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For Groups the Property of Having Finite Derivation Type is equivalent to the Homological Finiteness Condition FP_3

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The homological finiteness property FP_3 and the combinatorial property of having finite derivation type are both necessary conditions for finitely presented monoids to admit finite convergent presentations. For monoids in general, the property of having finite derivation type implies the property FP_3, and there even exist finitely presented monoids that are FP_3, but that do not have finite derivation type (Cremanns and Otto, 1994). Here, contrasting this result, we show that for groups these two properties are equivalent. The proof is based on the result that a group G, which is given through a finite presentation $\langle X; R\rangle$, has finite derivation type if and only if the $\mathbb{Z}G$-module of identities among relations that is associated with $\langle X; R\rangle$ is finitely generated. This result, which was announced in (Cremanns and Otto, 1994), is proved in a conceptually simple manner, greatly improving upon the original proof that was only outlined in (Cremanns and Otto, 1994). Then, using elementary algebraic arguments we derive our main result without using much of homology theory, thus making the proof easily accessible to computer scientists and mathematicians with some background in algebra and rewriting theory.

1. Introduction

In many instances rewriting systems that are *convergent*, that is *noetherian* and *confluent*, have been found to be quite useful for solving decision problems for algebraic structures effectively (LeChenadec, 1986). When the algebraic structures under consideration are monoids or groups, then the appropriate notion of rewriting systems is that of string-rewriting systems. A finite convergent string-rewriting system yields syntactically simple algorithms for solving the word problem, the order problem and the problem of deciding commutativity (Book and Otto, 1993), and in some instances such a system also induces algorithms for solving the conjugacy problem (Otto, 1984) and the generalized word problem (Kuhn and Madlener, 1989).

An important question that remained open for many years is the following: does every finitely presented monoid with a decidable word problem have a presentation through some finite convergent string-rewriting system? In 1987 this question was finally answered

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by C. Squier. He proved that a monoid which admits a presentation of this particular form satisfies the homological finiteness condition FP_3. Since there are known examples of finitely presented monoids (in fact, groups) that have decidable word problem but are not of type FP_3 (Bieri, 1976), this solved the above question in the negative.

Subsequently, it has been observed that a monoid that admits a finite convergent presentation even satisfies the stronger homological finiteness condition FP_∞ (Kobayashi, 1990). Each monoid of type FP_∞ is also of type FP_3, but there are finitely presented monoids (in fact, groups) with decidable word problem which are of type FP_3 but not of type FP_∞ (Bieri, 1976). Hence, the finiteness condition FP_3 is necessary, but not sufficient for a finitely presented monoid to admit a finite convergent presentation.

Naturally, this raises the question of whether the homological finiteness condition FP_∞ is sufficient for a finitely presented monoid with a decidable word problem to admit a presentation through some finite convergent string-rewriting system.

In a subsequent paper Squier introduces another, combinatorial, finiteness property for finitely presented monoids (Squier, 1994). He considers certain relations between paths in a graph associated with a finite monoid-presentation. These relations are called *homotopy relations*. If the set of all pairs of paths that have a common initial and a common terminal vertex is finitely generated as a homotopy relation, then the monoid-presentation is said to have *finite derivation type* (FDT). In fact, this property is an invariant of finite presentations, so that we can actually speak of monoids that have finite derivation type. Squier proves that a monoid has finite derivation type if it is presented by a finite convergent system. Exhibiting a monoid S_1 which is of homology type FP_∞ but does not have finite derivation type, he succeeds in showing that the property FP_∞ does not imply the existence of a finite convergent presentation.

Of course, this in turn raises the question of whether, for finitely presented monoids with decidable word problem, the property of having finite derivation type does imply the existence of finite convergent presentations, or whether it is just another necessary, but not sufficient condition for the existence of finite convergent presentations. And what is the exact relationship between the homological finiteness conditions FP_3 and FP_∞ on the one hand and the property of having finite derivation type on the other hand?

Concerning this last question Squier's example monoid S_1 shows that in general not even the homological finiteness condition FP_∞ suffices to imply the property of having finite derivation type. On the other hand, it has been shown recently, independently by various authors, that the condition of having finite derivation type implies the condition FP_3 (Cremanns and Otto, 1994, Lafont, 1995, Pride, 1995).

Here we settle the question of the exact relationship between these notions for the special case that the monoids under consideration are groups. Our main result states that a group has finite derivation type if it is of homology type FP_3. This is a consequence of the fact that a group G, which is given through a finite presentation $\langle X; R\rangle$, has finite derivation type if and only if the $\mathbb{Z}G$-module of identities among relations that is associated with $\langle X; R\rangle$ is finitely generated. This result, which was announced in (Cremanns and Otto, 1994), is proved in a conceptually simple manner, greatly improving upon the original proof that was only outlined in (Cremanns and Otto, 1994).

Thus, for finitely presented groups, the homological finiteness condition FP_3 is equivalent to the combinatorial condition of having finite derivation type. Since there are finitely presented groups with decidable word problem which are of type FP_3 but not of type FP_∞, this shows that the property of having finite derivation type is not sufficient to guarantee that a finitely presented group with decidable word problem admits a finite convergent presentation.

In summary, we have the following relationship between the three conditions considered: FDT and FP_∞ are both strictly stronger than FP_3. In general, FDT does not imply FP_∞, and FP_∞ does not imply FDT, either. Hence, none of these three conditions is in general sufficient for a finitely presented monoid with a decidable word problem to guarantee the existence of a finite convergent presentation.

However, the following question remains open: is the homological finiteness condition FP_∞ sufficient for a finitely presented group with a decidable word problem to admit a finite convergent presentation?

One-relator groups are known to have decidable word problem (Lyndon and Schupp, 1977), and they are of type FP_∞ (Bieri, 1976). But it is an open question whether each one-relator group has a finite convergent presentation. Also each *automatic group* has decidable word problem (Epstein, 1992) and is of type FP_∞ (Alonso, 1991), but it is not known whether each automatic group admits a finite convergent presentation. From our characterization result it follows that all these groups have finite derivation type.

This paper is organized as follows. In Section 2 we review the basic definitions on string-rewriting systems and group-presentations, and the definition and some results on the property of having finite derivation type are given. In Section 3 we give a new and fairly elementary proof for our result announced in (Cremanns and Otto, 1994), stating that a group has finite derivation type if and only if the $\mathbb{Z}G$-module π of identities among relations is finitely generated (a notion that was introduced by Peiffer (1949)). Then, in Section 4, we use this result and some well-known facts on resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ to prove our main result. Even this part uses mainly elementary algebraic arguments, thus making the whole paper easily accessible to computer scientists and mathematicians with some background in algebra and rewriting theory. Finally, in the concluding section we briefly discuss some undecidability results, answering two more questions of Squier (1994) in the negative.

2. Finite Derivation Type

We assume that the reader is familiar with the basics of combinatorial group theory and the theory of string-rewriting systems. Therefore, we only establish notation concerning string-rewriting systems, monoid-presentations, and group-presentations. For combinatorial group theory our main references are the monographs (Lyndon and Schupp, 1977) and (Magnus, Karrass and Solitar, 1976), while for the theory of string-rewriting systems we use Book and Otto (1993) as our main reference. Since we do not assume the reader to be familiar with homological algebra, all the necessary definitions will be given in short when they are encountered for the first time.

The notion of finite derivation type was introduced by Squier (1994). It is a property of a graph that is associated with a monoid-presentation $(\Sigma; R)$. Here we restate the basic definitions and results regarding this notion in short.

Let Σ be a finite alphabet. Then Σ^* denotes the set of words over Σ including the empty word λ. The length of a word w is written as $|w|$, and the concatenation of two words u and v is simply written as uv.

A *string-rewriting system* R on Σ is a subset of $\Sigma^* \times \Sigma^*$. Its elements are also called *(rewrite-)rules.* The string-rewriting system R generates a *reduction relation* on Σ^*:

$$\to_R := \{(x\ell y, xry) \mid (\ell, r) \in R, x, y \in \Sigma^*\}.$$

By $=_R$ we denote the reflexive, symmetric, transitive closure of $\to_R$. The relation $=_R$ is a *congruence* on Σ^*. Actually, it is the smallest congruence on the free monoid Σ^* that contains R. Therefore, it is called the congruence *generated by* R. The ordered pair $(\Sigma; R)$ is called a *monoid-presentation.* The monoid presented by $(\Sigma; R)$ is the factor monoid $\Sigma^*/{=_R}$.

A string-rewriting system R is called *convergent*, if it is *noetherian*, that is, there is no infinite sequence of $\to_R$-reductions, and if it is *confluent*, that is, each congruence class contains exactly one word that is *irreducible* with respect to $\to_R$. If R is a finite convergent string-rewriting system, then the word problem for R can be solved by rewriting. Actually, in this situation the irreducible words form a set of representatives for the congruence $=_R$, and the process of computing the representative of a word can be performed effectively.

With a monoid-presentation $(\Sigma; R)$ we associate a graph $\Gamma(\Sigma; R)$.

Definition 2.1. *Let $(\Sigma; R)$ be a monoid-presentation. With this presentation we associate an infinite graph $\Gamma := (V, E, \sigma, \tau, ^{-1})$ that is defined as follows:*

(a) $V := \Sigma^*$ *is the set of vertices,*
(b) $E := \{(u, (\ell, r), v, \varepsilon) \mid u, v \in \Sigma^*, (\ell, r) \in R, \varepsilon \in \{1, -1\}\}$
$= \Sigma^* \times R \times \Sigma^* \times \{1, -1\}$ *is the set of edges,*
(c) *the mappings $\sigma, \tau : E \to V$, which associate with each edge $e \in E$ its initial vertex $\sigma(e)$ and its terminal vertex $\tau(e)$, respectively, are defined through*

$$\sigma(u, (\ell, r), v, \varepsilon) := \begin{cases} u\ell v & \text{if } \varepsilon = 1 \\ urv & \text{if } \varepsilon = -1 \end{cases}$$

and

$$\tau(u, (\ell, r), v, \varepsilon) := \begin{cases} urv & \text{if } \varepsilon = 1 \\ u\ell v & \text{if } \varepsilon = -1, \end{cases}$$

(d) *and the mapping $^{-1} : E \to E$, which associates with each edge $e \in E$ an inverse edge $e^{-1} \in E$, is defined through*

$$(u, (\ell, r), v, \varepsilon)^{-1} := (u, (\ell, r), v, -\varepsilon).$$

A path p of Γ of length $n (n \geq 1)$ is a sequence $p = e_1 \circ e_2 \circ \cdots \circ e_n$ of edges satisfying $\tau(e_i) = \sigma(e_{i+1})$ for all i, $1 \leq i \leq n-1$. In this situation p is a path from $\sigma(e_1)$ to $\tau(e_n)$, and we extend the mappings σ and τ to paths by setting $\sigma(p) := \sigma(e_1)$ and $\tau(p) := \tau(e_n)$. By $|p|$ we denote the length n of the path p. The set of all paths in Γ is denoted by $P(\Gamma)$, where, for each vertex $w \in V$, we include a path (w) of length 0 from w to w. Also the mapping $^{-1}$ is extended to paths. If (w) is a path of length 0, then we take $(w)^{-1} := (w)$, and if $p = e_1 \circ \cdots \circ e_n$ is a path of length $n \geq 1$, then we take $p^{-1} := e_n^{-1} \circ \cdots \circ e_1^{-1}$. Observe that p^{-1} is indeed a path in Γ, and that $\sigma(p^{-1}) = \tau(p)$ and $\tau(p^{-1}) = \sigma(p)$ hold. Further, if p and q are paths in Γ with $\tau(p) = \sigma(q)$, then $p \circ q$ denotes the path which is obtained by concatenating p and q. Obviously, $p \circ p^{-1}$ is a path from $\sigma(p)$ back to $\sigma(p)$.

Finally, if $e = (u, (\ell, r), v, \varepsilon)$ is an edge of Γ and $x, y \in \Sigma^*$, then $xey := (xu, (\ell, r), vy, \varepsilon)$ is an edge of Γ satisfying $\sigma(xey) = x \cdot \sigma(e) \cdot y$ and $\tau(xey) = x \cdot \tau(e) \cdot y$, and $(xey)^{-1} = xe^{-1}y$. Thus, the free monoid Σ^* induces a two-sided action on the graph Γ. In fact, this action can be extended to paths as follows: if $p = e_1 \circ e_2 \circ \cdots \circ e_n$ and $x, y \in \Sigma^*$, then $xpy := xe_1y \circ xe_2y \circ \cdots \circ xe_ny$ is a path from $\sigma(xpy) = x \cdot \sigma(p) \cdot y$ to $\tau(xpy) = x \cdot \tau(p) \cdot y$, and $(xpy)^{-1} = xp^{-1}y$. Hence, Σ^* induces a two-sided action on $P(\Gamma)$.

By $P^{(2)}(\Gamma)$ we denote the following set of pairs of paths in Γ:

$$P^{(2)}(\Gamma) := \{(p, q) \mid p, q \in P(\Gamma) \text{ such that } \sigma(p) = \sigma(q) \text{ and } \tau(p) = \tau(q)\}.$$

Two particular subsets of $P^{(2)}(\Gamma)$ will play an important role.

Definition 2.2. *Let $(\Sigma; R)$ be a monoid-presentation, and let Γ be the associated graph. We define two subsets D_Γ and I_Γ of $P^{(2)}(\Gamma)$ as follows:*

$$D_\Gamma := \{(p\sigma(q) \circ \tau(p)q, \sigma(p)q \circ p\tau(q)) \mid p, q \in P(\Gamma)\},$$
$$I_\Gamma := \{(p \circ p^{-1}, (\sigma(p))) \mid p \in P(\Gamma)\}.$$

The set D_Γ is called the set of *disjoint derivations*, while I_Γ is the set of *inverse derivations.* Observe that both these sets are indeed subsets of $P^{(2)}(\Gamma)$.

Since groups can be seen as a special class of monoids, they can be presented through monoid-presentations. However, in combinatorial group theory they are usually presented through so-called group-presentations, which form a particular class of monoid-presentations. After introducing these presentations we will define the notion of finite derivation type only for the special case of group-presentations.

Let X be a finite alphabet, let $\bar{X} := \{\bar{x} \mid x \in X\}$ denote an alphabet in one-to-one correspondence to X such that $X \cap \bar{X} = \emptyset$, and let $\underline{X} := X \cup \bar{X}$. Further, let R_0 be the following string-rewriting system on $\underline{X}$:

$$R_0 := \{(x\bar{x}, \lambda), (\bar{x}x, \lambda) \mid x \in X\}.$$

Then R_0 is a finite and convergent system, and the monoid presented by $(\underline{X}; R_0)$ is the *free group* generated by X, which will be denoted by F. The reduction relation on $\underline{X}^*$ that is generated by R_0 is called the relation of *free reduction*. It is denoted by $\to_F$, and the congruence generated by R_0 is denoted by $=_F$. A word in $\underline{X}^*$ is called *freely reduced* if it is irreducible with respect to R_0. Each element of the free group F corresponds to a unique freely reduced word. Accordingly, we will usually identify the group F with the set of freely reduced words.

We define a function ${}^{-1} : \underline{X}^* \to \underline{X}^*$ through $\lambda^{-1} := \lambda$, $(wx)^{-1} := \bar{x}w^{-1}$, $(w\bar{x})^{-1} := xw^{-1}$ for all $w \in \underline{X}^*$ and $x \in X$. Observe that $ww^{-1} =_F w^{-1}w =_F \lambda$ holds for all words $w \in \underline{X}^*$, that is, w^{-1} is the *inverse* of the word w in the free group F. Finally, by Γ_F we denote the graph that is associated with the monoid-presentation $(\underline{X}; R_0)$.

For each subset $R \subseteq \underline{X}^*$, the ordered pair $\langle X; R\rangle$ is called a *group-presentation.* It is finite, if both the set X and the set R are finite. The group presented by $\langle X; R\rangle$ is defined as the monoid that is presented by the monoid-presentation $(\underline{X}; R')$, where $R' := R_0 \cup \{(r, \lambda) \mid r \in R\}$. This monoid is easily seen to be actually a group. In fact, this group is the factor group F/N of the free group F modulo the normal subgroup N of F that is generated by R. By $=_R$ we denote the congruence $=_{R'}$ on $\underline{X}^*$ that is generated by R'. Note that $=_F \subseteq =_R$. By R^{-1} we denote the set $\{r^{-1} \mid r \in R\}$.

With the group-presentation $\langle X; R\rangle$ we associate the graph Γ that is obtained from the monoid-presentation $(\underline{X}; R')$. The graph Γ_F is obviously a subgraph of Γ, since $R_0 \subseteq R'$.

Definition 2.3. *Let $\langle X; R\rangle$ be a group-presentation and let Γ be the associated graph.*

An equivalence relation $\simeq \subseteq P^{(2)}(\Gamma)$ *is called a* homotopy relation *on* $P(\Gamma)$ *if it satisfies the following conditions:*

(a) $D_\Gamma \cup I_\Gamma \subseteq \simeq$,
(b) *if* $p \simeq q$, *then* $xpy \simeq xqy$ *for all* $x, y \in \underline{X}^*$,
(c) *if* $p, q, r_1, r_2 \in P(\Gamma)$ *satisfy* $\tau(r_1) = \sigma(p) = \sigma(q)$, $\tau(p) = \tau(q) = \sigma(r_2)$, *and* $p \simeq q$, *then* $r_1 \circ p \circ r_2 \simeq r_1 \circ q \circ r_2$,
(d) *if* $(p, q) \in P^{(2)}(\Gamma_F)$, *then* $p \simeq q$.

The collection of all homotopy relations on $P(\Gamma)$ is closed under arbitrary intersection. Since the set $P^{(2)}(\Gamma)$ itself is a homotopy relation, this implies that, for each subset $B \subseteq P^{(2)}(\Gamma)$, there is a smallest homotopy relation $\simeq_B$ on $P(\Gamma)$ that contains the set B. We say that $\simeq_B$ is the *homotopy relation generated by* B. Actually, $\simeq_B$ is the equivalence relation that is generated by the following binary relation on $P^{(2)}(\Gamma)$: $p \Longrightarrow_B q$ if and only if there exist $(p_1, q_1) \in B \cup D_\Gamma \cup I_\Gamma \cup P^{(2)}(\Gamma_F)$, $x, y \in \Sigma^*$, and $r_1, r_2 \in P(\Gamma)$ such that $p = r_1 \circ xp_1y \circ r_2$ and $q = r_1 \circ xq_1y \circ r_2$ (Cremanns and Otto, 1994).

Definition 2.4. *Let* $\langle X; R\rangle$ *be a group-presentation, and let* Γ *be the associated graph. We say that* $\langle X; R\rangle$ *has* finite derivation type *if there exists a finite subset* $B \subseteq P^{(2)}(\Gamma)$ *which generates* $P^{(2)}(\Gamma)$ *as a homotopy relation, that is,* $P^{(2)}(\Gamma)$ *is the only homotopy relation on* $P(\Gamma)$ *that contains the set* B.

Our definition of homotopy relations differs slightly from Squier's definition (1994). He defines a homotopy relation as an equivalence relation $\simeq \subseteq P^{(2)}(\Gamma)$ that satisfies the conditions (a) to (c) of Definition 2.3, and his definition of homotopy relations deals with monoid-presentations in general. However, for the definition of finite derivation type in the special case of group-presentations, it does not matter which of the two definitions is used. In fact, if $\langle X; R\rangle$ is a group-presentation, then $\langle X; R\rangle$ has finite derivation type (using our definition of homotopy relations) if and only if $\langle X; R\rangle$ has finite derivation type (using Squier's definition of homotopy relations). Since the if-direction of this equivalence is obvious, the following proposition of (Squier, 1994) holds here, too.

Proposition 2.5. *If a group can be presented by a finite and convergent string-rewriting system, then it has finite derivation type.*

We also want to explain the only-if direction of the above equivalence. Let $\langle X; R\rangle$ be a group-presentation. The string-rewriting system R_0 on $\underline{X}^*$ is finite and convergent. Thus, there is a finite set $B_F \subseteq P^{(2)}(\Gamma_F)$ such that B_F generates $P^{(2)}(\Gamma_F)$ as a homotopy relation (Squier's definition). Assume that $\langle X; R\rangle$ has finite derivation type (our definition). Then there is a finite set $B \subseteq P^{(2)}(\Gamma)$ that generates $P^{(2)}(\Gamma)$ as a homotopy relation (our definition). Of course, this implies that the finite set $B \cup B_F$ generates $P^{(2)}(\Gamma)$ as a homotopy relation (Squier's definition). Thus, $\langle X; R\rangle$ also has finite derivation type (Squier's definition). Actually, for B_F we can take the set $\{((\lambda, (x^{-1}x, \lambda), x^{-1}, 1), (x^{-1}, (xx^{-1}, \lambda), \lambda, 1)) \mid x \in \underline{X}\}$.

Since our definition of the notion of having finite derivation type is equivalent to that given by Squier, Squier's results on the notion of finite derivation type remain valid for our definition. Specifically, the following result is proved in Squier (1994).

Proposition 2.6. *Let $\langle X_1; R_1\rangle$ and $\langle X_2; R_2\rangle$ be two finite group-presentations of the same group. Then $\langle X_1; R_1\rangle$ has finite derivation type if and only if $\langle X_2; R_2\rangle$ has finite derivation type.*

Thus, having finite derivation type is an invariant property of finitely presented groups, that is, we can talk about finitely presented groups that have finite derivation type.

We proceed with a technical result on homotopy relations. For each word $u \in \underline{X}^*$, we fix a path in Γ_F from u to the freely reduced form u_0 of u, where we take the empty path (u) in the case that u is freely reduced. We denote this path by $(u \to u_0)$, and by $(u_0 \to u)$ we denote the inverse path from u_0 back to u. If $u_0 = \lambda$, then we use the abbreviations $(u \to)$ and $(\to u)$ for $(u \to u_0)$ and $(u_0 \to u)$, respectively.

Lemma 2.7. *Let $\langle X; R\rangle$ be a group-presentation, let Γ be the associated graph, and let $\simeq$ be a homotopy relation on $P(\Gamma)$.*

(a) *For all $p, q, r_1, r_2 \in P(\Gamma)$, if $\tau(r_1) = \sigma(p) = \sigma(q)$ and $\tau(p) = \tau(q) = \sigma(r_2)$, then $p \simeq q$ if and only if $r_1 \circ p \circ r_2 \simeq r_1 \circ q \circ r_2$.*
(b) *For all $(p, q) \in P^{(2)}(\Gamma)$ and $x, y \in \underline{X}^*$, $p \simeq q$ if and only if $xpy \simeq xqy$.*
(c) *For all $(p, q) \in P^{(2)}(\Gamma)$ and $u \in \underline{X}^*$ with $\sigma(p) = \sigma(q) = u$, $p \simeq q$ if and only if $(\to uu^{-1}) \circ (p \circ q^{-1})u^{-1} \circ (uu^{-1} \to) \simeq (\lambda)$.*

Proof. (a): Let $p, q, r_1, r_2 \in P(\Gamma)$ such that $\tau(r_1) = \sigma(p) = \sigma(q)$ and $\tau(p) = \tau(q) = \sigma(r_2)$. By the definition of homotopy relations, $p \simeq q$ implies $r_1 \circ p \circ r_2 \simeq r_1 \circ q \circ r_2$. To show the converse assume that $r_1 \circ p \circ r_2 \simeq r_1 \circ q \circ r_2$. Then $p \simeq r_1^{-1} \circ r_1 \circ p \circ r_2 \circ r_2^{-1} \simeq r_1^{-1} \circ r_1 \circ q \circ r_2 \circ r_2^{-1} \simeq q$.

(b): Let $(p, q) \in P^{(2)}(\Gamma)$ and $x, y \in \underline{X}^*$. By the definition of homotopy relations, $p \simeq q$ implies $xpy \simeq xqy$. To show the converse assume that $xpy \simeq xqy$. This implies that $x^{-1}xpyy^{-1} \simeq x^{-1}xqyy^{-1}$. Let $u, v \in \underline{X}^*$ such that p and q are paths from u to v. Then we obtain the following sequence of homotopic paths:

$$\begin{aligned} p &\simeq p \circ (\to x^{-1}x)v \circ x^{-1}xv(\to yy^{-1}) \circ x^{-1}xv(yy^{-1} \to) \circ (x^{-1}x \to)v \\ &\simeq (\to x^{-1}x)u \circ x^{-1}xu(\to yy^{-1}) \circ x^{-1}xpyy^{-1} \circ x^{-1}xv(yy^{-1} \to) \circ (x^{-1}x \to)v \\ &\simeq (\to x^{-1}x)u \circ x^{-1}xu(\to yy^{-1}) \circ x^{-1}xqyy^{-1} \circ x^{-1}xv(yy^{-1} \to) \circ (x^{-1}x \to)v \\ &\simeq q \circ (\to x^{-1}x)v \circ x^{-1}xv(\to yy^{-1}) \circ x^{-1}xv(yy^{-1} \to) \circ (x^{-1}x \to)v \\ &\simeq q. \end{aligned}$$

(c): Let $(p, q) \in P^{(2)}(\Gamma)$ and $u \in \underline{X}^*$ such that $\sigma(p) = \sigma(q) = u$. Using (a) and (b), the following is easily verified:

$$\begin{aligned} p \simeq q \text{ iff } & p \circ q^{-1} \simeq q \circ q^{-1} \\ \text{iff } & p \circ q^{-1} \simeq (u) \qquad (\text{since } q \circ q^{-1} \simeq (u)) \\ \text{iff } & (p \circ q^{-1})u^{-1} \simeq (uu^{-1}) \\ \text{iff } & (\to uu^{-1}) \circ (p \circ q^{-1})u^{-1} \circ (uu^{-1} \to) \simeq (\to uu^{-1}) \circ (uu^{-1} \to) \\ \text{iff } & (\to uu^{-1}) \circ (p \circ q^{-1})u^{-1} \circ (uu^{-1} \to) \simeq (\lambda) \quad (\text{since } (\to uu^{-1}) \circ (uu^{-1} \to) \simeq (\lambda)). \end{aligned}$$

This proves the intended result. □

Part (c) of the lemma above implies that each homotopy relation is already determined by the set of paths which are homotopic to the empty path (λ). In fact, we can prove the following result. Here, $P_1(\Gamma)$ denotes the set of all paths from λ to λ.

Lemma 2.8. *Let $\langle X; R\rangle$ be a group-presentation, and let Γ be the associated graph. Then $\langle X; R\rangle$ has finite derivation type if and only if there is a finite subset $B \subseteq P_1(\Gamma)$ such that, for each path $p \in P_1(\Gamma)$, $p \simeq_{B'} (\lambda)$, where B' denotes the set $\{(p,(\lambda)) \mid p \in B\}$.*

Proof. Suppose that $\langle X; R\rangle$ has finite derivation type. Then there is a finite set $B_0 \subseteq P^{(2)}(\Gamma)$ such that $\simeq_{B_0} = P^{(2)}(\Gamma)$. Let $B = \{(\rightarrow uu^{-1}) \circ (p \circ q^{-1})u^{-1} \circ (uu^{-1} \rightarrow) \mid (p,q) \in B_0$ with $u = \sigma(p) = \sigma(q)\}$, and let $B' = \{(p,(\lambda)) \mid p \in B\}$. Lemma 2.7 (c) implies that, for each homotopy relation $\simeq$, $B_0 \subseteq \simeq$ if and only if $B' \subseteq \simeq$. Thus, $\simeq_{B_0} = \simeq_{B'}$, that is, $\simeq_{B'} = P^{(2)}(\Gamma)$. In particular, for each path $p \in P_1(\Gamma)$, we have $p \simeq_{B'} (\lambda)$.

To prove the converse implication, let $B \subseteq P_1(\Gamma)$ be a finite set, let B' denote the finite set $\{(p,(\lambda)) \mid p \in B\}$, and suppose that, for each path $p \in P_1(\Gamma)$, $p \simeq_{B'} (\lambda)$. Let $(p,q) \in P^{(2)}(\Gamma)$, and let $u = \sigma(p) = \sigma(q)$. By Lemma 2.7 (c), $p \simeq_{B'} q$ if and only if $(\rightarrow uu^{-1}) \circ (p \circ q^{-1})u^{-1} \circ (uu^{-1} \rightarrow) \simeq_{B'} (\lambda)$, which holds by our assumption. Thus, $\simeq_{B'} = P^{(2)}(\Gamma)$. Since B' is a finite set, this implies that $\langle X; R\rangle$ has finite derivation type.
□

Now we are prepared to prove the following characterization of group-presentations which have finite derivation type.

Lemma 2.9. *Let $\langle X; R\rangle$ be a group-presentation, and let Γ be the associated graph. By $\sim$ we denote the homotopy relation on $P(\Gamma)$ that is generated by the empty set. Then $\langle X; R\rangle$ has finite derivation type if and only if there is a finite subset $B \subseteq P_1(\Gamma)$ such that, for each path $p \in P_1(\Gamma)$, $p \sim \prod_{i=1}^{n} q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$ for some $n \geq 0$, $p_i \in B$, $x_i, y_i \in \underline{X}^*$, $\varepsilon_i \in \{1,-1\}$, $q_i \in P(\Gamma)$, where q_i is a path from λ to $x_i y_i$. Here the empty product denotes the empty path (λ).*

Proof. Let $B \subseteq P_1(\Gamma)$, and let $B' = \{(p,(\lambda)) \mid p \in B\}$. We define a relation $\approx_B$ on $P(\Gamma)$ as follows. $p \approx_B q$ if and only if $p \sim q$ or $p \sim q \circ \prod_{i=1}^{n} q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$ for some $n \geq 1$, $p_i \in B$, $x_i, y_i \in \underline{X}^*$, $\varepsilon_i \in \{1,-1\}$, $q_i \in P(\Gamma)$, where q_i is a path from $\tau(q)$ to $x_i y_i$.

Claim. $\approx_B = \simeq_{B'}$.

Proof. First we prove the inclusion $\approx_B \subseteq \simeq_{B'}$. Let $p,q \in P(\Gamma)$ such that $p \approx_B q$. Note that $\sim$ is contained in the homotopy relation $\simeq_{B'}$. Thus, if $p \sim q$, then $p \simeq_{B'} q$. Now assume that $p \sim q \circ \prod_{i=1}^{n} q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$ for some $n \geq 1$, $p_i \in B$, $x_i, y_i \in \underline{X}^*$, $\varepsilon_i \in \{1,-1\}$, $q_i \in P(\Gamma)$, where q_i is a path from $\tau(q)$ to $x_i y_i$. Then $p \simeq_{B'} q \circ \prod_{i=1}^{n} q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$, and by induction on n it is easily proved that $q \simeq_{B'} q \circ \prod_{i=1}^{n} q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$. Thus, $p \simeq_{B'} q$. Hence, $\approx_B \subseteq \simeq_{B'}$. To prove the converse inclusion, we show that $\approx_B$ is a homotopy relation on $P(\Gamma)$ that contains B'. It is easy to check that the relation $\approx_B$ is reflexive, symmetric and transitive. Thus, $\approx_B$ is an equivalence relation. Obviously, $\approx_B$ is contained in $P^{(2)}(\Gamma)$. To show that $\approx_B$ is a homotopy relation, we have to check conditions (a) to (d) of Definition 2.3. Since $\approx_B$ contains the homotopy relation $\sim$, $\approx_B$ satisfies (a) and (d). To prove (b), let $p,q \in P(\Gamma)$ and $x,y \in \underline{X}^*$. Suppose that $p \approx_B q$. If $p \sim q$, then $xpy \sim xqy$ implying that $xpy \approx_B xqy$. If $p \sim q \circ \prod_{i=1}^{n} q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$ for some $n \geq 1$, $p_i \in B$, $x_i, y_i \in \underline{X}^*$, $\varepsilon_i \in \{1,-1\}$, $q_i \in P(\Gamma)$, where q_i is a path from $\tau(q)$ to $x_i y_i$, then $xpy \sim xqy \circ \prod_{i=1}^n xq_i y \circ (xx_i p_i y_i y)^{\varepsilon_i} \circ xq_i^{-1} y$ implying that $xpy \approx_B xqy$. Hence, $\approx_B$ satisfies condition (b). To prove (c), let $p, q, r_1, r_2 \in P(\Gamma)$ with $\tau(r_1) = \sigma(p) = \sigma(q)$, $\tau(p) = \tau(q) = \sigma(r_2)$, and $p \approx_B q$. If $p \sim q$, then $r_1 \circ p \circ r_2 \sim r_1 \circ q \circ r_2$ implying that $r_1 \circ p \circ r_2 \approx_B r_1 \circ q \circ r_2$. If $p \sim q \circ \prod_{i=1}^n q_i \circ (x_i p_i y_i)^{\varepsilon_i} \circ q_i^{-1}$ for some $n \geq 1$, $p_i \in B$, $x_i, y_i \in \underline{X}^*$, $\varepsilon_i \in \{1,-1\}$, $q_i \in P(\Gamma)$, where q_i is a path from $\tau(q)$ to $x_i y_i$, then $r_1 \circ p \circ r_2 \sim r_1 \circ q \circ r_2 \circ \prod_{i=1}^n (r_2^{-1} \circ q_i) \circ (x_i p_i y_i)^{\varepsilon_i} \circ (q_i^{-1} \circ r_2)$ implying that $r_1 \circ p \circ r_2 \approx_B r_1 \circ q \circ r_2$. Hence, $\approx_B$ satisfies condition (c). Thus, $\approx_B$ is a homotopy relation. Obviously $\approx_B$ contains B'. Hence, $\simeq_{B'} \subseteq \approx_B$, and therewith $\approx_B = \simeq_{B'}$. □

This claim together with Lemma 2.8 implies the result. □

3. The Module of Identities Among Relations

With a group-presentation $\langle X; R\rangle$ of a group G we associate the $\mathbb{Z}G$-module π of identities among relations, where $\mathbb{Z}G$ denotes the integral group ring of G. A description of this module, which was introduced by Peiffer (1949), can also be found in Brown and Huebschmann (1982).

In (Cremanns and Otto, 1994) we stated the result that $\langle X; R\rangle$ has finite derivation type if and only if the $\mathbb{Z}G$-module π is finitely generated. The proof for this result, of which only an outline is given in (Cremanns and Otto, 1994), was technically rather involved. Here we present a new proof, that is much simpler, but which still uses fairly elementary methods.

We begin by restating a few basic definitions from module theory. The integral group ring $\mathbb{Z}G$ of G can be described as follows. Let $\mathbb{Z}G$ denote the set of all mappings $f : G \to \mathbb{Z}$ for which the set $\{g \in G \mid f(g) \neq 0\}$ is finite. An element $f \in \mathbb{Z}G$ will be written as $f = \sum_{g\in G} f(g)\cdot g$, that is, each $f \in \mathbb{Z}G$ is expressed formally as a polynomial $\sum_{g\in G} z_g \cdot g$, where $\{g \in G \mid z_g \neq 0\}$ is a finite set. On $\mathbb{Z}G$ the operations of *addition* and *multiplication* are defined as follows: if $f = \sum_{g\in G} z_g \cdot g$ and $f' = \sum_{g\in G} z'_g \cdot g$, then

$$f + f' := \sum_{g\in G} (z_g + z'_g)\cdot g \qquad \text{and} \qquad f \cdot f' := \sum_{g\in G} \left(\sum_{\substack{g_1, g_2 \in G \\ g_1 \cdot g_2 = g}} z_{g_1} \cdot z'_{g_2} \right) \cdot g.$$

It is easily verified that $\mathbb{Z}G$ is a ring with identity. It is called the *integral group ring* of G.

We proceed with some remarks on normal subgroups and congruences on groups. Let H be a group. An equivalence relation $\approx$ on H is called a *congruence* if, for all $x, y, x', y' \in H$, $x \approx y$ and $x' \approx y'$ imply that $xx' \approx yy'$. Each set of equations $B \subseteq H \times H$ generates a congruence on H, which is denoted by $=_B$, and which is defined as the smallest congruence on H containing B. It is well-known that there is a one-to-one correspondence between the congruences on H and the normal subgroups of H. If $\approx$ is a congruence on H, then $[1]_\approx$ is the corresponding normal subgroup of H.

For the following considerations let $\langle X; R\rangle$ be a fixed group-presentation, let F be the free group generated by X, and let N be the normal subgroup of F that is generated by R. Then G is the factor group F/N. To define the $\mathbb{Z}G$-module π of identities among relations we need some preparations.

With the group-presentation $\langle X; R\rangle$ we associate a new infinite *alphabet* Y that is to be in one-to-one correspondence to $F \times R$. To express this correspondence we use the notation $Y = F \times R$. Accordingly, elements of Y will be denoted as pairs (u, r), where $u \in F$ and $r \in R$, and elements of $\underline{Y}$ will be denoted as pairs $(u, r)^{\varepsilon}$, where $u \in F$, $r \in R$ and $\varepsilon \in \{1, -1\}$. We define a monoid-homomorphism $\theta : \underline{Y}^* \to F$ through

$$\theta((u, r)^{\varepsilon}) := ur^{\varepsilon}u^{-1}.$$

Also we define a *left action* of F on $\underline{Y}^*$ through

$${}^{x}((u, r)^{\varepsilon}) := (xu, r)^{\varepsilon}.$$

It is easily verified that, for all $y \in \underline{Y}^*$ and $x \in F$, $\theta({}^{x}y) = x\theta(y)x^{-1}$.

Let $\hat{N}$ be the free group that is generated by Y. It is easily seen that the function $\theta : \underline{Y}^* \to F$ induces a homomorphism $\theta : \hat{N} \to F$, and that the action of F on $\underline{Y}^*$ induces an action of F on $\hat{N}$. Note that the image of θ is N. Let $E \subseteq \hat{N}$ be the kernel of the homomorphism $\theta : \hat{N} \to F$. Then N is isomorphic to the factor group $\hat{N}/E$. The elements of E are called *identities*.

Next we define two sets D_Y and I_Y of pairs of words from $\underline{Y}^*$:

$$D_Y := \{(uv, {}^{\theta(u)}vu) \mid u, v \in \underline{Y}^*\},$$
$$I_Y := \{(uu^{-1}, \lambda) \mid u \in \underline{Y}^*\}.$$

We will show that the sets D_Y and I_Y are closely related to the sets of pairs of paths D_{Γ} and I_{Γ}, where Γ is the graph associated with the group-presentation $\langle X; R\rangle$. The free group $\hat{N}$ generated by Y is the factor monoid $\underline{Y}^*/{=_{I_Y}}$. Let $\sim\, := \,=_{D_Y \cup I_Y}$ be the congruence on $\underline{Y}^*$ that is generated by $D_Y \cup I_Y$, and let $C = \underline{Y}^*/{\sim}$ be the corresponding factor monoid. Obviously, C is a group. By $\sim$ we also denote the congruence on $\hat{N}$ that is generated by D_Y. Thus, C can also be described as the factor group $\hat{N}/{\sim}$. Finally, let D be the normal subgroup of $\hat{N}$ that corresponds to the congruence $\sim$ on $\hat{N}$. The elements of D are called *Peiffer identities*.

Since for all $u, v \in \underline{Y}^*$,

$$\theta({}^{\theta(u)}vu) = \theta({}^{\theta(u)}v)\theta(u) = \theta(u)\theta(v)(\theta(u))^{-1}\theta(u) = \theta(u)\theta(v) = \theta(uv),$$

θ induces a group-homomorphism $\theta : C \to F$.

Further, for all $u, v \in \underline{Y}^*$ and $x \in F$,

$${}^{x}(uv) = {}^{x}u\,{}^{x}v \sim {}^{\theta({}^{x}u)}({}^{x}v)\,{}^{x}u = {}^{x\theta(u)x^{-1}}({}^{x}v)({}^{x}u) = ({}^{x\theta(u)}v)({}^{x}u) = {}^{x}({}^{\theta(u)}vu).$$

Thus, the action of F on $\hat{N}$ induces an action of F on C. The ordered pair (C, θ) together with the action of F on C is called the *free F-crossed module that is generated by R.*

Lemma 3.1. *The congruence $\sim$ on $\hat{N}$ has the following properties:*

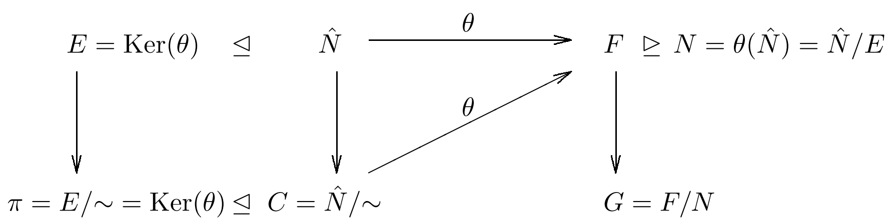
(a) *$uv \sim vu$ for all $u \in E, v \in \hat{N}$.*
(b) *${}^{x}u \sim u$ for all $u \in E$, $x \in N$.*

Proof. Part (a) is immediate from the definition of the set D_Y. To prove (b), let $u \in E$ and $x \in N$. Since $N = \theta(\hat{N})$, there exists an element $v \in \hat{N}$ such that $\theta(v) = x$. Hence, in $\hat{N}$ we have ${}^{x}u = {}^{\theta(v)}u = v \cdot v^{-1} \cdot {}^{\theta(v)}u \sim v \cdot {}^{\theta(v^{-1})\theta(v)}u \cdot v^{-1} = vuv^{-1} \sim u$ (by (a)). This proves part (b). □

Now we define the announced $\mathbb{Z}G$-module π as the kernel of the homomorphism $\theta : C \to F$. It remains to verify that π does indeed have the structure of a $\mathbb{Z}G$-module.

The natural homomorphism from $\hat{N}$ onto $\hat{N}/D$ maps the kernel E of the homomorphism $\theta : \hat{N} \to F$ onto π, that is $\pi = E/D$. Hence, Lemma 3.1 (a) implies that π is commutative, that is, π is an abelian group. In fact, π is contained in the center of C.

By Lemma 3.1 (b), N acts trivially on π. Therefore, the action of F on π induces an action of G on π. It is straightforward to extend this action to an action of $\mathbb{Z}G$ on π. In this way π obtains the structure of a left module over $\mathbb{Z}G$. It is called the *module of identities among relations.*



In the rest of this section we will show that the group-presentation $\langle X; R\rangle$ has finite derivation type if and only if the $\mathbb{Z}G$-module π of identities among relations is finitely generated. Let Γ be the graph that is associated with $\langle X; R\rangle$, and let $\sim$ denote the homotopy relation on $P(\Gamma)$ that is generated by the empty set. Note that we use the symbol $\sim$ also to denote the congruence on $\underline{Y}^*$ that is generated by $D_Y \cup I_Y$ and to denote the congruence on $\hat{N}$ that is generated by D_Y. However, this should not cause any problems, since it will always be apparent from the context, which of these relations is being meant.

Let $f : P(\Gamma) \to \underline{Y}^*$ be the function that associates a word from $\underline{Y}^*$ with each path in Γ, and that is defined inductively as follows:

$$f(p) := \lambda \qquad \text{if } p \text{ is a path of length } 0$$

$$f(p \circ e) := \begin{cases} f(p) & \text{if } p \in P(\Gamma) \text{ and } e \text{ is an edge in } \Gamma_F \\ f(p)(u,r)^{\varepsilon} & \text{if } p \in P(\Gamma) \text{ and } e = (u,(r,\lambda),v,\varepsilon) \text{ is an edge such that } r \in R. \end{cases}$$

For all paths p_1 and p_2 in Γ, we have $f(p_1^{-1}) = (f(p_1))^{-1}$, and if $\tau(p_1) = \sigma(p_2)$, then $f(p_1 \circ p_2) = f(p_1)f(p_2)$. For all paths p in Γ and $x, y \in \underline{X}^*$, $f(xpy) = {}^{x}f(p)$.

The function f can be extended to pairs of paths and to sets of pairs of paths. For example, if $B \subseteq P^{(2)}(\Gamma)$, then $f(B)$ denotes the set $\{(f(p), f(q)) \mid (p,q) \in B\}$.

Notice that $f(p) = \lambda$ for all $p \in P(\Gamma_F)$. Thus, for all $p \in P(\Gamma_F)$, $\theta(f(p)) =_F \lambda$. Moreover, using induction on the length of the path involved the following technical result can be verified easily.

LEMMA 3.2. *For all* $p \in P(\Gamma)$, $\sigma(p) =_F \theta(f(p))\tau(p)$.

Recall from Section 2, that, for each word $u \in \underline{X}^*$, we have fixed a path $(u \to u_0)$ in Γ_F from u to the freely reduced form u_0 of u, and by $(u_0 \to u)$ we denote the inverse path. Next we define a function $\delta : \underline{Y} \to P(\Gamma)$ as follows. For each $r \in R$ and each freely reduced word $u \in \underline{X}^*$, we take

$$\delta((u,r)) := (u,(r,\lambda),u^{-1},1) \circ (uu^{-1} \to),$$
$$\delta((u,r)^{-1}) := (u,(r,\lambda),r^{-1}u^{-1},-1) \circ (urr^{-1}u^{-1} \to).$$

Observe that $f(\delta((u,r)^\varepsilon)) = (u,r)^\varepsilon$, $\sigma(\delta((u,r)^\varepsilon)) =_F \theta((u,r)^\varepsilon)$, and $\tau(\delta((u,r)^\varepsilon)) = \lambda$ hold for each $r \in R$, each freely reduced word $u \in \underline{X}^*$, and each $\varepsilon \in \{1,-1\}$.

LEMMA 3.3. *If $w \in \underline{Y}^*$ and $x, y \in \underline{X}^*$ satisfy $x =_F \theta(w)y$, then there is a path p in Γ from x to y such that $f(p) = w$.*

PROOF. We proceed by induction on $|w|$. If $w = \lambda$, then $x =_F y$. Hence, there is a path p in Γ_F from x to y, and from the definition of f we obtain $f(p) = \lambda$. If $|w| > 0$, then there are $w' \in \underline{Y}^*$ and $(u,r)^\varepsilon \in \underline{Y}$ such that $w = (u,r)^\varepsilon w'$. Let $x' \in \underline{X}^*$ be the freely reduced word satisfying $x' =_F \theta(w')y$. By the induction hypothesis, there is a path p' from x' to y such that $f(p') = w'$. We take q to be the path $q = \delta((u,r)^\varepsilon)x'$. Then $\tau(q) = x'$, $f(q) = (u,r)^\varepsilon$, and $\sigma(q) =_F \theta((u,r)^\varepsilon)x' =_F \theta((u,r)^\varepsilon)\theta(w')y =_F \theta(w)y =_F x$, implying that there is a path q' in Γ_F from x to $\sigma(q)$. Hence, $p := q' \circ q \circ p'$ is a path from x to y such that $f(p) = (u,r)^\varepsilon w' = w$. □

Based on this technical lemma we can now establish the following close correspondence between the relation $\sim$ on $P(\Gamma)$ and the relation $\sim$ on $\underline{Y}^*$.

LEMMA 3.4. *For all $(p,q) \in P^{(2)}(\Gamma)$, $p \sim q$ if and only if $f(p) \sim f(q)$.*

PROOF. To prove this lemma, we need several auxiliary claims.

CLAIM 1. $f(D_\Gamma) = D_Y$, and $f(I_\Gamma) = I_Y$.

PROOF. It is easily seen that $f(I_\Gamma) = I_Y$ and $f(D_\Gamma) \subseteq D_Y$. Thus, it remains to prove $D_Y \subseteq f(D_\Gamma)$. Let $u, v \in \underline{Y}^*$. We have to verify that $(uv, {}^{\theta(u)}vu) \in f(D_\Gamma)$. By Lemma 3.3 there exist paths p and q in Γ such that $f(p) = u$ and $f(q) = {}^{\tau(p)^{-1}}v$. Then $(p\sigma(q) \circ \tau(p)q, \sigma(p)q \circ p\tau(q)) \in D_\Gamma$, and therewith, $(u\,{}^{\tau(p)\tau(p)^{-1}}v, {}^{\sigma(p)\tau(p)^{-1}}vu) \in f(D_\Gamma)$. Since $\tau(p)\tau(p)^{-1} =_F \lambda$, and $\sigma(p)\tau(p)^{-1} =_F \theta(u)$ by Lemma 3.2, we see that $u\,{}^{\tau(p)\tau(p)^{-1}}v = uv$ and ${}^{\sigma(p)\tau(p)^{-1}}vu = {}^{\theta(u)}vu$, that is, $(uv, {}^{\theta(u)}vu) \in f(D_\Gamma)$. □

CLAIM 2. Let $(u,r)^\varepsilon \in \underline{Y}$, and let p be a path in Γ such that $f(p) = (u,r)^\varepsilon$. Then there exists a path p' in Γ_F such that $p \sim p' \circ \delta((u,r)^\varepsilon)\tau(p)$.

PROOF. First we consider the case $\varepsilon = 1$. So let $r \in R$, let $u \in \underline{X}^*$ be a freely reduced word, and let p be a path in Γ such that $f(p) = (u,r)$. From the definition of the function f we conclude that there are paths p_1, p_2 in Γ_F and an edge $e = (x,(r,\lambda),y,1)$ such that $p = p_1 \circ e \circ p_2$ and $x =_F u$. Let e' denote the edge $e' = (u,(r,\lambda),y,1)$. Then $p = p_1 \circ e \circ p_2 \sim p_1 \circ e \circ (x \to u)y \circ (u \to x)y \circ p_2 \sim p_1 \circ (x \to u)ry \circ e' \circ (u \to x)y \circ p_2$, and

$$\begin{aligned} e' &= (u,(r,\lambda),y,1) \\ &\sim (u,(r,\lambda),y,1) \circ u(\to u^{-1}u)y \circ u(u^{-1}u \to)y \\ &\sim ur(\to u^{-1}u)y \circ (u,(r,\lambda),u^{-1}uy,1) \circ u(u^{-1}u \to)y \\ &\sim ur(\to u^{-1}u)y \circ (u,(r,\lambda),u^{-1}uy,1) \circ (uu^{-1} \to)uy \\ &= ur(\to u^{-1}u)y \circ \delta((u,r))uy. \end{aligned}$$

Thus, we get

$$\begin{aligned} p &\sim p_1 \circ (x \to u)ry \circ e' \circ (u \to x)y \circ p_2 \\ &\sim p_1 \circ (x \to u)ry \circ ur(\to u^{-1}u)y \circ \delta((u,r))uy \circ (u \to x)y \circ p_2 \\ &\sim p_1 \circ (x \to u)ry \circ ur(\to u^{-1}u)y \circ uru^{-1}(u \to x)y \circ uru^{-1}p_2 \circ \delta((u,r))\tau(p). \end{aligned}$$

Hence, if p' denotes the path $p' := p_1 \circ (x \to u)ry \circ ur(\to u^{-1}u)y \circ uru^{-1}(u \to x)y \circ uru^{-1}p_2$, then p' is a path in Γ_F that satisfies $p \sim p' \circ \delta((u,r)^\varepsilon)\tau(p)$.

The case $\varepsilon = -1$ is dealt with analogously. □

Using this claim, the following can now be proved.

Claim 3. For all $(p,q) \in P^{(2)}(\Gamma)$, if $f(p) = f(q)$, then $p \sim q$.

Proof. The proof is by induction on the length of the word $f(p)$. If $f(p) = f(q) = \lambda$, then $(p,q) \in P^{(2)}(\Gamma_F)$ implying that $p \sim q$.

Next suppose that $f(p) = f(q)$ consists of a single letter $(u,r)^\varepsilon \in \underline{Y}$. Then by Claim 2, there exist paths p' and q' in Γ_F such that $p \sim p' \circ \delta((u,r)^\varepsilon)\tau(p)$ and $q \sim q' \circ \delta((u,r)^\varepsilon)\tau(q)$. Since $\tau(p) = \tau(q)$, we have $(p',q') \in P^{(2)}(\Gamma_F)$ and so $p' \sim q'$. Thus,

$$p \sim p' \circ \delta((u,r)^\varepsilon)\tau(p) \sim q' \circ \delta((u,r)^\varepsilon)\tau(q) \sim q.$$

Finally suppose that the word $f(p) = f(q)$ consists of more than one letter. Then p can be written as $p = p_1 \circ p_2$ and q can be written as $q = q_1 \circ q_2$ such that $f(p_1) = f(q_1)$, $f(p_2) = f(q_2)$, $|f(p_1)| < |f(p)|$, and $|f(p_2)| < |f(p)|$. By Lemma 3.2, $\tau(p_1) =_F \tau(q_1)$, implying that there is a path $h \in P(\Gamma_F)$ leading from $\tau(p_1)$ to $\tau(q_1)$. By the induction hypothesis, we have $p_1 \sim q_1 \circ h^{-1}$ and $p_2 \sim h \circ q_2$. Thus,

$$p = p_1 \circ p_2 \sim q_1 \circ h^{-1} \circ h \circ q_2 \sim q_1 \circ q_2 = q.$$

This proves Claim 3. □

The sets D_Y and I_Y are subsets of $\underline{Y}^*$. So $D_Y \cup I_Y$ can be considered as a string-rewriting system on $\underline{Y}$ that generates the reduction relation $\to_{D_Y \cup I_Y}$. By $\leftrightarrow_{D_Y \cup I_Y}$ we denote the symmetric closure of $\to_{D_Y \cup I_Y}$. Note that the reflexive, transitive closure of $\leftrightarrow_{D_Y \cup I_Y}$ is just the congruence $\sim\; = \;=_{D_Y \cup I_Y}$ on $\underline{Y}^*$.

Claim 4. Let $u, v \in \underline{Y}^*$ and $x, y \in \underline{X}^*$ such that $u \leftrightarrow_{D_Y \cup I_Y} v$ and $x =_F \theta(u)y$. Then there are paths p and q from x to y such that $f(p) = u$, $f(q) = v$, and $p \sim q$.

Proof. Since $u \leftrightarrow_{D_Y \cup I_Y} v$, there is some pair $(u_0, v_0) \in D_Y \cup I_Y$ and $z_1, z_2 \in \underline{Y}^*$ such that $u = z_1 u_0 z_2$ and $v = z_1 v_0 z_2$. By Claim 1, there is a pair $(p_0, q_0) \in D_\Gamma \cup I_\Gamma$ such that $f(p_0) = u_0$ and $f(q_0) = v_0$. Trivially,

$$\tau(p_0)\tau(p_0)^{-1}\theta(z_2)y =_F \theta(z_2)y.$$

Thus by Lemma 3.3, there is a path p_2 from $\tau(p_0)\tau(p_0)^{-1}\theta(z_2)y$ to y such that $f(p_2) = z_2$. By Lemma 3.2, $\sigma(p_0)\tau(p_0)^{-1} =_F \theta(u_0)$. Thus,

$$x =_F \theta(u)y =_F \theta(z_1)\theta(u_0)\theta(z_2)y =_F \theta(z_1)\sigma(p_0)\tau(p_0)^{-1}\theta(z_2)y.$$

Hence by Lemma 3.3, there is a path p_1 leading from x to $\sigma(p_0)\tau(p_0)^{-1}\theta(z_2)y$ such that $f(p_1) = z_1$. Now let $p = p_1 \circ p_0\tau(p_0)^{-1}\theta(z_2)y \circ p_2$ and $q = p_1 \circ q_0\tau(p_0)^{-1}\theta(z_2)y \circ p_2$. We have $f(p) = f(p_1)f(p_0)f(p_2) = z_1u_0z_2 = u$ and $f(q) = f(p_1)f(q_0)f(p_2) = z_1v_0z_2 = v$. Since $(p_0, q_0) \in D_\Gamma \cup I_\Gamma$, we have $p \sim q$. $\Box$

Now we complete the proof of Lemma 3.4 as follows. Let $(p, q) \in P^{(2)}(\Gamma)$ such that $p \sim q$. By Claim 1, $f(D_\Gamma) \subseteq D_Y$ and $f(I_\Gamma) \subseteq I_Y$. Since $\sim$ is the homotopy relation that is generated by the empty set of pairs of paths, $f(p) \sim f(q)$ can now be derived easily using the characterization of homotopy relations that is given after Definition 2.3. To show the converse, let $(p, q) \in P^{(2)}(\Gamma)$ such that $f(p) \sim f(q)$. If $f(p) = f(q)$, then $p \sim q$ by Claim 3. So suppose that $f(p) \neq f(q)$. Since $\sim$ is the reflexive, transitive closure of $\leftrightarrow_{D_Y \cup I_Y}$, it follows that there are words $u_0, u_1, \ldots, u_n \in \underline{Y}^*$ for some $n \geq 1$ such that $f(p) = u_0 \leftrightarrow_{D_Y \cup I_Y} u_1 \leftrightarrow_{D_Y \cup I_Y} \cdots \leftrightarrow_{D_Y \cup I_Y} u_n = f(q)$. By Claim 4, it follows that, for each $1 \leq i \leq n$, there is a pair of paths (q_{i-1}, p_i) such that $\sigma(q_{i-1}) = \sigma(p_i) = \sigma(p)$ and $\tau(q_{i-1}) = \tau(p_i) = \tau(p)$, $f(q_{i-1}) = u_{i-1}$ and $f(p_i) = u_i$, and $q_{i-1} \sim p_i$. We have $f(p) = f(q_0)$, for each $1 \leq i < n$, $f(p_i) = f(q_i)$, and $f(p_n) = f(q)$. Thus by Claim 3, $p \sim q_0$, for each $1 \leq i < n$, $p_i \sim q_i$, and $p_n \sim q$. Hence $p \sim q$. $\Box$

The only-if part of the above lemma implies that the function $f : P(\Gamma) \to \underline{Y}^*$ induces a function $f : P_1(\Gamma)/\!\sim \to \pi$. Recall that $P_1(\Gamma)$ denotes the set of paths from λ to λ. Obviously, $P_1(\Gamma)/\!\sim$ is a group, and it is easy to check that the function $f : P_1(\Gamma)/\!\sim \to \pi$ is a group-homomorphism. By the if-part of the above lemma, this homomorphism is injective, and by Lemma 3.3, this homomorphism is surjective. Thus, $f : P_1(\Gamma)/\!\sim \to \pi$ is a group-isomorphism.

By Lemma 2.9, a group-presentation $\langle X; R\rangle$ has finite derivation type if and only if the following condition is satisfied:

(1) There is a finite subset $B \subseteq P_1(\Gamma)/\!\sim$ such that, for each $p \in P_1(\Gamma)/\!\sim$,

$$p = \prod_{i=1}^{n} q_i \circ (x_ip_iy_i)^{\varepsilon_i} \circ q_i^{-1}$$

for some $n \geq 0$, $p_i \in B$, $x_i, y_i \in \underline{X}^*$, $\varepsilon_i \in \{1, -1\}$, $q_i \in P(\Gamma)$, where q_i is a path from λ to x_iy_i.

For $1 \leq i \leq n$, $f(p_i) \in \pi$ implies that ${}^{x_i}(f(p_i)^{\varepsilon_i}) \in \pi$. Thus, since π is contained in the center of C, the following equations hold in C:

$$f(q_i \circ (x_ip_iy_i)^{\varepsilon_i} \circ q_i^{-1}) = f(q_i)\,{}^{x_i}(f(p_i)^{\varepsilon_i})f(q_i)^{-1} = {}^{x_i}(f(p_i)^{\varepsilon_i})f(q_i)f(q_i)^{-1} = {}^{x_i}(f(p_i)^{\varepsilon_i}).$$

Thus, Condition (1) implies the following condition.

(2) There is a finite subset $B \subseteq \pi$ such that, for each $u \in \pi$,

$$u = \prod_{i=1}^{n} {}^{z_i}(u_i^{\varepsilon_i})$$

for some $n \geq 0$, $u_i \in B$, $z_i \in F$, $\varepsilon_i \in \{1, -1\}$.

For each $1 \leq i \leq n$, if p_i is a path in $P_1(\Gamma)$ such that $f(p_i) = u_i$, then f maps the path $(\to z_iz_i^{-1}) \circ (z_ip_iz_i^{-1})^{\varepsilon_i} \circ (z_iz_i^{-1} \to)$ to ${}^{z_i}(u_i^{\varepsilon_i})$. Hence, the Conditions (1) and (2) are in fact equivalent. Obviously, Condition (2) holds if and only if the $\mathbb{Z}G$-module π is finitely generated. Thus, we have the following result.

THEOREM 3.5. (CREMANNS AND OTTO, 1994) *The group-presentation $\langle X; R\rangle$ has finite derivation type if and only if the $\mathbb{Z}G$-module π is finitely generated.*

Peiffer (1949) was interested in the existence of *finite defining systems* for these modules. In our notation the finite defining systems of Peiffer are just the finite generating sets. Peiffer proved that the property of admitting a finite set of generators for the module π is an invariant property of finitely presented groups. This also follows from Proposition 2.6 and Theorem 3.5.

4. The Homological Finiteness Condition FP_3

In this section we prove the main result of this paper which states that a finitely presented group has finite derivation type if and only if it is of type FP_3. For that we need some basic definitions from homology theory that we restate in short.

Let R be a ring, and let M be a (left) R-module. A *resolution* of M over R is a sequence $F_0, F_1, F_2, \ldots$ of R-modules together with a sequence $\delta_0 : F_0 \to M$, $\delta_i : F_i \to F_{i-1}$, $i = 1, 2, \ldots$, of R-module-homomorphisms such that the sequence

$$\cdots \longrightarrow F_2 \longrightarrow_{\delta_2} F_1 \longrightarrow_{\delta_1} F_0 \longrightarrow_{\delta_0} M \longrightarrow 0.$$

is *exact*, that is, the following conditions are satisfied:

(1) $\delta_i(F_i) = ker(\delta_{i-1})$ for all $i = 1, 2, \ldots$, where $ker(\delta_{i-1}) = \{x \in F_{i-1} \mid \delta_{i-1}(x) = 0\}$ is the *kernel* of δ_{i-1}, and
(2) δ_0 is surjective, that is, $\delta_0(F_0) = M$.

If each F_i is a free R-module of finite rank, then this resolution is called *free-of-finite-rank*.

Let G be a group, and let $\mathbb{Z}G$ be the integral group ring of G. The ring $\mathbb{Z}$ itself can be interpreted as a $\mathbb{Z}G$-module by simply defining $(\sum_{g\in G} z_g \cdot g) \cdot z := \sum_{g\in G} z_g \cdot z$ for all $\sum_{g\in G} z_g \cdot g \in \mathbb{Z}G$ and $z \in \mathbb{Z}$.

The group G is said to be of type FP_∞, if $\mathbb{Z}$ has a resolution over $\mathbb{Z}G$ which is free-of-finite-rank. The group G is said to be of type FP_n for some integer $n \geq 0$, if $\mathbb{Z}$ has a partial resolution

$$F_n \longrightarrow_{\delta_n} F_{n-1} \longrightarrow_{\delta_{n-1}} F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow_{\delta_1} F_0 \longrightarrow_{\delta_0} \mathbb{Z} \longrightarrow 0$$

over $\mathbb{Z}G$ which is free-of-finite-rank. Regarding these notions, the following results have been obtained.

PROPOSITION 4.1. *Let G be a finitely presented group.*

(a) *If G has a finite convergent presentation, then G is of type FP_3* (Squier, 1987).
(b) *If G has a finite convergent presentation, then G is of type FP_∞* (Kobayashi, 1990).

The above definitions can be generalized to monoids, and Proposition 4.1 holds for monoids in general. For a short introduction to the homology of monoids and a nice presentation of part (a) of Proposition 4.1 see Lafont and Prouté (1991). Several different proofs have been obtained for part (b) of Proposition 4.1, see Cohen (1993) for an overview.

The following result will be useful. It can be found in many text books on homology of groups, see, e.g., Brown (1982).

Proposition 4.2. *For every group G and every integer $n \geq 0$, the following conditions are equivalent:*

(i) *G is of type FP_n.*
(ii) *G is finitely generated, and for every partial free-of-finite-rank resolution*

$$F_k \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

with $k < n$, the kernel of the homomorphism $F_k \to F_{k-1}$ is finitely generated.

It follows that a group is of type FP_∞ if and only if it is of type FP_n for all integers $n \geq 0$.

With a finite group-presentation $\langle X; R\rangle$ of a group G, there is associated a standard partial free-of-finite-rank resolution

$$C_2 \longrightarrow_{\delta_2} C_1 \longrightarrow_{\delta_1} C_0 \longrightarrow_{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ over $\mathbb{Z}G$, see for example Lyndon and Schupp (1977). In the following we explain this resolution which we need to prove our result.

Let C_0 be the free $\mathbb{Z}G$-module generated by the single element $[\emptyset]$ which is isomorphic to $\mathbb{Z}G$.

For each $x \in X$, let $[x]$ denote a new symbol. Then we define C_1 to be the free $\mathbb{Z}G$-module generated by the set $\{[x] \mid x \in X\}$, that is,

$$C_1 = \Big\{\sum_{x\in X} h_x[x] \mid h_x \in \mathbb{Z}G\Big\} \cong \bigoplus_{x\in X} \mathbb{Z}G.$$

Here $\bigoplus_{x\in X} \mathbb{Z}G$ denotes the direct sum of $|X|$ copies of the $\mathbb{Z}G$-module $\mathbb{Z}G$.

For each $r \in R$, let $[r]$ denote another new symbol. The $\mathbb{Z}G$-module C_2 is to be the free $\mathbb{Z}G$-module generated by the set $\{[r] \mid r \in R\}$, that is,

$$C_2 = \Big\{\sum_{r\in R} h_r[r] \mid h_r \in \mathbb{Z}G\Big\} \cong \bigoplus_{r\in R} \mathbb{Z}G.$$

Next we define the $\mathbb{Z}G$-module-homomorphism $\varepsilon : C_0 \to \mathbb{Z}$. Since C_0 is the free $\mathbb{Z}G$-module that is generated by the single element $[\emptyset]$, it suffices to give the image of $[\emptyset]$ under ε:

$$\varepsilon([\emptyset]) := 1.$$

The homomorphism $\delta_1 : C_1 \to C_0$ is defined by setting

$$\delta_1([x]) := (x-1)[\emptyset].$$

In the expression $(x-1)$, the letter x denotes the element of G that is presented by x, and 1 denotes the identity element of the group G.

To define $\delta_2 : C_2 \to C_1$ we need an auxiliary function $\gamma_1 : \underline{X}^* \to C_1$, which we define as follows:

$$\begin{aligned} \gamma_1(\lambda) &:= 0 \\ \gamma_1(ux) &:= \gamma_1(u) + u[x] \qquad \text{for all } u \in \underline{X}^* \text{ and } x \in X \\ \gamma_1(u\bar{x}) &:= \gamma_1(u) - u\bar{x}[x] \qquad \text{for all } u \in \underline{X}^* \text{ and } x \in X. \end{aligned}$$

Now the $\mathbb{Z}G$-module-homomorphism $\delta_2 : C_2 \to C_1$ is defined by

$$\delta_2([r]) := \gamma_1(r) \qquad (r \in R).$$

It is well-known that the sequence

$$C_2 \longrightarrow_{\delta_2} C_1 \longrightarrow_{\delta_1} C_0 \longrightarrow_{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a partial resolution of $\mathbb{Z}$ (see, e.g., Lyndon and Schupp, 1977, a detailed proof can be found in Cremanns, 1995). This simply mirrors the well-known fact that each finitely presented group is of type FP_2.

The partial resolution

$$C_2 \longrightarrow_{\delta_2} C_1 \longrightarrow_{\delta_1} C_0 \longrightarrow_{\varepsilon} \mathbb{Z} \longrightarrow 0$$

can be extended one step further to a partial resolution

$$C_3 \longrightarrow_{\delta_3} C_2 \longrightarrow_{\delta_2} C_1 \longrightarrow_{\delta_1} C_0 \longrightarrow_{\varepsilon} \mathbb{Z} \longrightarrow 0$$

such that C_3 is again a finitely generated free $\mathbb{Z}G$-module if and only if the kernel of the homomorphism $\delta_2 : C_2 \to C_1$ is finitely generated. Thus, if $ker(\delta_2)$ is finitely generated, then G is of type FP_3. Since by Proposition 4.2 also the converse implication holds, we obtain the following equivalence.

Lemma 4.3. *The finitely presented group G is of type FP_3 if and only if the kernel of the homomorphism $\delta_2 : C_2 \to C_1$ is finitely generated as a $\mathbb{Z}G$-module.*

Thus, we have to investigate the kernel of the homomorphism $\delta_2 : C_2 \to C_1$. We will need the following properties of the function γ_1.

Lemma 4.4. (a) *For all $u, v \in \underline{X}^*$, $\gamma_1(uv) = \gamma_1(u) + u\gamma_1(v)$.*
(b) *For all $u, v \in \underline{X}^*$, if $u =_F v$, then $\gamma_1(u) = \gamma_1(v)$.*
(c) *For all $u \in \underline{X}^*$, $\gamma_1(u^{-1}) = -u^{-1}\gamma_1(u)$.*

Proof. Part (a) is immediate from the definition of γ_1. To prove part (b), let $u, v \in \underline{X}^*$ and $x \in X$. It suffices to show that $\gamma_1(ux\bar{x}v) = \gamma_1(u\bar{x}xv) = \gamma_1(uv)$. We have

$$\begin{aligned} \gamma_1(ux\bar{x}v) &= \gamma_1(u) + u[x] - ux\bar{x}[x] + ux\bar{x}\gamma_1(v) \\ &= \gamma_1(u) + u[x] - u[x] + u\gamma_1(v) = \gamma_1(u) + u\gamma_1(v) = \gamma_1(uv), \\ \gamma_1(u\bar{x}xv) &= \gamma_1(u) - u\bar{x}[x] + u\bar{x}[x] + u\bar{x}x\gamma_1(v) \\ &= \gamma_1(u) + u\gamma_1(v) = \gamma_1(uv). \end{aligned}$$

This proves part (b). To prove (c), let $u \in \underline{X}^*$. We have $\gamma_1(u^{-1}u) = \gamma_1(u^{-1}) + u^{-1}\gamma_1(u)$. By (b), $\gamma_1(u^{-1}u) = \gamma_1(\lambda) = 0$. It follows that $\gamma_1(u^{-1}) + u^{-1}\gamma_1(u) = 0$, implying that $\gamma_1(u^{-1}) = -u^{-1}\gamma_1(u)$. This proves part (c). □

Note that, by part (b) of the lemma, the function $\gamma_1 : \underline{X}^* \to C_1$ induces a function $\gamma_1 : F \to C_1$.

Further, we need an auxiliary function $\gamma_2 : \underline{Y}^* \to C_2$, which is defined as the monoid-homomorphism $\gamma_2 : (\underline{Y}^*, \cdot) \to (C_2, +)$ given through

$$\gamma_2((u, r)^{\varepsilon}) := \varepsilon u[r] \qquad \text{for all } (u, r)^{\varepsilon} \in \underline{Y}.$$

Here $(C_2, +)$ denotes the module C_2 considered as an abelian group with respect to $+$. Obviously, γ_2 induces a group-homomorphism $\gamma_2 : \hat{N} \to C_2$. For all $u, v \in \underline{Y}^*$, $\gamma_2(uv) = \gamma_2(vu) = \gamma_2({}^{\theta(u)}vu)$, since $\theta(u) \in N$ represents the identity of G. Hence, the group-homomorphism $\gamma_2 : \hat{N} \to C_2$ induces a group-homomorphism $\gamma_2 : C \to C_2$.

Let K_2 be the submodule of C_2 that is generated by the set $\{\gamma_2(w) \mid w \in \underline{Y}^*, \theta(w) = 1\}$. We want to show that $ker(\delta_2) = K_2$. We need the following auxiliary result.

LEMMA 4.5. *$\delta_2(\gamma_2(w)) = \gamma_1(\theta(w))$ for all $w \in \underline{Y}^*$.*

PROOF. The proof is by induction on the length of w. If $w = \lambda$, then $\delta_2(\gamma_2(w)) = 0 = \gamma_1(\theta(w))$. Next suppose that $w = (u, r)^\varepsilon \in \underline{Y}$. Then

$$\delta_2(\gamma_2(w)) = \delta_2(\gamma_2((u, r)^\varepsilon)) = \delta_2(\varepsilon u[r]) = \varepsilon u \delta_2([r]) = \varepsilon u \gamma_1(r).$$

On the other hand,

$$\begin{aligned}\gamma_1(\theta(w)) &= \gamma_1(\theta((u, r)^\varepsilon)) \\ &= \gamma_1(ur^\varepsilon u^{-1}) \\ &= \gamma_1(u) + u\gamma_1(r^\varepsilon) + ur^\varepsilon\gamma_1(u^{-1}) \\ &= \gamma_1(u) + \varepsilon u\gamma_1(r) + u\gamma_1(u^{-1}) \\ &= \gamma_1(u) + \varepsilon u\gamma_1(r) - \gamma_1(u) \\ &= \varepsilon u\gamma_1(r).\end{aligned}$$

If $w = uv$ with $|u| < |w|$ and $|v| < |w|$, then

$$\begin{aligned}\gamma_1(\theta(w)) &= \gamma_1(\theta(uv)) = \gamma_1(\theta(u)\theta(v)) \\ &= \gamma_1(\theta(u)) + \theta(u)\gamma_1(\theta(v)) \\ &= \gamma_1(\theta(u)) + \gamma_1(\theta(v)) \\ &= \delta_2(\gamma_2(u)) + \delta_2(\gamma_2(v)) \qquad \text{(by the induction hypothesis)} \\ &= \delta_2(\gamma_2(uv)) = \delta_2(\gamma_2(w))\end{aligned}$$

This proves the intended equality. □

For each $w \in \underline{Y}^*$ with $\theta(w) = 1$, we have

$$\delta_2(\gamma_2(w)) = \gamma_1(\theta(w)) = \gamma_1(1) = 0.$$

Thus, $K_2 \subseteq ker(\delta_2)$. It remains to prove the converse inclusion. We need some preparations.

For each element $g \in G$, let $\mathrm{w}(g) \in \underline{X}^*$ be some string representing g. For all $g \in G$ and all $x \in X$, the strings $\mathrm{w}(g)x$ and $\mathrm{w}(gx)$ both represent the same group element. Hence, there is a word $u \in \underline{Y}^*$ such that $\theta(u) \cdot \mathrm{w}(gx) = \mathrm{w}(g)x$ in F. For each $g \in G$ and each $x \in X$, let $\mathrm{u}(g, x)$ denote such a word. Further, for each $g \in G$ and each $x \in X$, let $\mathrm{u}(g, \bar{x}) := \mathrm{u}(g\bar{x}, x)^{-1}$. Then the following equations hold in F:

$$\begin{aligned}\theta(\mathrm{u}(g, \bar{x})) \cdot \mathrm{w}(g\bar{x}) &= \theta(\mathrm{u}(g\bar{x}, x)^{-1}) \cdot \mathrm{w}(g\bar{x}) \\ &= \theta(\mathrm{u}(g\bar{x}, x))^{-1} \cdot \mathrm{w}(g\bar{x}) \\ &= (\mathrm{w}(g\bar{x})x\mathrm{w}(g\bar{x}x)^{-1})^{-1} \cdot \mathrm{w}(g\bar{x}) \\ &= \mathrm{w}(g)\bar{x}\mathrm{w}(g\bar{x})^{-1}\mathrm{w}(g\bar{x}) = \mathrm{w}(g)\bar{x}.\end{aligned}$$

Thus, for all $a \in \underline{X}$, the equation $\theta(\mathrm{u}(g, a)) \cdot \mathrm{w}(ga) = \mathrm{w}(g)a$ holds in F.

Considered as an abelian group, the $\mathbb{Z}G$-module C_1 is the free abelian group generated by the set $\{g[x] \mid g \in G, x \in X\}$. Hence, we can define a group-homomorphism $\eta_2 : C_1 \to$

C_2 as follows:

$$\eta_2(g[x]) := \gamma_2(\mathrm{u}(g,x)).$$

Let $g \in G$ and $x \in X$. By definition,

$$\eta_2(g\gamma_1(x)) = \eta_2(g[x]) = \gamma_2(\mathrm{u}(g,x)).$$

For $\bar{x}$ we have

$$\begin{aligned}\eta_2(g\gamma_1(\bar{x})) &= \eta_2(-g\bar{x}[x]) = -\eta_2(g\bar{x}[x]) = -\gamma_2(\mathrm{u}(g\bar{x},x))\\ &= -\gamma_2(\mathrm{u}(g,\bar{x})^{-1}) = \gamma_2(\mathrm{u}(g,\bar{x})).\end{aligned}$$

Thus, for all $a \in \underline{X}$, we have

$$\eta_2(g\gamma_1(a)) = \gamma_2(\mathrm{u}(g,a)).$$

Let $g \in G$, and let $r \in R$, where $r = a_1a_2\cdots a_n$, $a_i \in \underline{X}$. Corresponding to g and r, we define

$$\mathrm{u}_1(g,r) := (\mathrm{w}(g),r)\cdot \mathrm{u}(ga_1\cdots a_{n-1},a_n)^{-1}\cdot\cdots\cdot \mathrm{u}(g,a_1)^{-1} \in \underline{Y}^*$$

In this situation we have

$$\begin{aligned}&\theta(\mathrm{u}(g,a_1)\mathrm{u}(ga_1,a_2)\cdots \mathrm{u}(ga_1\cdots a_{n-1},a_n))\\ &= \mathrm{w}(g)a_1\mathrm{w}(ga_1)^{-1}\mathrm{w}(ga_1)a_2\mathrm{w}(ga_1a_2)^{-1}\cdots \mathrm{w}(ga_1\cdots a_{n-1})a_n\mathrm{w}(ga_1\cdots a_{n-1}a_n)^{-1}\\ &= \mathrm{w}(g)a_1\cdots a_n\mathrm{w}(ga_1\cdots a_n)^{-1}\\ &= \mathrm{w}(g)r\mathrm{w}(gr)^{-1} = \mathrm{w}(g)r\mathrm{w}(g)^{-1} = \theta((\mathrm{w}(g),r)).\end{aligned}$$

Thus for all $g \in G$ and $r \in R$, $\theta(\mathrm{u}_1(g,r)) = 1$, that is, $\gamma_2(\mathrm{u}_1(g,r)) \in K_2$.

Considered as an abelian group the $\mathbb{Z}G$-module C_2 is freely generated by the set $\{g[r] \mid g \in G, r \in R\}$. Thus, we can define a group-homomorphism $\eta_3' : C_2 \to K_2$ as follows:

$$\eta_3'(g[r]) := \gamma_2(\mathrm{u}_1(g,r)).$$

Concerning η_3' we have the following result.

Lemma 4.6. *$\eta_3' + \eta_2\delta_2 = id_{C_2}$, that is, $\eta_3'(x) + \eta_2(\delta_2(x)) = x$ for all $x \in C_2$.*

Proof. Let $g \in G$ and $r \in R$. Then we have the following:

$$\begin{aligned}\eta_3'(g[r])) &= \gamma_2(\mathrm{u}_1(g,r))\\ &= g[r] - \gamma_2(\mathrm{u}(ga_1\cdots a_{n-1},a_n)) - \cdots - \gamma_2(\mathrm{u}(g,a_1)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\eta_2(\delta_2(g[r])) &= \eta_2(g\gamma_1(r))\\ &= \eta_2((g\gamma_1(a_1) + ga_1\gamma_1(a_2) + \cdots + ga_1\cdots a_{n-1}\gamma_1(a_n))\\ &= \gamma_2(\mathrm{u}(g,a_1)) + \cdots + \gamma_2(\mathrm{u}(ga_1\cdots a_{n-1},a_n)).\end{aligned}$$

Thus, $\eta_3'(g[r]) + \eta_2(\delta_2(g[r])) = g[r]$. Since as an abelian group, C_2 is freely generated by the set $\{g[r] \mid g \in G, r \in R\}$, this shows that indeed $\eta_3' + \eta_2\delta_2 = id_{C_2}$ holds. □

Based on this lemma the following result is now derived easily.

LEMMA 4.7. $ker(\delta_2) = K_2$.

PROOF. We have seen before that $K_2 \subseteq ker(\delta_2)$. Conversely, if $x \in ker(\delta_2)$, then $\delta_2(x) = 0$ implying that $\eta_2(\delta_2(x)) = 0$. Thus, $x = \eta_3'(x) + \eta_2(\delta_2(x)) = \eta_3'(x)$, and hence, $x \in K_2$. This proves the intended equality. □

Recall that the monoid-homomorphism $\gamma_2 : \underline{Y}^* \to C_2$ induces a group-homomorphism $\gamma_2 : \hat{N} \to C_2$ and a group-homomorphism $\gamma_2 : C \to C_2$. Further recall that π denotes the kernel of the group-homomorphism $\theta : C \to F$ and that π is actually a $\mathbb{Z}G$-module.

The restriction $\gamma_2 : \pi \to C_2$ of the group-homomorphism $\gamma_2 : C \to C_2$ is actually a $\mathbb{Z}G$-module-homomorphism. Since $ker(\delta_2) = K_2$, we see that γ_2 maps π onto $ker(\delta_2)$.

By Theorem 3.5, G has finite derivation type if and only if the $\mathbb{Z}G$-module π of identities among relations is finitely generated. Since γ_2 maps π onto $ker(\delta_2)$, it follows that, if G has finite derivation type, then the $\mathbb{Z}G$-module $ker(\delta_2)$ is finitely generated. Hence, every group is of type FP_3 if it has finite derivation type. In our previous paper we have shown that this fact holds for monoids in general (Cremanns and Otto, 1994). There, we use a partial resolution of $\mathbb{Z}$ over $\mathbb{Z}M$ that is associated with a finite presentation of a monoid M. This partial resolution is similiar to the partial resolution above, and this resolution is also used by Squier (1987) to show that, if M is given through a finite convergent presentation, then M is of type FP_3.

Now we consider the point that is different for the special case of groups. Namely, in the case of groups, it can be shown that the $\mathbb{Z}G$-module-homomorphism γ_2 that maps π onto $ker(\delta_2)$ is even injective. This implies that the $\mathbb{Z}G$-module π of identities among relations is isomorphic to the $\mathbb{Z}G$-module $ker(\delta_2)$ (see, e.g., Brown and Huebschmann, 1982).

We want to give a simple proof for the fact that the $\mathbb{Z}G$-module-homomorphism $\gamma_2 : \pi \to C_2$ is injective. To this end, we characterize the kernel of the homomorphism $\gamma_2 : \hat{N} \to C_2$ as follows, where $\hat{N}'$ denotes the commutator subgroup of $\hat{N}$.

LEMMA 4.8. *The kernel of the homomorphism* $\gamma_2 : \hat{N} \to C_2$ *is the normal subgroup* $D\hat{N}'$.

PROOF. Let K be the kernel of $\gamma_2 : \hat{N} \to C_2$, and let $\approx$ be the corresponding congruence relation on $\hat{N}$. Obviously, $\approx$ is generated by the commutator relations $\{ab = ba \mid a, b \in Y\}$ and the relations $\{a = {}^x a \mid a \in Y, x \in N\}$. It follows that $\approx$ is generated by the commutator relations and the relations $\{v = {}^x v \mid v \in \underline{Y}^*, x \in N\}$. Recall that θ maps $\hat{N}$ onto N. Hence, $\approx$ is generated by the commutator relations and the relations $\{v = {}^{\theta(u)} v \mid u, v \in \underline{Y}^*\}$. This in turn means that $\approx$ is generated by the commutator relations and the relations $\{uv = {}^{\theta(u)} vu \mid u, v \in \underline{Y}^*\}$, which is the set of relations D_Y. Hence, we conclude that K is the normal subgroup of $\hat{N}$ that is generated by the normal subgroups $\hat{N}'$ and D of $\hat{N}$. Since the product of two normal subgroups is again a normal subgroup, $K = D\hat{N}'$. □

From this observation we now derive the intended result.

LEMMA 4.9. (Reidemeister, 1949) *The* $\mathbb{Z}G$*-module-homomorphism* $\gamma_2 : \pi \to C_2$ *is injective.*

PROOF. Consider the homomorphism $\theta : \hat{N} \to F$. We have $N = \theta(\hat{N})$ and $E = ker(\theta)$, that is, N is isomorphic to the factor group $\hat{N}/E$. By the Nielsen-Schreier subgroup theorem, every subgroup of a free group is free. Thus, as a subgroup of the free group F, N is a free group as well. Let X_2 be a set of free generators for N, and let X_1 be a subset of $\hat{N}$ such that the restriction of θ to X_1 is a bijection from X_1 onto X_2. Let A be the subgroup of $\hat{N}$ that is generated by X_1. Then it is easy to check that the restriction of θ to A is an isomorphism from A onto N. Hence, each element x of $\hat{N}$ has a unique factorization of the form $x = uv$ with $u \in A$ and $v \in E$. Note that $A \cap E = \{1\}$.

The homomorphism $\gamma_2 : \pi \to C_2$ is injective if and only if, for all $x \in \pi$, $\gamma_2(x) = 0$ implies $x = 1$. Since $\pi = E/\sim$, we see that $\gamma_2 : \pi \to C_2$ is injective if and only if, for all $x \in E$, $\gamma_2(x) = 0$ implies $x \sim 1$. So let $x \in E$ such that $\gamma_2(x) = 0$. Then by Lemma 4.8, $x \in D\hat{N}'$. We have to verify that $x \sim 1$. Since $x \in D\hat{N}'$, there is some element $y \in \hat{N}'$ such that $x \sim y$. This element y is a product of commutators of $\hat{N}$. If $a^{-1}b^{-1}ab$ is one of these commutators, then each of the four factors a^{-1}, b^{-1}, a, and b can be factored into an A-component and an E-component as described above. By Lemma 3.1 (a), the E-components commute (modulo $\sim$) with all the other components. Thus, the E-components cancel, implying that $y \sim z$ for some $z \in A$. However, $x \in E$ and $x \sim z$ yield $z \in E$. But $A \cap E = \{1\}$, and so $z = 1$. Thus, $x \sim 1$. □

It follows that the $\mathbb{Z}G$-module π of identities among relations is isomorphic to the kernel of the $\mathbb{Z}G$-module-homomorphism $\delta_2 : C_2 \to C_1$. By Theorem 3.5, G has finite derivation type if and only if π is finitely generated. By Lemma 4.3, $ker(\delta_2)$ is finitely generated if and only if G is of type FP_3. Hence we obtain the following characterization.

Theorem 4.10. (Main Result) *A finitely presented group is of type FP_3 if and only if it has finite derivation type.*

5. Concluding Remarks

Squier (1994) raised the question of whether the property of having finite derivation type is decidable for finitely presented monoids (that are given through finite presentations). From our main result a negative answer to this question follows from the fact that it is undecidable whether a finitely presented group (given through some finite group-presentation) is of type FP_3. This latter result follows from the construction used to prove the Adian-Rabin Theorem which states that each Markov property of finitely presented groups is undecidable. Actually we need the following lemma (see, e.g., Miller, 1991).

Lemma 5.1. *Let $\langle X; R\rangle$ be a finite group-presentation of a group G. For any word $w \in \underline{X}^*$, a finite group-presentation $\langle X'; R'\rangle$ of a group G' can be constructed effectively such that*

(a) *if $w \neq_R 1$, then G' is a free product with amalgamation with factors $G * F_2$ and F_2, where the amalgamated subgroup is free of finite rank. Here, F_2 denotes the free group of rank 2, and $G * F_2$ denotes the free product of G and F_2;*
(b) *if $w =_R 1$, then G' is the trivial group.*

Moreover, we need the following result on the condition FP_3 (Bieri, 1976).

Lemma 5.2. *Let G be a free product with amalgamation with factors G_1 and G_2 where the amalgamated subgroup is free of finite rank. Then G is of type FP_3 if and only if G_1 and G_2 are both of type FP_3.*

Now the abovementioned undecidability result can be proved as follows. Let $\langle X; R\rangle$ be a finite group-presentation of a group G which has an undecidable word problem, and which is not of type FP_3. For example, we can take G to be the free product of a finitely presented group with an undecidable word problem and a finitely presented group which is not of type FP_3. For any word $w \in \underline{X}^*$, we construct a finite presentation $\langle X'; R'\rangle$ of a group G' as in Lemma 5.1. By Lemma 5.1 and Lemma 5.2, if $w \neq_R 1$, then G' is not of type FP_3, and if $w =_R 1$, then G' is the trivial group and thus of type FP_∞. It follows that $w =_R 1$ if and only if G' is of type FP_∞. Also for each $n \geq 3$, $w =_R 1$ if and only if G' is of type FP_n. Since the word problem for G is undecidable, it follows that the property of being of type FP_∞ is undecidable for finitely presented groups. Also for each $n \geq 3$, the property of being of type FP_n is undecidable for finitely presented groups.

Also there exist finitely presented groups of cohomological dimension 2 that have undecidable word problems (Miller, 1991). Since cohomological dimension 2 implies the property FP_∞ (Brown, 1982), which in turn implies the property of having finite derivation type, we see that a finitely presented group can have an undecidable word problem, even if it has finite derivation type. This answers another one of Squier's questions (1994) in the negative.

We conclude with some remarks on aspherical groups. Aspherical groups have extensively been studied in the literature. Actually, there are various different notions of asphericity. There are important classes of groups which are aspherical, among them the one-relator groups and the small cancellation groups. In Chiswell, Collins and Huebschmann (1981) an overview is given, and it is shown how these notions are related.

Aspherical groups can be characterized through certain finite generating sets for the module π of identities among relations. Using the relationship between homotopy relations and submodules of π described in Section 3, we obtain characterizations of aspherical groups in terms of homotopy relations (Cremanns and Otto, 1994a). In this way they arise naturally as special cases of groups with finite derivation type.

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