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Some properties of m-polar fuzzy graphs

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ABSTRACT

In many real world problems, data sometimes comes from n agents ($n \geq 2$), i.e., “multipolar information” exists. This information cannot be well-represented by means of fuzzy graphs or bipolar fuzzy graphs. Therefore, m-polar fuzzy set theory is applied to graphs to describe the relationships among several individuals. In this paper, some operations are defined to formulate these graphs. Some properties of strong m-polar fuzzy graphs, self-complementary m-polar fuzzy graphs and self-complementary strong m-polar fuzzy graphs are discussed.

1. Introduction

The origin of graph theory started with the Konigsberg bridge problem in 1735. This problem led to the concept of the Eulerian graph. Euler studied the Konigsberg bridge problem and constructed a structure that solves the problem that is referred to as an Eulerian graph. In 1840, Möbius proposed the idea of a complete graph and a bipartite graph and Kuratowski proved that they are planar by means of recreational problems. Currently, concepts of graph theory are highly utilized by computer science applications, especially in areas of computer science research, including data mining, image segmentation, clustering, and networking. The introduction of fuzzy sets by Zadeh [21] in 1965 greatly changed the face of science and technology. Fuzzy sets paved the way for a new method of philosophical thinking, “Fuzzy Logic” which is now an essential concept in artificial intelligence. The most important feature of a fuzzy set is that it consists of a class of objects that satisfy a certain property or several properties. In 1994, Zhang [24,25] initiated the concept of bipolar fuzzy sets. Juanjuan Chen et al. [1] introduced the notion of the m-polar fuzzy set as a generalization of bipolar fuzzy sets. The first definition of fuzzy graphs was proposed by Kafmann [7] from Zadeh's fuzzy relations [21–23]. However, Rosenfeld [11] introduced another group of elaborated definitions, including the fuzzy vertex, fuzzy edges, and several fuzzy analogues of theoretical graph concepts, such as paths, cycles, connectedness, and so on. Mordeson and Nair [10] defined the complement of a fuzzy graph. McAllister [9] characterized fuzzy intersection graphs. Samanta and Pal studied fuzzy tolerance graphs [14], fuzzy threshold graphs [15], bipolar fuzzy hypergraphs [16], irregular bipolar fuzzy graphs [17], fuzzy k-competition graphs, m step fuzzy competition graphs [18,19], and fuzzy planar graphs [20]. Later, Rashmanlou et al. [12,13] studied bipolar fuzzy graphs with categorical properties and Ghorai and Pal introduced product bipolar fuzzy line graphs [3]. In 2014, Juanjuan Chen et al. [1] defined m-polar fuzzy graphs. Ghorai and Pal introduced some operations and the density of m-polar fuzzy graphs [2], studied m-polar fuzzy planar graphs [4] and defined faces and the dual nature of m-polar fuzzy planar graphs [5]. In this paper the Cartesian product, composition, union and join of two m-polar fuzzy graphs are defined. Some important properties of isomorphisms, strong m-polar fuzzy graphs, self-complementary m-polar fuzzy graphs and self-complementary strong m-polar fuzzy graphs are discussed.

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2. Preliminaries

In this section, we briefly recall some definitions of undirected graphs, the notions of fuzzy sets, bipolar fuzzy sets and m-polar fuzzy sets. For further reference, see Refs. [6,8,10].

Definition 2.1. [6] *A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of all edges of G^*. Two vertices x and y in an undirected graph G^* are said to be adjacent in G^* if xy is an edge of G^*. A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices. A subgraph of a graph $G^* = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.*

Definition 2.2. [6] *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs.*

The Cartesian product $G^* = G_1^* \times G_2^* = (V, E)$ of graphs G_1^* and G_2^*. Then, $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2) : x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1 y_1 \in E_1\}$.

Then, the composition of the graph G_1^* with G_2^* is denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where $E^0 = E \cup \{(x_1, x_2)(y_1, y_2) : x_1 y_1 \in E_1, x_2 \neq y_2\}$ and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

The union of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1^* and G_2^* is denoted by $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.

The join of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and assume that $V_1 \cap V_2 = \emptyset$. The join of G_1^* and G_2^* is denoted by $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$.

Definition 2.3. [1] *Throughout the paper, $[0,1]^m$ (the m-power of $[0,1]$) is considered to be a poset with point-wise order $\leq$, where m is an natural number. $\leq$ is defined by $x \leq y \Leftrightarrow$ for each $i = 1,2,\ldots,m$; $p_i(x) \leq p_i(y)$, where $x, y \in [0,1]^m$ and $p_i : [0,1]^m \to [0,1]$ is the i^{th} projection mapping.*

An m-polar fuzzy set (or a $[0,1]^m$-*set*) *on X is a mapping $A : X \to [0,1]^m$. The set of all m-polar fuzzy sets on X is denoted* by $m(X)$.

Definition 2.4. *Let A and B be two m-polar fuzzy sets in X. Then, $A \cup B$ and $A \cap B$ are also m-polar fuzzy sets in X defined by: for $i = 1, 2,\ldots,m$ and $x \in X$*

$p_i \circ (A \cup B)(x) = max\{p_i \circ A(x), p_i \circ B(x)\}$ and $p_i \circ (A \cap B)(x) = min\{p_i \circ A(x), p_i \circ B(x)\}$. $A \subseteq B$ if and only if $p_i \circ A(x) \leq p_i \circ B(x)$ and $A = B$ if and only if $p_i \circ A(x) = p_i \circ B(x)$.

Definition 2.5. *Let A be an m-polar fuzzy set on a set X. An m-polar fuzzy relation on A is an m-polar fuzzy set B of $X \times X$ such that $B(x,y) \leq min\{A(x), A(y)\}$ for all $x, y \in X$, i.e., for each $i = 1, 2,\ldots,m$, for all $x, y \in X$, $p_i \circ B(x, y) \leq min\{p_i \circ A(x), p_i \circ A(y)\}$. An m-polar fuzzy relation B on X is called symmetric if $B(x, y) = B(y, x)$ for all $x, y \in X$.*

We assume the following: For a given set V, define an equivalence relation $\sim$ on $V \times V - \{(x, x) : x \in V\}$ as follows: $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow$ either $(x_1, y_1) = (x_2, y_2)$ or $x_1 = y_2$ and $y_1 = x_2$. The quotient set obtained in this way is denoted by $\widetilde{V^2}$, and the equivalence class that contains the element (x, y) is denoted as xy or yx.

Throughout this paper, $G^* = (V, E)$ represents a crisp graph and G is an m-polar fuzzy graph of G^*.

3. Generalized *m*-polar fuzzy graphs

Juanjuan Chen et al. [1] defined the m-polar fuzzy graph in the following way: An m-polar fuzzy graph with an underlying pair (V, E) (where $E \subseteq V \times V$ is symmetric) is defined to be a pair $G = (A, B)$, where $A : V \to [0,1]^m$ and $B : E \to [0,1]^m$, satisfying $B(xy) \leq min\{A(x), A(y)\}$ for all $xy \in E$.

According to the above definition, B is actually an m-polar fuzzy set in $E \subseteq V \times V$. However, when the definition is used, B is actually an m-polar fuzzy set defined in $\widetilde{V^2}$, satisfying $B(xy) = 0 = (0,0,\ldots,0)$ for all $xy \in (\widetilde{V^2} - E)$. The above definition will cause some problems by calculating the complement of an m-polar fuzzy graphs. Therefore, a generalized m-polar fuzzy graph is defined below.

Definition 3.1. *A generalized m-polar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$, where $A : V \to [0,1]^m$ is an m-polar fuzzy set in V and $B : \widetilde{V^2} \to [0, 1]^m$ is an m-polar fuzzy set in $\widetilde{V^2}$ such that $B(xy) \leq min\{A(x), A(y)\}$ for all $xy \in \widetilde{V^2}$ and $B(xy) = 0$ for all $xy \in (\widetilde{V^2} - E)$ ($0 = (0, 0,\ldots,0)$ is the smallest element in $[0,1]^m$). A is called the m-polar fuzzy vertex set of G, and B is called the m-polar fuzzy edge set of G.*

Example 3.2. *Let $X = \{F_1, F_2, F_3, F_4\}$ and $M = \{M_1, M_2, M_3\}$ be the set of four friends and three movies, respectively. Suppose they planned to watch a movie. This situation can be represented as a 4-polar fuzzy graph G by considering the vertex set as M and the edge set as $M \times M$. Let A be a 4-polar fuzzy set of M. The membership value of M_i represents the preference degrees of the movie M_i corresponding to the friends. Suppose $A(M_1) = <0.9, 0.4, 0.6, 0.1>$, $A(M_2) = <0.5, 0.3, 0.8, 0.1>$, $A(M_3) = <0.8, 0.9, 0.8, 0.2>$. This means that the preference degrees of M_1 corresponding to F_1, F_2, F_3 and F_4 are 0.9, 0.4, 0.6 and 0.1, respectively, and is similar for the others. An edge between any two nodes represents the degrees of common features (i.e., love story, comedy, fighting, and horror) of the nodes. Let $B(M_1M_2) = <0.4, 0.2, 0.2, 0.1>$, $B(M_2M_3) = <0.4, 0.2, 0.2, 0.2>$, $B(M_3M_1) = <0.4, 0.2, 0.3, 0.1>$. This means that the degrees of common features (i.e., love story, comedy, fighting, and horror) of the movies M_1 and M_2 are 0.4, 0.2, 0.2 and 0.1. In other words, both movies M_1 and M_2 have 40% love story, 20% comedy, 20% fighting and 10% horror. Similar to the others. It is easy to verify that G of.* It Fig. 1 *is a 4-polar fuzzy graph.*

Hereafter, we assume an m-polar fuzzy graph to be a generalized m-polar fuzzy graph.

4. Cartesian product, composition, union and join on *m*-polar fuzzy graphs

In this section, four types of operations, such as the Cartesian product, composition, union and join have been defined on m-polar fuzzy graphs to construct new types of m-polar fuzzy graphs.

Definition 4.1. *The Cartesian product $G_1 \times G_2$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, is defined as a pair $(V_1 \times V_2, A_1 \times A_2, B_1 \times B_2)$, such that for $i = 1, 2,\ldots,m$*

(i) $p_i \circ (A_1 \times A_2)(x_1, x_2) = min\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\}$ *for all* $(x_1, x_2) \in V_1 \times V_2$.

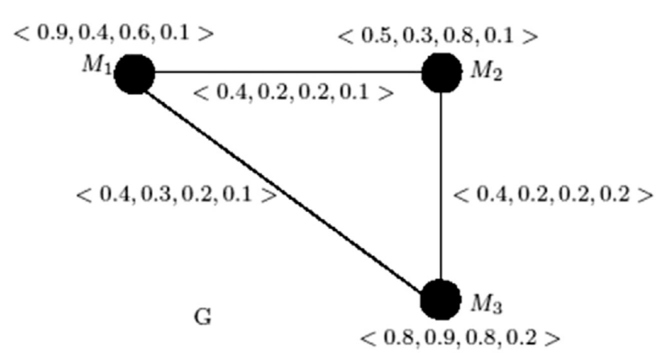

Fig. 1. Example of 4-polar fuzzy graph *G*.

(ii) $p_i \circ (B_1 \times B_2)((x, x_2)(x, y_2)) = min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\}$ for all $x \in V_1$, for all $x_2y_2 \in E_2$.
(iii) $p_i \circ (B_1 \times B_2)((x_1, z)(y_1, z)) = min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\}$ for all $z \in V_2$, for all $x_1y_1 \in E_1$.
(iv) $p_i \circ (B_1 \times B_2)((x_1, y_1)(x_2, y_2)) = 0$ for all $(x_1, y_1)(x_2, y_2) \in \widetilde{V_1 \times V_2}^2 - E$.

Example 4.2. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Consider the 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively where $A_1 = \left\{\frac{<0.3,0.4,0.6>}{a}, \frac{<0.3,0.5,0.7>}{b}\right\}$, $B_1 = \left\{\frac{<0.1,0.2,0.5>}{ab}\right\}$ and $A_2 = \left\{\frac{<0.1,0.4,0.5>}{c}, \frac{<0.2,0.6,0.6>}{d}\right\}$, $B_2 = \left\{\frac{<0.1,0.3,0.4>}{cd}\right\}$. Then, it is easy to verify the following:*

$$(B_1 \times B_2)((a,c)(a,d)) = <0.1, 0.3, 0.4>, \ (B_1 \times B_2)((a,c)(b,c)) = <0.1, 0.2, 0.5>,$$

$$(B_1 \times B_2)((b,c)(b,d)) = <0.1, 0.3, 0.4>, \ (B_1 \times B_2)((a,d)(b,d)) = <0.1, 0.2, 0.5>,$$

$$(B_1 \times B_2)((a,c)(b,d)) = <0,0,0>, \ (B_1 \times B_2)((b,c)(a,d)) = <0,0,0>$$

Hence, $G_1 \times G_2$ is a 3-polar fuzzy graph of $G_1^* \times G_2^*$ (see Fig. 2).

Proposition 4.3. *The Cartesian product $G_1 \times G_2 = (V_1 \times V_2, A_1 \times A_2, B_1 \times B_2)$ of two m-polar fuzzy graphs of the graphs G_1^* and G_2^* is an m-polar fuzzy graph of $G_1^* \times G_2^*$.*

Proof. Let $x \in V_1$, $x_2y_2 \in E_2$. Then, for $i = 1, 2, \ldots, m$

$$\begin{aligned} &p_i \circ (B_1 \times B_2)((x, x_2)(x, y_2)) \\ &= min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\} \\ &\le min\{p_i \circ A_1(x), min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}\} \\ &= min\{min\{p_i \circ A_1(x), p_i \circ A_2(x_2)\}, min\{p_i \circ A_1(x), p_i \circ A_2(y_2)\}\} \\ &= min\{p_i \circ (A_1 \times A_2)(x, x_2), p_i \circ (A_1 \times A_2)(x, y_2)\}. \end{aligned}$$

Let $z \in V_2$, $x_1y_1 \in E_1$. Then, for $i = 1, 2, \ldots, m$

$$\begin{aligned} &p_i \circ (B_1 \times B_2)((x_1, z)(y_1, z)) \\ &= min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\} \\ &\le min\{min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1)\}, p_i \circ A_2(z)\}\} \\ &= min\{min\{p_i \circ A_1(x_1), p_i \circ A_2(z)\}, min\{p_i \circ A_1(y_1), p_i \circ A_2(z)\}\} \\ &= min\{p_i \circ (A_1 \times A_2)(x_1, z), p_i \circ (A_1 \times A_2)(y_1, z)\}. \end{aligned}$$

Let $(x_1, y_1)(x_2, y_2) \in \widetilde{V_1 \times V_2}^2 - E$. Then, for $i = 1, 2, \ldots, m$

$$p_i \circ (B_1 \times B_2)((x_1, y_1)(x_2, y_2)) = 0 \le min\{p_i \circ (A_1 \times A_2)(x_1, y_1), p_i \circ (A_1 \times A_2)(x_2, y_2)\}.$$

Definition 4.4. *The composition $G_1[G_2] = (V_1 \times V_2, A_1 \circ A_2, B_1 \circ B_2)$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as follows: for $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \circ A_2)(x_1, x_2) = min\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\}$ for all $(x_1, x_2) \in V_1 \times V_2$.
(ii) $p_i \circ (B_1 \circ B_2)((x, x_2)(x, y_2)) = min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\}$ for all $x \in V_1$, for all $x_2y_2 \in E_2$.
(iii) $p_i \circ (B_1 \circ B_2)((x_1, z)(y_1, z)) = min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\}$ for all $z \in V_2$, for all $x_1y_1 \in E_1$.
(iv) $p_i \circ (B_1 \circ B_2)((x_1, x_2)(y_1, y_2)) = min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), p_i \circ B_1(x_1y_1)\}$ for all $(x_1, x_2)(y_1, y_2) \in E^0 - E$.
(v) $p_i \circ (B_1 \circ B_2)((x_1, y_1)(x_2, y_2)) = 0$ for all $(x_1, y_1)(x_2, y_2) \in \widetilde{V_1 \times V_2}^2 - E^0$.

Example 4.5. *Let G_1^* and G_2^* be the same as in Example 4.2. Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two 3-polar fuzzy graphs of the graphs G_1^* and G_2^*, respectively, where $A_1 = \left\{\frac{<0.2,0.4,0.5>}{a}, \frac{<0.3,0.5,0.4>}{b}\right\}$, $B_1 = \left\{\frac{<0.2,0.3,0.4>}{ab}\right\}$, $A_2 = \left\{\frac{<0.1,0.4,0.5>}{c}, \frac{<0.2,0.7,0.6>}{d}\right\}$, $B_2 = \left\{\frac{<0.1,0.2,0.3>}{cd}\right\}$. Then, we have,*

$$(B_1 \circ B_2)((a,c)(a,d)) = <0.1, 0.2, 0.3>, \ (B_1 \circ B_2)((b,c)(b,d)) = <0.1, 0.2, 0.3>,$$

$$(B_1 \circ B_2)((a,c)(b,c)) = <0.1, 0.3, 0.4>, \ (B_1 \circ B_2)((a,d)(b,d)) = <0.2, 0.3, 0.4>$$

$$(B_1 \circ B_2)((a,c)(b,d)) = <0.1, 0.3, 0.4>, \ (B_1 \circ B_2)((b,c)(a,d)) = <0.1, 0.3, 0.4>.$$

It can be easily determined that $G_1[G_2]$ is a 3-polar fuzzy graph of $G_1^*[G_2^*]$ (see Fig. 3).

Proposition 4.6. *The composition $G_1[G_2]$ of two m-polar fuzzy graphs G_1 and G_2 is an m-polar fuzzy graph.*

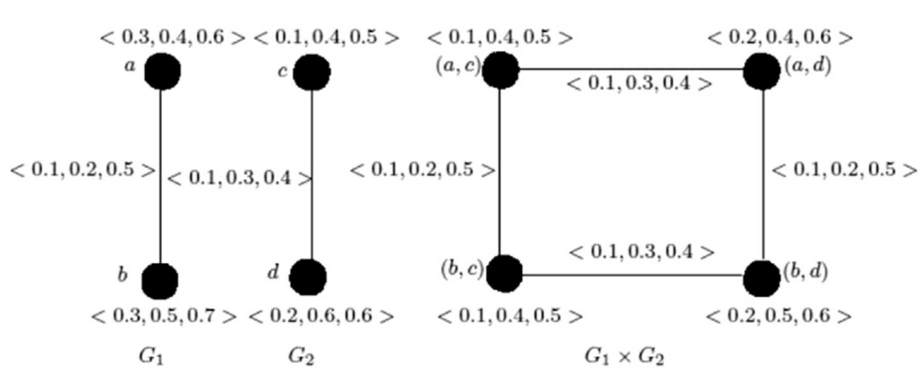

Fig. 2. Product of two 3-polar fuzzy graphs G_1 and G_2.

Proof. Let $x \in V_1$, $x_2y_2 \in E_2$. Then, for $i = 1, 2, \ldots, m$,

$$\begin{aligned} & p_i \circ (B_1 \times B_2)((x, x_2)(x, y_2)) \\ & = min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\} \\ & \leq min\{p_i \circ A_1(x), min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}\} \\ & = min\{min\{p_i \circ A_1(x), p_i \circ A_2(x_2)\}, min\{p_i \circ A_1(x), p_i \circ A_2(y_2)\}\} \\ & = min\{p_i \circ (A_1 \times A_2)(x, x_2), p_i \circ (A_1 \times A_2)(x, y_2)\}. \end{aligned}$$

Let $z \in V_2$, $x_1y_1 \in E_1$. The proof is similar to the above.

Let $(x_1, x_2)(y_1, y_2) \in E^0 - E$. Therefore, $x_1y_1 \in E_1$ and $x_2 \neq y_2$. Then, we have for each $i = 1, 2, \ldots, m$,

$$\begin{aligned} & p_i \circ (B_1 \circ B_2)((x_1, x_2)(y_1, y_2)) \\ & = min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), p_i \circ B_1(x_1y_1)\} \\ & \leq min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1)\}\} \\ & = min\{min\{\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\}, min\{p_i \circ A_1(y_1), p_i \circ A_2(y_2)\}\} \\ & = min\{p_i \circ (A_1 \times A_2)(x_1, x_2), p_i \circ (A_1 \times A_2)(y_1, y_2)\}. \end{aligned}$$

Hence, $G_1[G_2]$ is an m-polar fuzzy graph.

Definition 4.7. *The union $G_1 \cup G_2 = (V_1 \cup V_2, A_1 \cup A_2, B_1 \cup B_2)$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively is defined as follows: for $i = 1, 2, \ldots, m$*

1. $p_i \circ (A_1 \cup A_2)(x) = \begin{cases} p_i \circ A_1(x) & \text{if} \quad x \in V_1 - V_2 \\ p_i \circ A_2(x) & \text{if} \quad x \in V_2 - V_1 \\ max\{p_i \circ A_1(x), p_i \circ A_2(x)\} & \text{if} \quad x \in V_1 \cap V_2. \end{cases}$
2. $p_i \circ (B_1 \cup B_2)(xy) = \begin{cases} p_i \circ B_1(xy) & \text{if} \quad xy \in E_1 - E_2 \\ p_i \circ B_2(xy) & \text{if} \quad xy \in E_2 - E_1 \\ max\{p_i \circ B_1(xy), p_i \circ B_2(xy)\} & \text{if} \quad xy \in E_1 \cap E_2. \end{cases}$
3. $p_i \circ (B_1 \cup B_2)(xy) = 0$ if $xy \in \widetilde{V_1 \times V_2}^2 - E_1 \cup E_2$.

Example 4.8. *Let G_1^* and G_2^* be graphs such that $V_1 = \{a, b, c, d\}$, $E_1 = \{ab, bc, ad, bd\}$, $V_2 = \{a, b, c, f\}$ and $\{ab, bc, bf, cf\}$. Consider the two 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$, where $A_1 = \left\{ \frac{<0.2,0.4,0.3>}{a}, \frac{<0.4,0.5,0.6>}{b}, \frac{<0.3,0.6,0.2>}{c}, \frac{<0.3,0.7,0.8>}{d} \right\}$, $B_1 = \left\{ \frac{<0.1,0.3,0.2>}{ab}, \frac{<0.2,0.5,0.1>}{bc}, \frac{<0.2,0.3,0.2>}{ad}, \frac{<0.3,0.4,0.5>}{bd}, \frac{<0,0,0>}{cd}, \frac{<0,0,0>}{ac} \right\}$, $A_2 = \left\{ \frac{<0.2,0.4,0.7>}{a}, \frac{<0.2,0.5,0.6>}{b}, \frac{<0.3,0.6,0.7>}{c}, \frac{<0.4,0.5,0.3>}{f} \right\}$, and $B_2 = \left\{ \frac{<0.2,0.3,0.5>}{ab}, \frac{<0.2,0.5,0.4>}{bc}, \frac{<0.2,0.5,0.3>}{cf}, \frac{<0.1,0.4,0.3>}{bf}, \frac{<0,0,0>}{af}, \frac{<0,0,0>}{ac} \right\}$. Clearly, $G_1 \cup G_2$ is a 3-polar fuzzy graph* (See Fig. 4).

Proposition 4.9. *The union $G_1 \cup G_2 = (V_1 \cup V_2, A_1 \cup A_2, B_1 \cup B_2)$ of two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is an m-polar fuzzy graph.*

Proof. Let $xy \in E_1 \cap E_2$. Then, for $i = 1, 2, \ldots, m$

$$\begin{aligned} & p_i \circ (B_1 \cup B_2)(xy) = max\{p_i \circ B_1(xy), p_i \circ B_2(xy)\} \\ & \leq max\{min\{p_i \circ A_1(x), p_i \circ A_1(y)\}, min\{p_i \circ A_2(x), p_i \circ A_2(y)\}\} \\ & = min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}. \end{aligned}$$

Similarly, if $xy \in E_1 - E_2$, then $p_i \circ (B_1 \cup B_2)(xy) \leq min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}$ and if $xy \in E_2 - E_1$, then $p_i \circ (B_1 \cup B_2)(xy) \leq min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}$. This completes the proof.

Definition 4.10. *The join $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, is defined as follows:*

(i) $p_i \circ (A_1 + A_2)(x) = p_i \circ (A_1 \cup A_2)(x)$ if $x \in V_1 \cup V_2$
(ii) $p_i \circ (B_1 + B_2)(xy) = p_i \circ (B_1 \cup B_2)(xy)$ if $xy \in E_1 \cup E_2$
(iii) $p_i \circ (B_1 + B_2)(xy) = min\{p_i \circ A_1(x), p_i \circ A_2(y)\}$ if $xy \in E'$, where E' is the set of all of the edges joining the nodes of V_1 and V_2 and assuming that $V_1 \cap V_2 = \emptyset$.
(iv) $p_i \circ (B_1 + B_2)(xy) = 0$ if $xy \in \widetilde{V_1 \times V_2}^2 - E_1 \cup E_2 \cup E'$.

Proposition 4.11. *The join $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ of two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is an m-polar fuzzy graph of $G_1^* + G_2^*$.*

Proof. Follows from the definition.

Proposition 4.12. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs and let $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be m-polar fuzzy subsets of $V_1, V_2, \widetilde{V_1^2}$ and $\widetilde{V_2^2}$, respectively. Then, $G_1 \cup G_2 = (V_1 \cup V_2, A_1 \cup A_2, B_1 \cup B_2)$ is an m-polar fuzzy graph of $G_1^* \cup G_2^*$ if and only if $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are m-polar fuzzy graphs of G_1^* and G_2^*, respectively.*

Proof. Suppose $G_1 \cup G_2$ is an m-polar fuzzy graph of $G_1^* \cup G_2^*$.

Let $xy \in E_1$. Then, $xy \notin E_2$ and $x, y \in V_1 - V_2$, and for $i = 1, 2, \ldots, m$ $p_i \circ B_1(xy) = p_i \circ (B_1 \cup B_2)(xy) \leq min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\} \leq min\{p_i \circ A_1(x), p_i \circ A_1(y)\}$. Let $xy \in \widetilde{V_1^2} - E_1$. Then, for $i = 1, 2, \ldots, m$, $p_i \circ B_1(xy) = p_i \circ B_1 \cup B_2(xy) = 0$. This shows that $G_1 = (V_1, A_1, B_1)$ is an m-polar fuzzy graph of G_1^*. Similarly, we can show that $G_2 = (V_2, A_2, B_2)$ is an m-polar fuzzy graph of G_2^*. The converse follows from proposition 4.9.

Proposition 4.13. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs and let $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be m-polar fuzzy subsets of $V_1, V_2, \widetilde{V_1^2}$ and $\widetilde{V_2^2}$, respectively. Then, $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ is an m-polar fuzzy graph of $G_1^* + G_2^*$ if and only if $G_1 = (V_1, A_1,$*

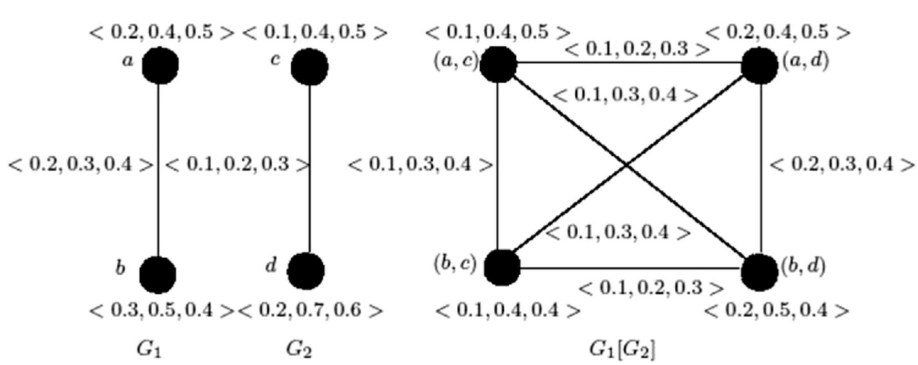

Fig. 3. Composition of two 3-polar fuzzy graphs G_1 and G_2.

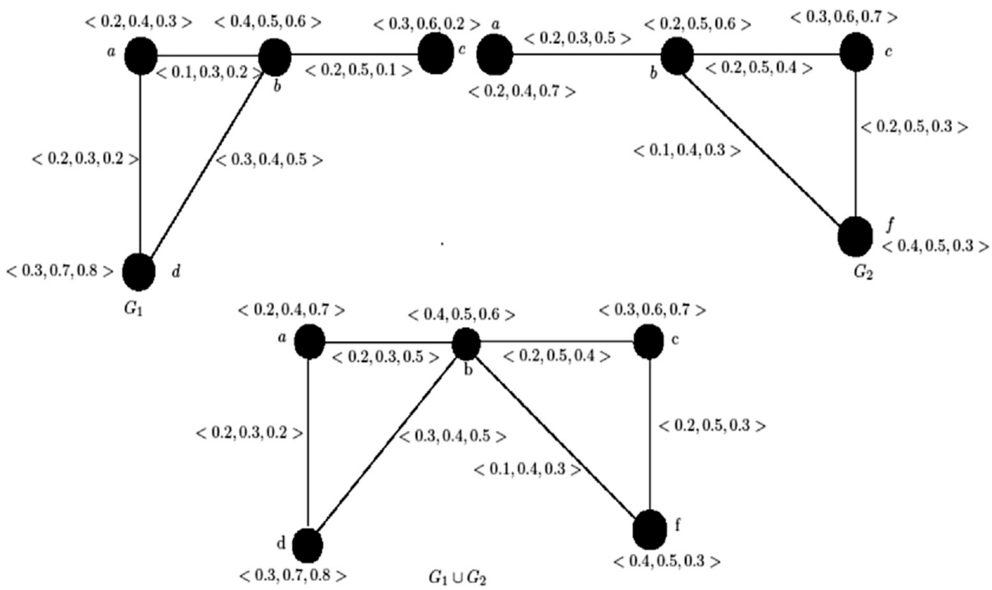

Fig. 4. Union of two 3-polar fuzzy graphs G_1 and G_2.

B_1) and $G_2 = (V_2, A_2, B_2)$ are m-polar fuzzy graphs of G_1^* and G_2^*, respectively.

Proof. Follows from propositions 4.11 *and* 4.12.

5. Isomorphisms of *m*-polar fuzzy graphs

In this section, different types of isomorphisms of *m*-polar fuzzy graphs are defined.

Definition 5.1. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. A homomorphism between G_1 and G_2 is a mapping $\phi: V_1 \to V_2$ such that for $i = 1, 2, \ldots, m$*

(i) $p_i \circ A_1(x_1) \le p_i \circ A_2(\phi(x_1))$ *for all* $x_1 \in V_1$,
(ii) $p_i \circ B_1(x_1y_1) \le p_i \circ B_2(\phi(x_1)\phi(y_1))$ *for all* $x_1y_1 \in \widetilde{V_1^2}$.

Definition 5.2. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. An isomorphism between G_1 and G_2 is a bijective mapping $\phi: V_1 \to V_2$ such that for $i = 1, 2, \ldots, m$*

(i) $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ *for all* $x_1 \in V_1$,
(ii) $p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1)\phi(y_1))$ *for all* $x_1y_1 \in \widetilde{V_1^2}$. *In this case, we write* $G_1 \cong G_2$.

Definition 5.3. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. A weak isomorphism between G_1 and G_2 is a bijective mapping $\phi: V_1 \to V_2$, which satisfies the following conditions:*

(i) *ϕ is a homomorphism, and*
(ii) *for each $i = 1, 2, \ldots, m$, $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ for all $x_1 \in V_1$. In other words, a weak isomorphism preserves the weights of the nodes, but not necessarily the weights of the arcs.*

Example 5.4. *Consider the two 3-polar fuzzy graphs G_1 and G_2 (see Fig. 5) of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, where $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Let us define a map $\phi: V_1 \to V_2$ to be defined by $\phi(a) = d$, $\phi(b) = c$. Then, we have*
$p_1 \circ A_1(a) = 0.2 = p_1 \circ A_2(\phi(a) = d)$, $p_2 \circ A_1(a) = 0.4 = p_2 \circ A_2(\phi(a) = d)$, $p_3 \circ A_1(a) = 0.5 = p_3 \circ A_2(\phi(a) = d)$. $p_1 \circ A_1(b) = 0.3 = p_1 \circ A_2(\phi(b) = c)$, $p_2 \circ A_1(b) = 0.5 = p_2 \circ A_2(\phi(b) = c)$, $p_3 \circ A_1(b) = 0.7 = p_3 \circ A_2(\phi(b) = c)$. $p_1 \circ B_1(ab) = 0.1 < 0.2 = p_1 \circ B_2(\phi(a)\phi(b) = dc)$, $p_2 \circ B_1(ab) = 0.4 = p_2 \circ B_2(\phi(a)\phi(b) = dc)$, $p_3 \circ B_1(ab) = 0.3 < 0.4 = p_3 \circ B_2(\phi(a)\phi(b) = dc)$.Hence $B_1(ab) \ne B_2(\phi(a)\phi(b))$. This shows that the map ϕ is a weak isomorphism but not an isomorphism.

Definition 5.5. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. A co-weak isomorphism between G_1 and G_2 is a bijective mapping $\phi: V_1 \to V_2$ which satisfies the following:*

(i) *ϕ is a homomorphism,*
(ii) *for each $i = 1, 2, \ldots, m$, $p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1y_1))$ for all $x_1y_1 \in \widetilde{V_1^2}$. In other words, a co-weak isomorphism preserves the weight of the arcs but not necessarily the weights of the nodes.*

Example 5.6. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be as in Example 5.4. Consider the 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of G_1^* and G_2^* (see Fig. 6). Consider the map $\phi: V_1 \to V_2$ defined by $\phi(a) = d$, $\phi(b)$=c. Then, we have the following:*
$p_1 \circ A_1(a) = 0.2 < 0.3 = p_1 \circ A_2(\phi(a) = d)$, $p_2 \circ A_1(a) = 0.4 < 0.6 = p_2 \circ A_2(\phi(a) = d)$, $p_3 \circ A_1(a) = 0.5 = 0.5 = p_3 \circ A_2(\phi(a) = d)$. Therefore, $A_1(a) \ne A_2(\phi(a) = d)$. Similarly, $A_1(b) \ne A_2(\phi(b) = c)$. However, $p_1 \circ B_1(ab) = 0.1 = p_1 \circ B_2(\phi(a)\phi(b) = dc)$, $p_2 \circ B_1(ab) = 0.4 = p_2 \circ B_2(\phi(a)\phi(b) = dc)$, and $p_3 \circ B_1(ab) = 0.2 = p_3 \circ B_2(\phi(a)\phi(b) = dc)$. Therefore, $B_1(ab) = B_2(\phi(a)\phi(b) = dc)$. Hence, the map ϕ is a co-weak isomorphism, but not an isomorphism.

6. Some properties of *m*-polar fuzzy graphs

The strong *m*-polar fuzzy graph is defined below.

Definition 6.1. *An m-polar fuzzy graph $G = (V, A, B)$ of the graph $G^* = (V, E)$ is called strong if $p_i \circ B(xy) = \min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in E$, $i = 1, 2, \ldots, m$.*

Example 6.2. *Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let $G = (V, A, B)$ be the 3-polar fuzzy graph of G^* where*

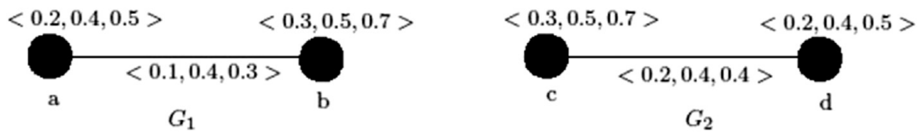

Fig. 5. Weak isomorphism of G_1 and G_2.

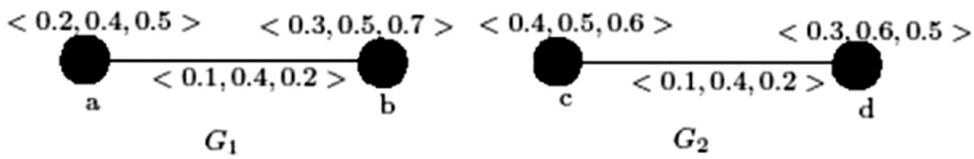

Fig. 6. Co-weak isomorphism of G_1 and G_2.

$A=\left\{\frac{<0.2,0.4,0.5>}{x},\frac{<0.3,0.5,0.6>}{y},\frac{<0.4,0.3,0.1>}{z}\right\}$,

$B=\left\{\frac{<0.2,0.4,0.5>}{xy},\frac{<0.3,0.3,0.1>}{yz},\frac{<0.2,0.3,0.1>}{zx}\right\}$. *Hence, G is a strong* 3-*polar fuzzy graph* (*see* Fig. 7).

Proposition 6.3. *If G_1 and G_2 are the strong m-polar fuzzy graphs of the graphs $G_1^*=(V_1,E_1)$ $G_2^*=(V_2,E_2)$, respectively, then $G_1\times G_2$, $G_1[G_2]$ and G_1+G_2 are strong m-polar fuzzy graphs of the graphs $G_1^*\times G_2^*$, $G_1^*[G_2^*]$ and $G_1^*+G_2^*$.*

Proof. Follows from the Proposition 4.3, 4.6 *and* 4.11.

Remark 6.4. *The union of two strong m-polar fuzzy graphs is not necessarily a strong m-polar fuzzy graph. For example, let us consider the* 3-*polar fuzzy graphs G_1 and G_2, as shown in* Fig. 8.

Proposition 6.5. *If $G_1\times G_2$ is strong m-polar fuzzy graph, then at least G_1 or G_2 must be strong.*

Proof. Suppose that both G_1 and G_2 are not strong m-polar fuzzy graphs. Then, there exists at least one $x_1y_1\in E_1$ and at least one $x_2y_2\in E_2$ such that

(i) $B_1(x_1y_1)<min\{A_1(x_1),\ A_1(y_1)\}$, *and* $B_2(x_2y_2)<min\{A_2(x_2), A_2(y_2)\}$.

Without loss of generality, we assume that

(ii) $B_2(x_2y_2)\le B_1(x_1y_1)<min\{A_1(x_1),A_1(y_1)\}\le A_1(x_1)$.

Let $E=\{(x,x_2)(x,y_2):x\in V_1,x_2y_2\in E_2\}\cup\{(x_1,z)(y_1,z):z\in V_2,x_1y_1\in E_1\}$. *Consider* $(x,x_2)(x,y_2)\in E$. *Then, by definition of $G_1\times G_2$ and inequality* (*i*) *we have,*

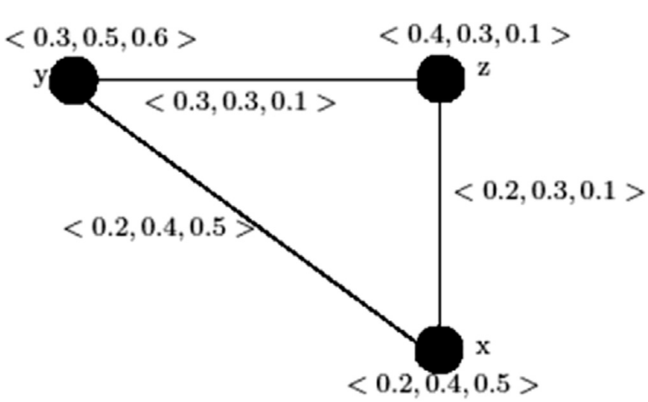

Fig. 7. Strong 3-polar fuzzy graph G.

$$(B_1\times B_2)((x,x_2)(x,y_2))=min\{A_1(x),B_2(x_2y_2)\}<min\{A_1(x),A_2(x_2),A_2(y_2)\}$$

and $(A_1\times A_2)(x,x_2)=min\{A_1(x),A_2(x_2)\}$,$(A_1\times A_2)(x,y_2)=min\{A_1(x),A_2(y_2)\}$.

Thus, $min\{(A_1\times A_2)(x,x_2),(A_1\times A_2)(x,y_2)\}=min\{A_1(x),A_2(x_2),A_2(y_2)\}$.

Hence, $(B_1\times B_2)((x,x_2)(x,y_2))=min\{A_1(x),B_2(x_2y_2)\}<min\{(A_1\times A_2)(x,x_2),(A_1\times A_2)(x,y_2)\}$, *i.e., $G_1\times G_2$ is not a strong m-polar fuzzy graph, which is a contradiction. Hence, if $G_1\times G_2$ is a strong m-polar fuzzy graph, then at least G_1 or G_2 must be a strong m-polar fuzzy graph.*

Proposition 6.6. *If $G_1[G_2]$ is a strong m-polar fuzzy graph, then at least G_1 or G_2 must be strong.*

Proof. Follows from previous propositions.

Proposition 6.7. *Let $G=(V,A,B)$ be a strong m-polar fuzzy graph of a graph $G^*=(V,E)$. If $\overline{G}=(V,\overline{A},\overline{B})$ satisfies $\overline{A}=A$ and $\overline{B}$ is defined by, for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$*

$$p_i\circ\overline{B}(xy)=\begin{cases}0 & \text{if}\quad 0<p_i\circ B(xy)\le 1\\ min\{p_i\circ A(x),p_i\circ A(y)\} & \text{if}\quad p_i\circ B(xy)=0.\end{cases}$$

Then, $\overline{G}$ is a strong m-polar fuzzy graph of $\overline{G^*}=(V,\widetilde{V^2}-E)$.

Proof. Obviously, the m-polar fuzzy sets $\overline{A}$ and $\overline{B}$ satisfy $p_i\circ\overline{B}(xy)\le min\{p_i\circ A(x),p_i\circ A(y)\}$ for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$.

Now, let $xy\in\widetilde{V^2}-(\widetilde{V^2}-E)=E$. As G is a strong m-polar fuzzy graph, therefore we have for $i=1,2,\ldots,m$, $p_i\circ\overline{B}(xy)=min\{p_i\circ A(x),p_i\circ A(y)\}$.

If $B(xy)=0$, then for each $i=1,2,\ldots,m$, $p_i\circ B(xy)=0$. Therefore, $p_i\circ\overline{B}(xy)=min\{p_i\circ A(x),p_i\circ A(y)\}=p_i\circ B(xy)=0$, $i=1,2,\ldots,m$. Hence, $\overline{B}(xy)=0$.

If for $i=1,2,\ldots,m$, $0<p_i\circ B(xy)\le 1$ then $p_i\circ\overline{B}(xy)=0$, i.e., $\overline{B}(xy)=0$. Hence, for all $xy\in\widetilde{V^2}-(\widetilde{V^2}-E)=E$, $\overline{B}(xy)=0$. Therefore, $\overline{G}=(V,\overline{A},\overline{B})$ is an m-polar fuzzy graph of $G^*=(V,\widetilde{V^2}-E)$.

On the other hand, for all $xy\in\widetilde{V^2}-E$, we have by Definition 3.1, *$B(xy)=0$, i.e., for each $i=1,2,\ldots,m$, $p_i\circ B(xy)=0$. Then, we have for each $i=1,2,\ldots,m$, $p_i\circ\overline{B}(xy)=min\{p_i\circ A(x),p_i\circ A(y)\}$. Therefore, $\overline{G}$ is a strong m-polar fuzzy graph of $G^*=(V,\widetilde{V^2}-E)$.*

Definition 6.8. *The strong m-polar fuzzy graph $\overline{G}=(V,\overline{A},\overline{B})$ defined above, is called the complement of the strong m-polar fuzzy graph $G=(V,A,B)$.*

Definition 6.9. *A strong m-polar fuzzy graph G is called self-complementary if $G\cong\overline{G}$.*

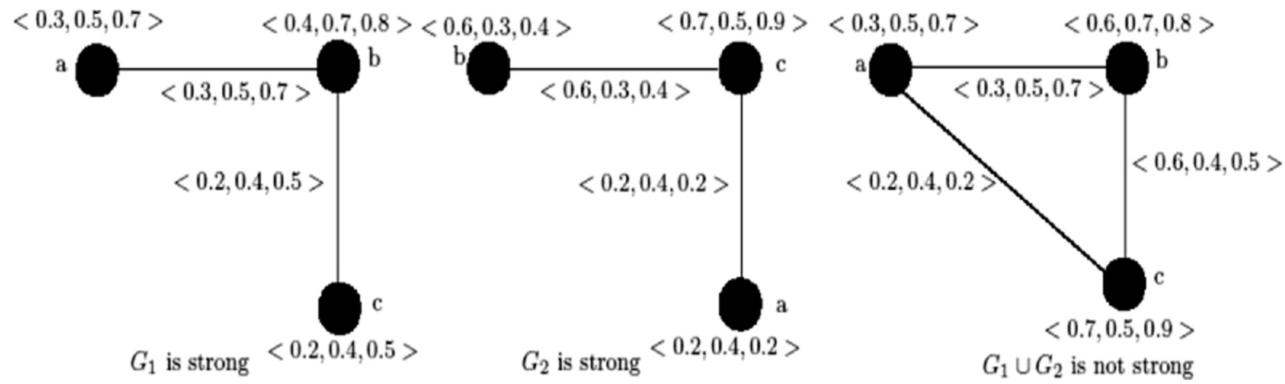

Fig. 8. The union of the two strong 3-polar graphs G_1 and G_2 is not strong.

Example 6.10. *Let* $G^* = (V, E)$ *be a graph where* $V = \{a, b, c, d\}$, $E = \{ab, ac, cd\}$ *and* $G = (V, A, B)$*(see Fig. 9) be a strong 3-polar fuzzy graph of* G^* *where* $A = \left\{\frac{<0.1,0.2,0.3>}{a}, \frac{<0.1,0.2,0.3>}{b}, \frac{<<0.1,0.2,0.3>}{c}, \frac{<0.1,0.2,0.3>}{d}\right\}$, $B = \left\{\frac{<0.1,0.2,0.3>}{ab}, \frac{<0.1,0.2,0.3>}{ac}, \frac{<0.1,0.2,0.3>}{cd}, \frac{<0,0,0>}{bd}, \frac{<0,0,0>}{ad}, \frac{<0,0,0>}{bc}\right\}$. *Then, G is self complementary. Let* $\overline{G} = (V, \overline{A}, \overline{B})$ *be the complement of G, where* $\overline{A} = A$, $\overline{B} = \left\{\frac{<0,0,0>}{ab}, \frac{<0,0,0>}{ac}, \frac{<0,0,0>}{cd}, \frac{<0.1,0.2,0.3>}{bd}, \frac{<0.1,0.2,0.3>}{ad}, \frac{<0.1,0.2,0.3>}{bc}\right\}$. *Let us now define a mapping* $\phi: V \to V$ *by* $\phi(a) = b$, $\phi(b) = c$, $\phi(c) = d$, $\phi(d) = a$. *Then, clearly,* ϕ *is a bijective mapping and* $A(a) = \overline{A}(\phi(a) = b)$, $A(b) = \overline{A}(\phi(b) = c)$, $A(c) = \overline{A}(\phi(c) = d)$, $A(d) = \overline{A}(\phi(d) = a)$. *Additionally,*

$$B(ab) = <0.1, 0.2, 0.3> = \overline{B}(\phi(a)\phi(b) = bc),\ B(ac) = <0.1, 0.2, 0.3> = \overline{B}(\phi(a)\phi(c) = bd),$$

$$B(cd) = <0.1, 0.2, 0.3> = \overline{B}(\phi(c)\phi(d) = ad),\ B(bc) = <0, 0, 0> = \overline{B}(\phi(b)\phi(c) = cd),$$

$$B(bd) = <0, 0, 0> = \overline{B}(\phi(b)\phi(d) = ac),\ B(ad) = <0, 0, 0> = \overline{B}(\phi(a)\phi(d) = ab).$$

Hence ϕ *is an isomorphism from G onto* $\overline{G}$*, i.e.,* $G \cong \overline{G}$*, which means that G is self complementary.*

Proposition 6.11. *Let* $G = (V, A, B)$ *be a strong m-polar fuzzy graph of the graph* $G^* = (V, E)$ *and* $\overline{G} = (V, \overline{A}, \overline{B})$ *be the complement of G. Then,* $p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$ *for all* $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Proof. Let $xy \in \widetilde{V^2}$. *If* $0 < p_i \circ B(xy) \leq 1$ *for each* $i = 1, 2, \ldots, m$*; then,* $xy \in E$ *by Definition 3.1. As G is strong, for* $i = 1, 2, \ldots, m$, $min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy) = 0 = p_i \circ \overline{B}(xy)$.

If for $i = 1, 2, \ldots, m$, $p_i \circ B(xy) = 0$, *then* $min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} = p_i \circ \overline{B}(xy)$. *Hence the result.*

Proposition 6.12. *Let G be a self-complementary strong m-polar fuzzy graph. Then, for all* $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$

$$\sum_{x \neq y} p_i \circ B(xy) = \frac{1}{2} \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}.$$

Proof. Let $G = (V, A, B)$ *be a self-complementary strong m-polar fuzzy graph. Then, for all* $xy \in E$, $i = 1, 2, \ldots, m$, $p_i \circ B(xy) = min\{p_i \circ A(x), p_i \circ A(y)\}$ *and there exists an isomorphism* $\phi: G \to \overline{G}$ *such that* $p_i \circ A(x) = p_i \circ \overline{A}(x)$ *for all* $x \in V$ *and* $p_i \circ B(xy) = p_i \circ \overline{B}(\phi(x)\phi(y))$ *for all* $xy \in \widetilde{V^2}$.

Let $xy \in \widetilde{V^2}$. *Then, by Proposition 6.11, for* $i = 1, 2, \ldots, m$,

$$p_i \circ \overline{B}(\phi(x)\phi(y)) = min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\} - p_i \circ B(\phi(x)\phi(y))$$

i.e., $p_i \circ B(xy) = min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\} - p_i \circ B(\phi(x)\phi(y))$.

Therefore,

$$\sum_{x \neq y} p_i \circ B(xy) + \sum_{x \neq y} p_i \circ B(\phi(x)\phi(y) = \sum_{x \neq y} min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\} = \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}$$

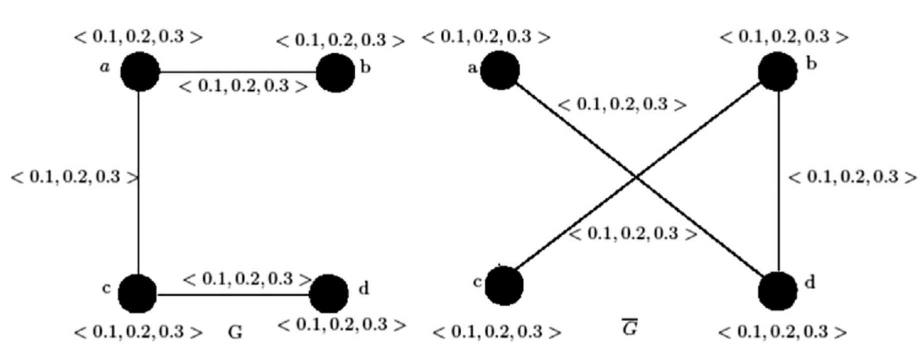

Fig. 9. Self-complementary 3-polar fuzzy graphs.

i.e.,

$$2\sum_{x\neq y} p_i\circ B(xy)=\sum_{x\neq y} min\{p_i\circ A(x),p_i\circ A(y)\}$$

i.e.,

$$\sum_{x\neq y} p_i\circ B(xy)=\frac{1}{2}\sum_{x\neq y} min\{p_i\circ A(x),p_i\circ A(y)\}$$

Proposition 6.13. *Let $G=(V,A,B)$ be a strong m-polar fuzzy graph of $G^*=(V,E)$. If $p_i\circ B(xy)=\frac{1}{2}min\{p_i\circ A(x),p_i\circ A(y)\}$ for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$, then G is self complementary.*

Proof. If $G=(V,A,B)$ is a strong m-polar fuzzy graph satisfying $p_i\circ B(xy)=\frac{1}{2}min\{p_i\circ A(x),p_i\circ A(y)\}$ for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$, then the identity mapping $I:V\to V$ is an isomorphism from G to $\overline{G}$. Clearly, I satisfies the first condition for isomorphism, i.e., $A(x)=\overline{A}(I(x))$ for all $x\in V$, and by Proposition 6.11, we have for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$,

$$p_i\circ\overline{B}(I(x)(y))=p_i\circ\overline{B}(xy)=min\{p_i\circ A(x),p_i\circ A(y)\}-p_i\circ B(xy)$$
$$=min\{p_i\circ A(x),p_i\circ A(y)\}-\frac{1}{2}min\{p_i\circ A(x),p_i\circ A(y)\}$$
$$=\frac{1}{2}min\{p_i\circ A(x),p_i\circ A(y)\}=p_i\circ B(xy).$$

i.e., $p_i\circ\overline{B}(xy)=p_i\circ B(xy)$ for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$ i.e., I also satisfies the second condition for isomorphism. Therefore, $G\cong\overline{G}$, i.e., G is self complementary.

From Proposition 6.12 and 6.13, we have the following result.

***Corollary 1** Let $G=(V,A,B)$ be a strong m-polar fuzzy graph of $G^*=(V,E)$. Then, G is self complementary if and only if $p_i\circ B(xy)=\frac{1}{2}min\{p_i\circ A(x),p_i\circ A(y)\}$ for all $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$.*

Proposition 6.14. *Let G_1 and G_2 be two strong m-polar fuzzy graphs. Then, $G_1\cong G_2$ if and only if $\overline{G_1}\cong\overline{G_2}$.*

Proof. Assume that $G_1\cong G_2$. Then, there exists a bijective mapping $\phi:V_1\to V_2$ satisfying $A_1(x)=A_2(\phi(x))$ for all $x\in V_1$ and $p_i\circ B_1(xy)=p_i\circ B_2(\phi(x)\phi(y))$ for all $xy\in\widetilde{V_1^2}$, $i=1,2,\ldots,m$.

Let $xy\in\widetilde{V_1^2}$. If for $i=1,2,\ldots,m$, $p_i\circ B_1(xy)=0$, then $p_i\circ\overline{B_1}(xy)=min\{p_i\circ A_1(x),p_i\circ A_1(y)\}=min\{p_i\circ A_2(\phi(x)),p_i\circ A_2(\phi(y))\}=p_i\circ\overline{B_2}(\phi(x)\phi(y))$.

If for, $0<p_i\circ B_1(xy)\le 1$, then $0<p_i\circ B_2(\phi(x)\phi(y))\le 1$. Therefore, $p_i\circ\overline{B_1}(xy)=0=p_i\circ\overline{B_2}(\phi(x)\phi(y))$. So, $G_1\cong G_2$.

Conversely, let $\overline{G_1}\cong\overline{G_2}$. Then, there exists a bijective mapping $\psi:V_1\to V_2$ satisfying $\overline{A_1}(x)=\overline{A_2}(\psi(x))$ for all $x\in V_1$ and $p_i\circ\overline{B_1}(xy)=p_i\circ\overline{B_2}(\psi(x)\psi(y))$ for all $xy\in\widetilde{V_1^2}$.

Let $xy\in\widetilde{V_1^2}$. If for $i=1,2,\ldots,m$, $p_i\circ B_1(xy)=0$, then

$$p_i\circ\overline{B_2}(\psi(x)\psi(y))=p_i\circ\overline{B_1}(xy)=min\{p_i\circ A_1(x),p_i\circ A_1(y)\}=min\left\{p_i\circ\overline{A_1}(x),p_i\circ\overline{A_1}(y)\right\}$$
$$=min\left\{p_i\circ\overline{A_2}(\psi(x)),p_i\circ\overline{A_2}(\psi(y))\right\}=min\{p_i\circ A_2(\psi(x)),p_i\circ A_2(\psi(y))\}$$

Again, $p_i\circ\overline{B_2}(\psi(x)\psi(y))=min\{p_i\circ A_2(\psi(x)),p_i\circ A_2(\psi(y))\}-p_i\circ B_2(\psi(x)\psi(y))$

So, $p_i\circ B_2(\psi(x)\psi(y))=0=p_i\circ B_1(xy)$, $i=1,2,\ldots,m$. If for $i=1,2,\ldots,m$, $0<p_i\circ B_1(xy)\le 1$, then $p_i\circ\overline{B_2}(\psi(x)\psi(y))=p_i\circ\overline{B_1}(\psi(x)\psi(y))=0$.

Thus we have,

$p_i\circ B_2(\psi(x)\psi(y))=min\{p_i\circ A_2(\psi(x)),p_i\circ A_2(\psi(y))\}-0=min\{p_i\circ\overline{A_2}(\psi(x)),p_i\circ\overline{A_2}(\psi(y))\}=min\{p_i\circ\overline{A_1}(\psi(x)),p_i\circ\overline{A_1}(\psi(y))\}=p_i\circ B_1(xy)$. Hence $G_1\cong G_2$.

7. Applications

Fuzzy graphs of the 1-polar type are nothing more than the most familiar fuzzy graphs and have many applications for cluster analysis and solving fuzzy intersection equations, database theory, problems concerning group structure, and so on. The further possible applications of *m*-polar fuzzy graphs in real-world problems can be viewed in the case of bipolar fuzzy graphs, i.e., 2-polar fuzzy graphs. Bipolar fuzzy graphs have many applications in social networks, engineering, computer science, database theory, expert systems, neural networks, artificial intelligence, signal processing, pattern recognition, robotics, computer networks, medical diagnosis and so on. Additionally, *m*-polar fuzzy graphs ($m>2$) are very useful in many decision-making situations. This occurs when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leader. For instance, consider the case of a company. In a company, a group of people decides which product design to manufacture. In such a case, different product designs can be taken as nodes. An edge is drawn between two nodes if there is some *m*-polar fuzzy relationship between them. We assume that the membership value of each node represents the degrees of preference of the product design corresponding to the group of people of the company. The degrees of preference (within [0,1]) represent the individual preferences of the people. Thus, a node has multi-preference degrees corresponding to a product design. Similarly, the degree of relationship between the nodes measures the edge relationship value. Between two product designs, one design may have a better appearance, may be in very high demand, may be cheap, and so on. Therefore, there is multipolar information between two product designs. This type of network is an ideal example of *m*-polar fuzzy graphs. It is very important for a company to decide which product design to manufacture so that they can make as great a profit as possible. A very good product design is readily accepted by customers if it is also inexpensive. The determination of which product design to manufacture is called the decision-making problem. By taking the very good decision (very good product design), one company can spread their product all over the world, keeping in mind that the product design is very good, in demand, cheap, easily accessible, and so on. Moreover, the results of *m*-polar fuzzy graphs can be applicable in various areas of engineering, computer science, artificial intelligence, neural networks, social networks, and so on.

8. Conclusions

Graph theory is an extremely useful tool for solving combinatorial problems in different areas, including algebra, number theory, geometry, topology, operations research, optimisation and computer science. Because research on or modelling of real world problems often involve multi-agent, multi-attribute, multi-object, multi-index, multi-polar information, uncertainty, and/or process limits, m-polar fuzzy graphs are very useful. The m-polar fuzzy models give increasing precision, flexibility, and comparability to the system compared to the classical, fuzzy and bipolar fuzzy models. Therefore, we have studied several important results of m-polar fuzzy graphs. Our next plan is to extend our research work to m-polar fuzzy intersection graphs, isomorphisms on m-polar fuzzy graphs, m-polar fuzzy interval graphs, m-polar fuzzy hypergraphs, and so on.

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