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Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations ☆

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Abstract

This paper is concerned with boundary value problems for systems of nonlinear second-order differential equations. Under the suitable conditions, the existence and multiplicity of positive solutions are established by using abstract fixed-point theorems.

Keywords: Boundary value problems; Positive solutions; Multiplicity; Green's functions; Cones

1. Introduction

Recently, existence and multiplicity of solutions for boundary value problems of ordinary differential equations have been of great interest in mathematics and its applications to engineering sciences (see [4,6–10] and references cited therein). To our knowledge, most existing results on this topic are concerned with single equation and simple boundary conditions.

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It should be pointed out that Erbe and Wang [3] discussed the boundary value problem as follows

$$\begin{cases} -u'' = f(x,u), \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases} \tag{1}$$

By using a Krasnosel'skii fixed-point theorem, the existence of solutions of (1) is obtained in the case when, either f is superlinear, or f is sublinear. Yang and Sun [9] considered the boundary value problem of the system of differential equations

$$\begin{cases} -u'' = f(x,v), \\ -v'' = g(x,u), \\ u(0) = u(1) = 0, \\ v(0) = v(1) = 0. \end{cases} \tag{2}$$

By appealing to the degree theory, the existence of solutions of (2) is established. Note that, there is only one differential equation in (1) and BVP (2) holds simple boundary conditions.

Motived by the works of [3] and [9], this paper is concerned with the existence and multiplicity of positive solutions for boundary value problems

$$\begin{cases} -u'' = f(x,v), \\ -v'' = g(x,u), \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \\ \alpha v(0) - \beta v'(0) = 0, \quad \gamma v(1) + \delta v'(1) = 0, \end{cases} \tag{3}$$

where $f, g \in C([0,1] \times R^+, R^+)$, $f(x,0) \equiv 0$, $g(x,0) \equiv 0$, $\alpha, \beta, \gamma, \delta \geqslant 0$, $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The arguments for establishing the existence of solutions of (3) involve properties of Green's functions that play a key role in defining some cones. A fixed point theorem due to Krasnosel'skii [5] is applied to yield the existence of positive solutions of (3). Another fixed point theorem about multiplicity is applied to obtain the multiplicity of positive solutions of (3).

This paper is organized as follows. In the next section, we present some notation and preliminaries. The main results, existence and multiplicity of positive solutions of BVP(3), are given in Section 3. Some examples are given to illustrate our main results.

2. Preliminaries

Obviously, $(u,v) \in C^2[0,1] \times C^2[0,1]$ is the solution of (3) if and only if $(u,v) \in C[0,1] \times C[0,1]$ is the solution of the system of integral equations

$$\begin{cases} u(x) = \displaystyle\int_0^1 k(x,y) f\big(y, v(y)\big)\,dy, \\ v(x) = \displaystyle\int_0^1 k(x,y) g\big(y, u(y)\big)\,dy, \end{cases} \tag{4}$$

where $k(x, y)$ is the Green's function defined as follows:

$$k(x, y) = \begin{cases} \frac{1}{\rho}(\gamma + \delta - \gamma x)(\beta + \alpha y), & 0 \leqslant y \leqslant x \leqslant 1, \\ \frac{1}{\rho}(\beta + \alpha x)(\gamma + \delta - \gamma y), & 0 \leqslant x \leqslant y \leqslant 1. \end{cases}$$

Integral equations (4) can be transferred to the nonlinear integral equation

$$u(x) = \int_0^1 k(x, y) f\left(y, \int_0^1 k(y, z) g\big(z, u(z)\big)\, dz\right) dy. \tag{5}$$

Lemma 1. *The Green's function $k(x, y)$ satisfies*

(i) $k(x, y) \leqslant k(y, y)$ *for* $0 \leqslant x, y \leqslant 1$;
(ii) $k(x, y) \geqslant Mk(y, y)$ *for* $\frac{1}{4} \leqslant x \leqslant \frac{3}{4}$, $0 \leqslant y \leqslant 1$,

where $M = \min\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\} \leqslant 1$.

The proof of this lemma is standard and omitted.

Let $E = C[0, 1]$. For $u \in E$, define $\|u\| = \max_{0 \leqslant x \leqslant 1} |u(x)|$. Then $(E, \|\cdot\|)$ is a Banach space. Denote

$$P = \left\{u \in E \;\middle|\; u(x) \geqslant 0,\ \min_{\frac{1}{4} \leqslant x \leqslant \frac{3}{4}} u(x) \geqslant M\|u\|\right\}.$$

It is obvious that P is a positive cone in E. Define

$$Au(x) = \int_0^1 k(x, y) f\left(y, \int_0^1 k(y, z) g\big(z, u(z)\big)\, dz\right) dy, \quad u \in P. \tag{6}$$

Lemma 2. *If the operator A is defined as* (6)*, then $A : P \to P$ is completely continuous.*

Proof. From the continuity of f and g, we know $Au \in E$ for each $u \in P$. It follows from Lemma 1 that for $u \in P$,

$$\begin{aligned} Au(x) &= \int_0^1 k(x, y) f\left(y, \int_0^1 k(y, z) g\big(z, u(z)\big)\, dz\right) dy \\ &\leqslant \int_0^1 k(y, y) f\left(y, \int_0^1 k(y, z) g\big(z, u(z)\big)\, dz\right) dy. \end{aligned}$$

Note that by the nonnegativity of f and g, one has

$$\|Au\| \leqslant \int_0^1 k(y, y) f\left(y, \int_0^1 k(y, z) g\big(z, u(z)\big)\, dz\right) dy,$$

from which we have

$$\begin{aligned}\min_{\frac{1}{4}\leqslant x\leqslant\frac{3}{4}} Au(x) &= \min_{\frac{1}{4}\leqslant x\leqslant\frac{3}{4}} \int_0^1 k(x,y) f\left(y, \int_0^1 k(y,z) g\big(z,u(z)\big)\,dz\right) dy \\ &\geqslant M \int_0^1 k(y,y) f\left(y, \int_0^1 k(y,z) g\big(z,u(z)\big)\,dz\right) dy \\ &\geqslant M\|Au\|, \quad u \in P.\end{aligned}$$

Therefore $A : P \to P$. Since $k(x,y)$, $f(x,u)$ and $g(x,u)$ are continuous, it is easily known that $A : P \to P$ is completely continuous. The proof is complete. □

From above arguments, we know that the existence of positive solutions of (3) can be transferred to the existence of positive fixed points of the operator A.

Lemma 3. *(See [1,2,5].) Let $(E, \|\cdot\|)$ be a Banach space, and $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E such that $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$$

is a completely continuous operator such that either

(i) $\|Au\| \leqslant \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Au\| \geqslant \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|$, $u \in P \cap \partial\Omega_2$,

then A has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Lemma 4. *(See [1,2,5].) Let $(E, \|\cdot\|)$ be a Banach space, and $P \subset E$ be a cone in E. Assume that Ω_1, Ω_2 and Ω_3 are open subsets of E such that $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, $\bar{\Omega}_2 \subset \Omega_3$. If*

$$A : P \cap (\bar{\Omega}_3 \setminus \Omega_1) \to P$$

is a completely continuous operator such that:

$$\begin{aligned}&\|Au\| \geqslant \|u\|, \quad \forall u \in P \cap \partial\Omega_1;\\ &\|Au\| \leqslant \|u\|, \quad Au \neq u, \quad \forall u \in P \cap \partial\Omega_2;\\ &\|Au\| \geqslant \|u\|, \quad \forall u \in P \cap \partial\Omega_3,\end{aligned}$$

then A has at least two fixed points x^*, x^{**} in $P \cap (\bar{\Omega}_3 \setminus \Omega_1)$, and furthermore $x^* \in P \cap (\Omega_2 \setminus \Omega_1)$, $x^{**} \in P \cap (\bar{\Omega}_3 \setminus \bar{\Omega}_2)$.

3. Main results

First we give the following assumptions:

$$(\mathrm{A}_1) \qquad \lim_{u\to 0^+} \sup_{x\in[0,1]} \frac{f(x,u)}{u} = 0, \qquad \lim_{u\to 0^+} \sup_{x\in[0,1]} \frac{g(x,u)}{u} = 0;$$

$$(\mathrm{A}_2) \qquad \lim_{u\to \infty} \inf_{x\in[0,1]} \frac{f(x,u)}{u} = \infty, \qquad \lim_{u\to \infty} \inf_{x\in[0,1]} \frac{g(x,u)}{u} = \infty;$$

(A3) $$\lim_{u\to 0^+}\inf_{x\in[0,1]}\frac{f(x,u)}{u}=\infty,\qquad \lim_{u\to 0^+}\inf_{x\in[0,1]}\frac{g(x,u)}{u}=\infty;$$

(A4) $$\lim_{u\to \infty}\sup_{x\in[0,1]}\frac{f(x,u)}{u}=0,\qquad \lim_{u\to \infty}\sup_{x\in[0,1]}\frac{g(x,u)}{u}=0;$$

(A5) $f(x,u)$, $g(x,u)$ are increasing functions with respect to u and, there is a number $N>0$, such that

$$f\left(x,\int_0^1 N' g(y,N)\,dx\right)<\frac{N}{N'},\quad \forall x\in[0,1],\ y\in[0,1],$$

where $N'=(\alpha+\beta)(\gamma+\delta)/\rho$.

Theorem 1. *If* (A_1) *and* (A_2) *are satisfied, then* (3) *has at least one positive solution* $(u,v)\in C^2([0,1],R^+)\times C^2([0,1],R^+)$ *satisfying* $u(x)>0$, $v(x)>0$.

Proof. From (A_1) there is a number $H_1\in(0,1)$, such that for each $(x,u)\in[0,1]\times(0,H_1)$, one has

$$f(x,u)\leqslant \eta u,\qquad g(x,u)\leqslant \eta u,$$

where $\eta>0$ satisfies

$$\eta\int_0^1 k(x,x)\,dx\leqslant 1.$$

For every $u\in P$ and $\|u\|=H_1/2$, note that

$$\int_0^1 k(y,z)g\big(z,u(z)\big)\,dz\leqslant \eta\int_0^1 k(z,z)u(z)\,dz\leqslant \|u\|=\frac{H_1}{2}<H_1,$$

thus

$$\begin{aligned}Au(x)&\leqslant \int_0^1 k(y,y)f\left(y,\int_0^1 k(y,z)g\big(z,u(z)\big)\,dz\right)dy\\&\leqslant \eta^2\|u\|\int_0^1 k(y,y)\int_0^1 k(z,z)\,dz\,dy\\&\leqslant \|u\|.\end{aligned}$$

Let $\Omega_1=\{u\in E:\ \|u\|<H_1/2\}$ then

$$\|Au\|\leqslant\|u\|,\quad u\in P\cap\partial\Omega_1. \tag{7}$$

On the other hand, from (A_2) there exist four positive numbers μ, μ', C_1 and C_2 such that

$$f(x,u)\geqslant \mu u-C_1,\quad \forall(x,u)\in[0,1]\times R^+,$$
$$g(x,u)\geqslant \mu' u-C_2,\quad \forall(x,u)\in[0,1]\times R^+,$$

where μ and μ' satisfy

$$\mu M \int_{\frac{1}{4}}^{\frac{3}{4}} k\left(\frac{1}{2}, y\right) dy \geqslant 2, \qquad \mu' M \int_{\frac{1}{4}}^{\frac{3}{4}} k(y, y)\, dy \geqslant 1.$$

By direct computation, for $u \in P$,

$$\begin{aligned}
Au\left(\frac{1}{2}\right) &\geqslant \int_0^1 k\left(\frac{1}{2}, y\right)\left[\mu \int_0^1 k(y,z) g\big(z, u(z)\big)\, dz - C_1\right] dy \\
&\geqslant \mu \int_0^1 k\left(\frac{1}{2}, y\right) \int_0^1 k(y,z) g\big(z, u(z)\big)\, dz\, dy - C_1 \int_0^1 k\left(\frac{1}{2}, y\right) dy \\
&\geqslant \mu \int_0^1 k\left(\frac{1}{2}, y\right) \int_0^1 k(y,z)\big[\mu' u(z) - C_2\big]\, dz\, dy - C_1 \int_0^1 k\left(\frac{1}{2}, y\right) dy \\
&\geqslant \mu\mu' \int_0^1 k\left(\frac{1}{2}, y\right) \int_0^1 k(y,z) u(z)\, dz\, dy - C_3\left(\frac{1}{2}\right),
\end{aligned}$$

where

$$\begin{aligned}
C_3\left(\frac{1}{2}\right) &= \mu C_2 \int_0^1 k\left(\frac{1}{2}, y\right) \int_0^1 k(y,z)\, dz\, dy + C_1 \int_0^1 k\left(\frac{1}{2}, y\right) dy \\
&\leqslant \mu C_2 \int_0^1 k\left(\frac{1}{2}, y\right) \int_0^1 k(z,z)\, dz\, dy + C_1 \int_0^1 k\left(\frac{1}{2}, y\right) dy \\
&= C_3.
\end{aligned}$$

Therefore

$$Au\left(\frac{1}{2}\right) \geqslant \mu \int_{\frac{1}{4}}^{\frac{3}{4}} k\left(\frac{1}{2}, y\right) dy \cdot M\mu' \int_{\frac{1}{4}}^{\frac{3}{4}} k(z,z) u(z)\, dz - C_3 \geqslant 2\|u\| - C_3,$$

from which it follows that $\|Au\| \geqslant Au(\frac{1}{2}) \geqslant \|u\|$ as $\|u\| \to \infty$.

Let $\Omega_2 = \{u \in E \colon \|u\| < H_2\}$. Then for $u \in P$ and $\|u\| = H_2 > 0$ sufficient large, we have

$$\|Au\| \geqslant \|u\|, \quad \forall u \in P \cap \partial\Omega_2. \tag{8}$$

Thus, from (7), (8) and Lemma 3, we know that the operator A has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The proof is complete. □

Theorem 2. *If* (A$_3$) *and* (A$_4$) *are satisfied, then* (3) *has at least one positive solution* $(u, v) \in C^2([0,1], R^+) \times C^2([0,1], R^+)$ *satisfying* $u(x) > 0$, $v(x) > 0$.

Proof. From (A_3) there is a number $\hat{H}_3 \in (0,1)$ such that for each $(x,u) \in [0,1] \times (0, \hat{H}_3)$, one has

$$f(x,u) \geqslant \lambda u, \qquad g(x,u) \geqslant \lambda' u,$$

where λ and λ' satisfy

$$\lambda M \int_{\frac{1}{4}}^{\frac{3}{4}} k\left(\frac{1}{2}, x\right) dx \geqslant 1, \qquad \lambda' M \int_{\frac{1}{4}}^{\frac{3}{4}} k(y,y)\, dy \geqslant 1.$$

From $g(x,0) \equiv 0$ and the continuity of $g(x,u)$, we know that there exists a number $H_3 \in (0, \hat{H}_3)$ small enough such that

$$g(x,u) \leqslant \frac{\hat{H}_3}{\int_0^1 k(x,x)\, dx}, \quad \forall (x,u) \in [0,1] \times (0, H_3).$$

For every $u \in P$ and $\|u\| = H_3$, note that

$$\int_0^1 k(y,z) g\big(z, u(z)\big)\, dz \leqslant \int_0^1 k(y,z) \frac{\hat{H}_3}{\int_0^1 k(z,z)\, dz}\, dz < \hat{H}_3,$$

thus

$$\begin{aligned} Au\left(\frac{1}{2}\right) &= \int_0^1 k\left(\frac{1}{2}, y\right) f\left(y, \int_0^1 k(y,z) g\big(z,u(z)\big)\, dz\right) dy \\ &\geqslant \int_{\frac{1}{4}}^{\frac{3}{4}} k\left(\frac{1}{2}, y\right) \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} k(y,z) \lambda' u(z)\, dz\, dy \\ &\geqslant M^2 \|u\| \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} k\left(\frac{1}{2}, y\right) \lambda' \int_{\frac{1}{4}}^{\frac{3}{4}} k(z,z)\, dz\, dy \\ &\geqslant \|u\|. \end{aligned}$$

Let $\Omega_3 = \{u \in E\colon \|u\| < H_3\}$ we have

$$\|Au\| \geqslant \|u\|, \quad u \in P \cap \partial \Omega_3. \tag{9}$$

On the other hand, we know from (A_4) that there exist three positive numbers η', C_4, and C_5 such that for every $(x,u) \in [0,1] \times R^+$,

$$f(x,u) \leqslant \eta' u + C_4, \qquad g(x,u) \leqslant \eta' u + C_5,$$

where

$$\eta' \int_0^1 k(x,x)\, dx \leqslant \frac{1}{2}.$$

Thus we have

$$\begin{aligned}
Au(x) &\leqslant \int_0^1 k(y,y)\left[\eta' \int_0^1 k(z,z)g\big(z,u(z)\big)\,dz + C_4\right]dy \\
&\leqslant \eta' \int_0^1 k(y,y)\,dy \int_0^1 k(z,z)\big[\eta' u(z) + C_5\big]\,dz + C_4 \int_0^1 k(y,y)\,dy \\
&\leqslant \frac{1}{4}\|u\| + C_6,
\end{aligned}$$

where

$$C_6 = C_5\eta' \int_0^1 k(y,y)\,dy \int_0^1 k(z,z)\,dz + C_4 \int_0^1 k(y,y)\,dy,$$

from which it follows $Au(x) \leqslant \|u\|$ as $\|u\| \to \infty$. Let $\Omega_4 = \{u \in E\colon\ \|u\| < H_4\}$. For each $u \in P$ and $\|u\| = H_4 > 0$ large enough, we have

$$\|Au\| \leqslant \|u\|, \quad \forall u \in P \cap \partial\Omega_4. \tag{10}$$

From (9), (10) and Lemma 3, we know that the operator A has a fixed point in $P \cap (\bar{\Omega}_4 \setminus \Omega_3)$. The proof is complete. □

Theorem 3. *If* (A_2), (A_3) *and* (A_5) *are satisfied, then* (3) *has at least two distinct positive solutions* $(u_1, v_1), (u_2, v_2) \in C^2([0,1], R^+) \times C^2([0,1], R^+)$ *satisfying* $u_i(x) > 0$, $v_i(x) > 0$ $(i = 1, 2)$.

Proof. Note that $k(x,y) \leqslant \frac{1}{\rho}(\alpha+\beta)(\gamma+\delta) = N'$. Let $B_N = \{u \in E\colon\ \|u\| < N\}$. Then from (A_5), for every $u \in \partial B_N \cap P$, $x \in [0,1]$, we have

$$\begin{aligned}
Au(x) &\leqslant N' \int_0^1 f\left(y, \int_0^1 N' g(z,N)\,dz\right)dz \\
&< N' \frac{N}{N'} = N.
\end{aligned}$$

Thus

$$\|Au\| < \|u\|, \quad \forall u \in \partial B_N \cap P. \tag{11}$$

And from (A_2) and (A_3) we have

$$\|Au\| \geqslant \|u\|, \quad u \in \partial\Omega_2 \cap P, \tag{12}$$

$$\|Au\| \geqslant \|u\|, \quad u \in \partial\Omega_3 \cap P. \tag{13}$$

We can choose H_2, H_3 and N such that $H_3 \leqslant N \leqslant H_2$ and (11)–(13) are satisfied. From Lemma 4, A has at least two fixed points in $P \cap (\bar{\Omega}_2 \setminus B_N)$ and $P \cap (\bar{B}_N \setminus \Omega_2)$, respectively. The proof is complete. □

Examples. Some examples are given to illustrate our main results.

(i) Let $f(x, v) = v^2$, $g(x, u) = u^3$, then conditions of Theorem 1 are satisfied. From Theorem 1, BVP(3) has at least one positive solution.

(ii) Let $f(x, v) = v^{\frac{1}{2}}$, $g(x, u) = u^{\frac{1}{2}}$, then conditions of Theorem 2 are satisfied. From Theorem 2, BVP(3) has at least one positive solution.

(iii) Let $f(x, v) = \frac{v^{\frac{1}{2}} + v^2}{12}$, $g(x, u) = u^{\frac{1}{2}} + u^2$, $\alpha = \beta = \gamma = \delta = 1$. Thus $N' = \frac{4}{3}$. We can choose $N = 1$, then conditions of Theorem 3 are satisfied. From Theorem 3, BVP(3) has at least two positive solutions.

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