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A sequence of generalizations of the geometric series

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We consider the following initial members of a sequence of generalizations of the geometric series:

$$S(v;z) \equiv {}_1\mathrm{F}_0(v+1;z) = \sum_{n=0}^{\infty} \frac{(n+v)!}{n!v!} z^n = \frac{1}{(1-z)^{v+1}}, \tag{1}$$

$$\begin{aligned} S(\mu,v;z) &\equiv {}_2\mathrm{F}_1(\mu+1,v+1;1;z) \\ &= \sum_{n=0}^{\infty} \frac{(n+\mu)!}{n!\mu!} \frac{(n+v)!}{n!v!} z^n, \\ &= \frac{1}{(1-z)^{\mu+v+1}} \sum_{k=0}^{\{\mu,v\}} \frac{\mu!v!}{k!(\mu-k)!k!(v-k)!} z^k, \end{aligned} \tag{2}$$

$$\begin{aligned} S(\lambda,\mu,v;z) &\equiv {}_3\mathrm{F}_2(\lambda+1,\mu+1,v+1;1,1;z) \\ &= \sum_{n=0}^{\infty} \frac{(n+\lambda)!}{n!\lambda!} \frac{(n+\mu)!}{n!\mu!} \frac{(n+v)!}{n!v!} z^n \\ &= \frac{1}{(1-z)^{\lambda+\mu+v+1}} \sum_{k=0}^{\{\lambda+\mu,\lambda+v,\mu+v\}} \frac{(\lambda+\mu+v+1)!}{\lambda!\mu!v!} \\ &\quad \times \left(\sum_{j=0}^{k} \frac{(-1)^j(\lambda+k-j)!(\mu+k-j)!(v+k-j)!}{j!(k-j)!^3(\lambda+\mu+v+1-j)!} \right) z^k. \end{aligned} \tag{3}$$

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The first equation follows by differentiation of the geometric series. The second equation can be obtained as a special case of the transformation relations of the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$. For nonnegative integer μ or (and) v, it is at once a summation formula because the sum on the right-hand side terminates in these cases. The third equation is also a summation formula, for nonnegative integer (λ, μ, v), of the series on the left-hand side. Similarly to the preceding cases, the series can be represented as a special case of the hypergeometric function ${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2; z)$ and the transformation to the right-hand side should find an embedding into a transformation formula of this hypergeometric function. However, in contrast to the preceding relations, it cannot be a transformation into the same type of hypergeometric functions and it seems to us that these relations and the class of functions are unknown up to now. In contrast to (1) and (2), the right-hand side of (3) contains an inner finite number sum for which we could not find a closed formula of multiplicative type and are almost sure that it does not exist. However, in the special case $\lambda = 0$ it has to make the transition to formula (2) and the inner sum can be evaluated by a closed formula.

Therefore, a main problem is to embed the summation formulae (3) and possible continuations of the mentioned sequence into transformation formulae for the corresponding hypergeometric functions. Some subsidiary problems are the limiting transition to the special case $\lambda = 0$ and the proof of the necessary evaluation of the sums that appear. In the case of formula (2), all corresponding similar problems are solved.