with respect to local root u such that $u, v \in C_2$ but $w \notin C_2$, or (2) there is a coarsening with respect to local root u such that $w \in C_1$ and $v \in C_3$, or (3) for all levels of coarsening we have $u, v, w \in C_2$ but with v a child of u and w the parent of u .

Definition 5.6. Let μ, ν be two labels in a labelled tree whose directed paths to their destinations intersect, letting $e_{x,y}$ be the final shared edge, when there is such an edge, and otherwise letting x be the unique shared vertex. Then μ has *higher priority* than ν at x , or in other words μ *travels farther* than ν from x , if any of the following conditions hold:

- (1) The path from x to $dest(\mu)$ properly contains the path from x to $dest(\nu)$.
- (2) There is a shared edge $e_{x,y}$ as above, and the coarsening-path-distance of μ is greater than that of ν with respect to some coarsening with y as local root.
- (3) There is a shared edge $e_{x,y}$ as above, and the edge $e_{y,u}$ in the path of μ is a higher priority edge out of y (cf. Definition 5.5) than the edge $e_{y,v}$ traversed by ν .
- (4) There is a shared edge $e_{x,y}$ as above, $dest(\mu) = dest(\nu)$, and either (a) $\mu < \nu$ with $e_{x,y}$ proceeding from later to earlier position in depth-first-search order, or (b) $\mu > \nu$ with $e_{x,y}$ proceeding from earlier to later position in depth-first-search order.
- (5) μ traverses a higher priority edge out of x than ν does.

Note that in Definition 5.7, given next, two labels will never form a label inversion unless their paths share at least a vertex. V1–V4 below will deal with the various ways two paths may meet in just a vertex, while E1–E3 handle the ways two paths may share an edge or have the starting point of one path be contained in the other path.

Definition 5.7. A pair of labels (λ_i, λ_j) with $\lambda_i \in v_i$ and $\lambda_j \in v_j$ forms a *label inversion pair* in a tree labelling σ , denoted $(\lambda_i, \lambda_j) \in I_{val}(\sigma)$, if any of the following conditions are met:

- E1. λ_i and λ_j traverse the same edge in opposite directions on the paths from v_i to $dest(\lambda_i)$ and from v_j to $dest(\lambda_j)$.
- E2. λ_j is on the path of λ_i , and λ_i fills a lexicographically smaller list of unfilled capacity channels at v_j (i.e. ones not filled by higher priority labels according to the label prioritization scheme described below) outward from v_j than λ_j does, or there is a tie and λ_i travels farther from v_j than λ_j does. The lexicographic order is on lists of indexing positions for sublists of the ordering on capacity channels given within our label prioritization scheme below.
- E3. The paths of λ_i and λ_j share at least one directed edge $e_{u,v}$ for $u \neq v_j$, and if we let $e_{x,y}$ denote the final shared edge, then λ_j is closer than λ_i to y (in the sense of Definition 5.6), but λ_i travels farther than λ_j does from x (again as in Definition 5.6).

Additionally, $(\lambda_i, \lambda_j) \in I_{val}(\sigma)$ if the paths of λ_i and λ_j share a vertex w but no common edge, and one of the following conditions is met:

- V1. v_i, v_j are in parts C_r, C_s , respectively, of some coarsening with respect to a common ancestor w , while λ_i and λ_j have destination parts $C_{d(r)}$ and $C_{d(s)}$, respectively, with $r < s$ and $d(r) > d(s)$.

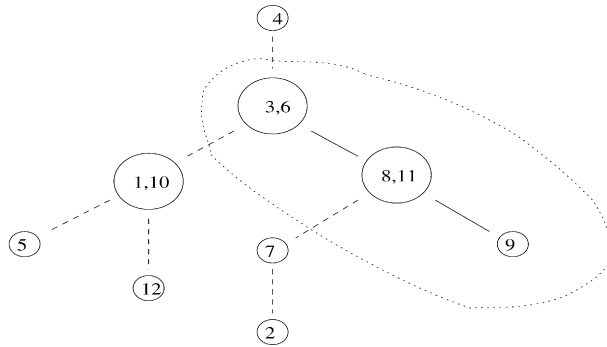


Fig. 4. A labelling of a distributable capacity tree.

- V2. $v_i \in C_r$ and $v_j \in C_s$ with $r < s$ for C_r, C_s parts in some coarsening with respect to a common ancestor w , while $dest(\lambda_i), dest(\lambda_j)$ are in the same part with respect to all possible coarsenings, and $dest(\lambda_j)$ is an ancestor of w which is an ancestor of $dest(\lambda_i)$.
- V3. v_j is a descendent of v_i , w is an ancestor of $dest(\lambda_i)$ and $dest(\lambda_j)$, but $dest(\lambda_j)$ precedes $dest(\lambda_i)$ in depth-first-search order.
- V4. $dest(\lambda_i) = dest(\lambda_j)$, $\lambda_i < \lambda_j$, and v_i comes later than v_j in depth-first-search order.

Definition 5.8. A node pair (v_i, v_j) forms an *inversion pair* in a tree labeling σ , denoted $(v_i, v_j) \in I(\sigma)$, if there are labels $\lambda_i \in v_i$ and $\lambda_j \in v_j$ forming a label inversion pair.

The tree labelling in Fig. 4 has inversion pairs (v_1, v_2) , (v_2, v_3) and (v_6, v_7) between neighboring nodes resulting from label inversions of type E1, but has no inversion pairs in which both nodes belong to the set $\{v_2, v_6, v_9\}$.

Label prioritization scheme. Given a vertex w and a collection of labels, first we prioritize edges, then will use the resulting ordering on the edges to prioritize labels. Begin by ordering the edges incident to w according to Definition 5.5. Then apply the list augmentation procedure below to add to the edge list all remaining tree edges. Now use this edge list to order the labels as follows. Repeatedly give highest priority among labels not yet chosen to the label whose path from w to its destination includes the lexicographically smallest sublist of capacity channels in the edge list that are not filled by any earlier labels in the label list. Keep repeating to obtain an ordering on all the labels.

List augmentation procedure. Proceed through a given ordered list of edges sequentially. Upon encountering edge $e_{x,y}$, append to the end of the list all $e_{y,y'}$ for this y which are not yet in the list, ordering those travelling farthest from y (as in Definition 5.5) earliest. Continue through the list until no further such augmentation is possible.

Lemma 5.9. For each pair of vertices v_i, v_j in a labelled tree, there is a unique redistribution of the set of labels collectively assigned to v_i and v_j such that (v_i, v_j) is not an inversion pair in the relabelled tree.

Proof. Let P be the path from v_i to v_j . For each capacity channel of v_i or v_j not involving any edges in P , the inversion-free labeling will assign to v_i (resp. v_j) the highest priority label using

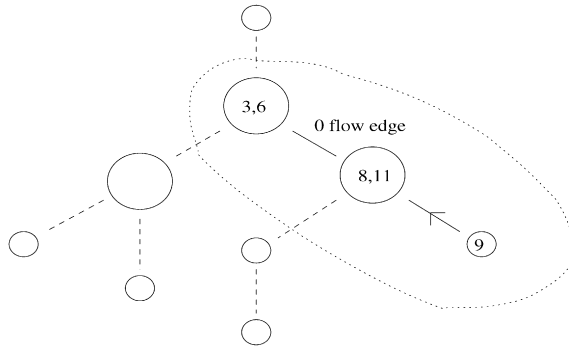


Fig. 5. Base labelling of a component with a prescribed set of labels.

that capacity channel, if any such label is available. The remaining labels must be assigned so as to avoid two labels traversing the same edge in opposite directions, which means that the vertex among v_i, v_j having excess capacity will receive all the labels destined for it or outward from it across capacity channels not involving any edges in P , as well as perhaps some additional labels.

For each label μ under consideration, there is a unique vertex $v \in P$ such that the path from v to $dest(\mu)$ only intersects P in v . This enables us to partition the labels to be assigned to v_i, v_j according to their associated vertices in P . By Definition 5.7, part E1, there is some vertex $x \in P$ such that all labels associated to $x' \in P$ with x' closer than x to v_i must be assigned to v_i in our inversion-free redistribution, and on the other hand all labels associated to $x'' \in P$ having x'' closer than x to v_j must be assigned to v_j in the inversion-free labelling. This just leaves the labels associated to x , but then conditions E3 and V1–V4 of Definition 5.7 determine uniquely how to distribute to v_i and v_j these remaining labels. \square

The unique inversion-free labelling of a component with a prescribed set of labels is its base-labelling, as defined next. Fig. 5 gives an example of a base-labelling of a component. Notice that this is a part of the tree labelling given in Fig. 4.

Definition 5.10. The *base-labelling* of a component C (of a forest obtained by deleting edges from a distributable capacity tree T) with a specified set of labels is defined as follows. If C consists of a single vertex of T , then all labels are assigned to the unique vertex of C , so henceforth assume $|C| > 1$. Let S be the maximal set of edges of C which may be removed to yield a forest, each of whose components C' has exactly as much capacity as the number of labels destined either for C' or across edges from C' to components outside C . For each C' , call this set of labels $S(C')$. Assign to each such C' exactly the labels in $S(C')$, associating them to specific nodes within C' as described next.

Obtain a directed graph on vertex set $V(C')$ by orienting each edge of the induced subgraph on $V(C')$ from the portion of C' with excess capacity towards the portion with insufficient capacity. Now we specify for each sink v in this digraph the set of labels to be assigned to v ; this will be a subset of the labels either destined for v or across edges $e_{v,w}$ for $w \notin C'$. For each capacity channel $e_{v,w}$ for $w \notin C'$, assign to v the highest priority label traversing $e_{v,w}$ according to the label prioritization scheme above. By Remark 5.2, we always have enough capacity to do this. Then exhaust as much of the remaining capacity at v as possible by greedily assigning additional labels to v whose paths involve edges from v to other components, using the label prioritization

scheme above to prioritize which labels are chosen. Break ties by choosing labels traveling farthest as in Definition 5.6. Fill any remaining vacancies at v with labels destined for v , doing so in the unique way that is inversion-free according to Definition 5.8. Since v is a sink, there will be enough such labels available to saturate v .

Once we thereby saturate v , remove v from consideration, causing new nodes to become sinks and potentially splitting what remains of C' into multiple components; keep repeating until each vertex in C' has been assigned as many labels as its capacity. If $v \in C'$ has multiple edges in C' leading to it just prior to its deletion, distribute as follows the unassigned labels of C' to the various components into which C' splits upon deleting v . Each component receives exactly as many of v 's excess labels as the capacity discrepancy across the edge from v to that component; specific excess labels are distributed to these components in the unique way that is inversion-free according to Definition 5.8. This is made possible by the fact that each label is an excess label for at most one sink at any given step in this process – because at any stage at most one sink is on the label's path to its destination within the entire tree (i.e. not just within C').

The *base-labelling of a face* is comprised of the base-labellings of its various components with their prescribed sets of labels. The *inversions* of a face are the inversions in its base-labelling.

Theorem 5.11. *Definition 5.8 specifies a full inversion function, hence its restriction to neighboring pairs of nodes is an inversion function.*

Proof. It is straightforward to check the consistency and completeness of Definitions 5.10, 5.7, and that base-labellings are inversion free; this can be seen by drawing a few diagrams of different types of trees showing how Definition 5.7 covers the various ways the paths of two labels may intersect. What remains is to verify that Definition 5.8 satisfies the requirements from Definition 3.4 for a full inversion function.

The fact that an inversion (v_i, v_k) forces an inversion of two neighboring vertices on the path from v_i to v_k follows directly from the fact that traditional inversions in permutations have this property, together with our base-labelling algorithm (i.e. Definition 5.10) which puts labels headed across the various capacity channels at nodes incident to these channels. This gives property (1) of Definition 3.4. The first part of our base-labelling algorithm, i.e. the splitting into components C' by deleting all possible edges having 0 flow, ensures that for each $G \subseteq F$, i.e. any face G obtained by merging components of F , that if F is inversion free on the restriction of F to any component of G , then the base-labellings of F and G coincide on the restriction of F to any component of G . This confirms property (2) of Definition 3.4.

Next, we show that the transitive closure of $<_{\text{weak}}$ is a partial order, i.e. property (3) of Definition 3.4. This is accomplished by verifying that the following integer-valued function f on tree labellings satisfies $f(\sigma) < f(\tau)$ for each $\sigma <_{\text{weak}} \tau$. For σ a tree labelling, let $N_{i,j}(\sigma)$ be the number of labels at v_i in σ that are assigned to v_j in the unique redistribution σ' of labels between v_i and v_j satisfying $(v_i, v_j) \notin I(\sigma')$. Lemma 5.9 guarantees the existence of such a σ' . Now let

$$f(\sigma) = \sum_{\substack{(v_i, v_j) \in I(\sigma) \\ i < j}} N_{i,j}(\sigma) \cdot d(v_i, v_j),$$

with $d(v_i, v_j)$ denoting distance in the usual graph-theoretic sense. Let $d_\sigma(\lambda_i, \lambda_j) = d(v_i, v_j)$ for $\lambda_i \in v_i$ and $\lambda_j \in v_j$ in the tree labelling σ .

Consider $\sigma \prec_{\text{weak}} \tau$ for $\sigma = (r, r + 1)\tau$. If $\{i, j\} \cap \{r, r + 1\} = \emptyset$, then $N_{i,j}(\tau) = N_{i,j}(\sigma)$, so we need only consider the contribution of pairs (i, j) with $\{i, j\} \cap \{r, r + 1\} \neq \emptyset$. For notational convenience, say $i = r$ throughout the following argument; other cases are similar.

Suppose $j \neq r + 1$. Consider $(\lambda_{i_m}, \lambda_{j_n}) \in I_{\text{val}}(\sigma)$ for $\lambda_{i_m}, \lambda_{j_n}$ located at v_r, v_j , respectively. Then $(\lambda_{i_m}, \lambda_{j_n}) \in I_{\text{val}}(\tau)$ with $d_\tau(\lambda_{i_m}, \lambda_{j_n}) \geq d_\sigma(\lambda_{i_m}, \lambda_{j_n})$ unless $(r, r + 1)$ swaps λ_{i_m} with some μ , and we have $d_\tau(\lambda_{i_m}, \lambda_{j_n}) + d_\tau(\mu, \lambda_{j_n}) = d_\sigma(\lambda_{i_m}, \lambda_{j_n}) + d_\sigma(\mu, \lambda_{j_n})$. There may be a choice of such μ , but any choice will work, provided our collection of such choices comprises a matching on pairs of values in $I_{\text{val}}(\tau)$ being swapped between nodes v_r and v_{r+1} . If $(\lambda_{i_m}, \lambda_{j_n}) \in I_{\text{val}}(\tau)$, then $(\mu, \lambda_{j_n}) \in I_{\text{val}}(\sigma)$ and $(\mu, \lambda_{j_n}) \in I_{\text{val}}(\tau)$. If $(\lambda_{i_m}, \lambda_{j_n}) \notin I_{\text{val}}(\tau)$, then λ_{i_m} is on the path from μ to v_j in σ with $(\mu, \lambda_{j_n}) \notin I_{\text{val}}(\sigma)$ and $(\mu, \lambda_{j_n}) \in I_{\text{val}}(\tau)$. In each case, the pairs $(\lambda_{i_m}, \lambda_{j_n})$ and (μ, λ_{j_n}) together contribute at least as much to $f(\tau)$ as to $f(\sigma)$.

Summing over all choices of $\{i, j\}$ such that $\{i, j\} \neq \{r, r + 1\}$ yields $f(\sigma) - N_{r,r+1}(\sigma) \leq f(\tau) - N_{r,r+1}(\tau)$. Finally, $N_{r,r+1}(\sigma) = 0$ while $N_{r,r+1}(\tau) > 0$, yielding $f(\sigma) < f(\tau)$, as desired. Property (4) of Definition 3.4, namely restriction to subtrees of the other properties, is immediate again from our definition of base-labelling. \square

6. Variations on the connectivity bound for Δ_T

A different choice of edge set E which yields a capacity distributable tree will give a shelling of $\Delta_T^{(j)}$ for

$$j = \sum_{v \in T} \left\lfloor \frac{\text{deg}(v) - 1}{2} \right\rfloor.$$

Namely, let E consist of the edges from each vertex v to its first $\lfloor \frac{\text{deg}(v)-1}{2} \rfloor$ children, together with the edge from the root to its only child. Now apply Theorem 3.5. In cases such as chessboard complexes with nodes of high degree, this gives an improved connectivity lower bound. Next we recover a shellability result from [16].

Proposition 6.1. *The shelling for $\Delta_{(T, \mathbf{m})}$ when (T, \mathbf{m}) has distributable capacity yields a shelling for $M_{m,n}$ for $n \geq 2m - 1$.*

Proof. Choose any edge set $E = \{e_1, \dots, e_k\}$ for T a star with m leaves of capacity one and central vertex of capacity $n - m$. Consider the subcomplex of $\Delta_{(T, \mathbf{m})} \cong M_{m,n}$ generated by those faces F with $E^C(F) = \{e_1, \dots, e_k\}$. These F give rise to distributable capacity trees if and only if $(m - k) + (n - m) \geq k - 1$. Thus, results of earlier sections yield shellability of $M_{m,n}^{(k-1)}$ for $k \leq \frac{n+1}{2}$, hence shellability of $M_{m,n}$ for $n \geq 2m - 1$. \square

Our results do not yield the optimal connectivity bound for chessboard complexes in general, regardless of our choice of E . It is interesting to note that an i -dimensional type-selected complex $\Delta_{(T', \mathbf{m})}$ may not be shellable while the full i -skeleton of Δ_T may still be shellable. It would be interesting to find the optimal connectivity bound for each Δ_T , or more generally for each $\Delta_{(T, \mathbf{m})}$, the latter of which would include the chessboard complexes as a special case.

In [9], we generalize Ziegler’s proof of vertex decomposability of skeleta to trees with one non-leaf vertex as well as long exact sequences of Shareshian and Wachs from [13]. This enables a sharpness result, in the sense that we characterize exactly which trees (T, \mathbf{m}) have shellable

Coxeter-like complexes. However, the question of a sharp connectivity bound remains open for general Coxeter-like complexes.

Let $\max\deg(T)$ be the largest degree of any vertex in a tree T . Then the fact that Ziegler's vertex decomposition for the $v_{m,n}$ -skeleton of a chessboard complex generalizes to the link of each face $F(v)$ together with the above theorem also suggests the following question.

Question 6.2. Is the $(n - \max\deg(T))$ -skeleton of Δ_T shellable?

Earlier sections prove a conjecture of Babson and Reiner by showing that the $(n - b)$ -skeleton is shellable, where b is the number of leaves in T . Notice that $b \geq \max\deg(T)$ holds, since each edge outward from a fixed vertex v of maximal degree leads to a subtree with at least one leaf in it. Thus, an affirmative answer would give an improved connectivity bound for Δ_T .

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