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SPLINE ANALYSIS OF HYDROGRAPHIC DATA

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Abstract—The basic problem involved in determining where the ship can not go is an attempt to reconstruct the sea bed. The interpolation of points necessary to reconstruct the sea bed was done using a bicubic spline. This method was chosen because of the similarities between the boundary conditions believed to be characteristic of the modeling problem and those of the natural spline. These include the continuity of the first and second derivatives, and the minimum curvature exhibited by the spline method which is characteristic of the sea bottom. The major problem faced in modeling the sea bed was selecting the extra data points needed in order to find a meaningful solution. This selection was done both by intuition and by constructing splines to model the possible behavior along a straight line. The results were two different models: a ridge model, characterized by a single shallow ridge in the center of the region; and a hill model, characterized by two smaller ridges. By varying one of these extra data points (called critical points), several models of both these extremes as well as intermediate models were generated. However, it was found that the number of given points did not permit a definitive model. Data was needed inside the region, especially at the critical points and at the exterior points in order to better define the boundary. The boundary could not be reliably determined since our spline model does not allow for accurate extrapolation. Thus, the model, although close to what is believed to be the correct model, is not good enough to allow for navigation because of the limited number of given data points.

INTRODUCTION

The model studied in this paper is the detailed determination of the depth of a body of water. Initial data consists of 14 (x, y) water surface coordinates and respective z coordinate depths. Applying spline interpolation techniques to this data, a detailed three-dimensional construction of the sea bed is obtained. The construction is obtained in the form of contour maps which can be used as depth navigational charts. We will focus our attention on a particular boat with a 5 ft draft.

ASSUMPTIONS

In constructing the contour map, an interpolation, based on 14 given depths must be made to other regions within the (x, y) domain of the initial data. Thus we are faced with the problem of finding a surface $z = F(x, y)$ which represents the sea bed. This function does not have to be smooth or continuous (i.e. there can exist sharp peaks, rock formations, coral reefs, etc). These possibilities will be excluded and it will be assumed that the sea bed is smooth and continuous. To be more rigorous, the first and second derivatives of $F(x, y)$ are continuous with no jump discontinuities. Physically, this would correspond to a fine gravel or sand bottom, which is common in shallow water. It now remains to find a physical principle that gives us insight into the general shape of a surface acted upon by water. Since the region being considered is shallow, it will be assumed that the water is flowing over the sea bed. Therefore the water will erode any structures which perturb the flow from its minimum lateral kinetic energy (i.e. the kinetic energy of motion perpendicular to the direction of flow). This implies that the most favorable sea bed that minimizes this function is a plane surface. The direction of motion is not perturbed except by otherwise negligible friction effects between water and the sea bed. Therefore any sea bed which is not a plane surface will tend towards one through erosion effects. This model is of course idealized and does not take into account such factors as sedimentation or large unerodible structures. Thus, there are two conditions imposed on the sea bed:

(1) At least the first two derivatives are continuous.

(2) Given the initial structure, the surface constructed is of minimum curvature.

CHOICE OF INTERPOLATING SCHEME

Interpolation is the fitting of a curve through a given set of data points; in this instance, by the use of polynomial approximations. Such a polynomial would have to be of fairly low degree to reflect the fact that the sea bed is flat, and the method of cubic splines does this well. The idea of a spline through a set of data points is best illustrated by imagining a straight flexible rod being placed over a set of points and putting weights on the rod to force it to pass through all the points. It is the straightest line through the points. This is precisely one of the conditions of the model. Cubic splines are differentiable piecewise polynomial functions on an interval $[X_0, X_n]$ where $X_0 < X_1 < \cdots < X_n$ are called nodes. The function is obtained by fitting a cubic polynomial between each successive pair of nodes. This is accomplished by fitting a cubic on $[X_0, X_1]$ agreeing with the function at X_0 and X_1, another cubic on $[X_1, X_2]$ agreeing with the function at X_1 and X_2, etc. A

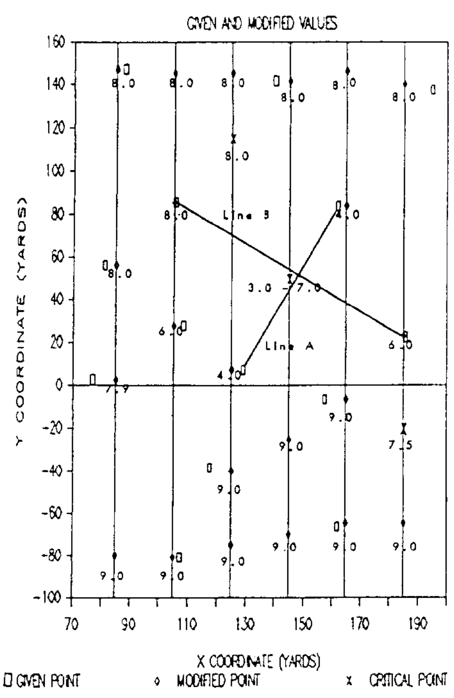


Fig. 1. Initial data coordinates for spline analysis.

general cubic polynomial involves four constants. Thus there is sufficient flexibility to ensure that not only the interpolant is continuously differentiable on the interval, but also that it has continuous second derivatives on the interval, even at the nodes. Thus, cubic splines satisfy the two conditions of the model; they have continuous first and second derivatives and can furnish the straightest curve fitting the given data points. For this model the natural spline boundary condition is chosen. This corresponds to the second derivatives being zero at the end points X_0 and X_n. Such a condition allows for the maximum flexibility in determining the straightest curve through the given nodes.

BICUBIC SPLINE PROCEDURE

Since the sea bed has two independent variables, it requires generalization of the one-dimensional spline method to two dimensions. This is called a bicubic spline. It is necessary to arrange the data points in an $m \times n$ grid formation $= \{(X_i, Y_j) \mid 0 < i < n, 0 < j < m\}$. From this it is possible to find the value of the interpolating surface $F(X, Y)$ at (X', Y') by running one-dimensional cubic spline interpolations along the grid lines $X = X_i$

Table 1. Initial grid data (nodes)

(a) The data are given as (y coordinate, depth).
The points A, B and C are critical points which determine the form of the model. A = CR (145,50) is the dominant critical point, with $3 < Z < 7$ the depth taken at the critical point.
The points B = CR (125,115) and C = CR (185, −20) are milder constraints on the model. BZ may vary between 5 and 7.5, and C is generally taken as shown (although it can be moved and have its z coordinate changed).
The columns correspond to the x coordinates: Col. 1: X = 85; Col. 2: X = 105; Col. 3: X = 125; Col. 4: X = 145; Col. 5: X = 165; Col. 6 X = 185

No\Col no	1	2	3	4	5	6
1	(−80,9)	(−81,9)	(−75,9)	(−70,9)	(−65,9)	(−65,9)
2	(3,7,9)	(28,6)	(−40,9)	(−25,9)	(−6,5,9)	C
3	(56,5,8)	(85,5,8)	(7,5,4)	A	(84,4)	(22,5,6)
4	(147,8)	(145,8)	B	(141,5,8)	(146,8)	(140,8)
5			(145,8)			

(b) Example: of spline calculations for determination of depth of A(50,Z)

The three one-dimensional spline calculations:
1. Line A: Ridge model result CR (145,50), Z = 3; 2. Line B: Hill model result CR (145,50), Z = 6.9; 3. Line C: result CR (145,60), Z = 4. 3 or CR (145,70) = Z = 4

Line	Coordinates X	Y	Depth	Description
A	117.5	−38.5	9	Given value
	129	7.5	4	Given value
	162	84	4	Given value
	185	140	8	Modified value
	145	50	3.02	Spline result
B	185.5	22.5	6	Given value
	105	85.5	8	Given value
	145	50	6.87	Spline result
C	75	−5	8	Modified value
	108.5	28	6	Given value
	162	84	4	Given value
	215	140	8	Modified value
	145	40	4.01	Spline result

through the data $F(X_i, Y_j)$, $0 < j < m$ and evaluating each spline at $Y = Y'$. Then a cubic spline interpolation is performed along $Y = Y'$, through the values obtained above, and then this is evaluated at $X = X'$.

A one-dimensional cubic spline program[1] was modified so as to perform the bicubic spline interpolation as outlined above. The program listing is given in the Appendix. The region (75,200) × (−50, 150) was divided into six columns parallel to the y-axis. This orientation was chosen because the maximum number of nodes (given data points) could be obtained and they were better distributed along this direction [refer to Fig. 1 and Table 1(a)]. The node grid input into the program consists of initial data points along the appropriate columns. The position values assigned to the columns are given in Table 1, along with the y and z coordinates of the nodes and their grouping into columns. Most of the nodes used are given data points, or given data points slightly extrapolated to correspond to the center of the column. The program then performed a spline analysis down the x columns and interpolated 41 equally spaced points per column. These points were then used to construct a spline along y = constant with 20 equally spaced points interpolated.

The output was in the form of a contour map with a resolution of 20 × 41 extrapolated points. The contour map resolution 20 × 41 was selected because it is the compromise between tolerable computation time and maximum resolution.

MODEL PARAMETERS

The data given is not sufficient. Extra points must be given in order to clarify the possible nature of the sea bed surface. Observing the general form of the surface based of the given data, it becomes apparent that two possible structures exist. These structures are centered about the two given data points with depths $z = 4$. These structures can be either a long ridge or two smaller hills each centered about the points (129, 75) and (162, 84). The point that discriminates between these two models is the critical point CR(145, 50). A variation in depth of this point will decisively alter the results. To determine the

Table 2. Impassable area sensitivity analysis [for depths (5ft)]
(Total grid area = 820 sq. units)

Model	Depth %	Central peak	Other	Total area	Percent of 820
Ridge	3.0				
CR(185, −20)	−8.0	140	6	146	17.80
CR(185, −20)	9.0	141	14	155	18.90
CR(185, −85)	8.0				
CR(192,84)	6.4	137	22	157	19.15
CR(192,84)	6.0	113	9	122	14.88
CR(192,84)	7.0	108	14	122	14.88
CR(185, −20)	7.5				0.00
CR(192,85)	6.0	113	7	120	14.63
CR(125,5.5)	5.5	157	5	163	19.88
Intermediate					
CR(145 50)	4.0	130	4	134	16.34
CR(145 50)	5.0	113	3	116	14.15
CR(145 50)	6.0	85	0	85	10.37
Hill					
CR(145 50)	7.0	86	0	86	10.49

**CR(X, Y) = critical point at position x,y Other peak centralized at (200,25) *Standard model CR(185, −20) % = 8, CR(125,115) % = 7.5

possible range of depths for this point a one-dimensional spline analysis was carried out along the major trend lines of the sea bed. Trend lines for the critical point CR(145, 50) are the lines between the two points where $z = 4$ (trend A) and the line that is perpendicular to this direction and passes through (185.5, 22.5) and (105, 85.5) (trend B, see Fig. 1). A spline analysis is performed along these two lines to determine the range of variation of depth at the critical point [see Table 1(b)]. Two other milder critical points exist at CR(125, 115) and CR(185, −20). Analysis similar to the one performed on CR(145, 50) gave depth ranges as indicated in Table 1(a). These two critical points are less sensitive to variations.

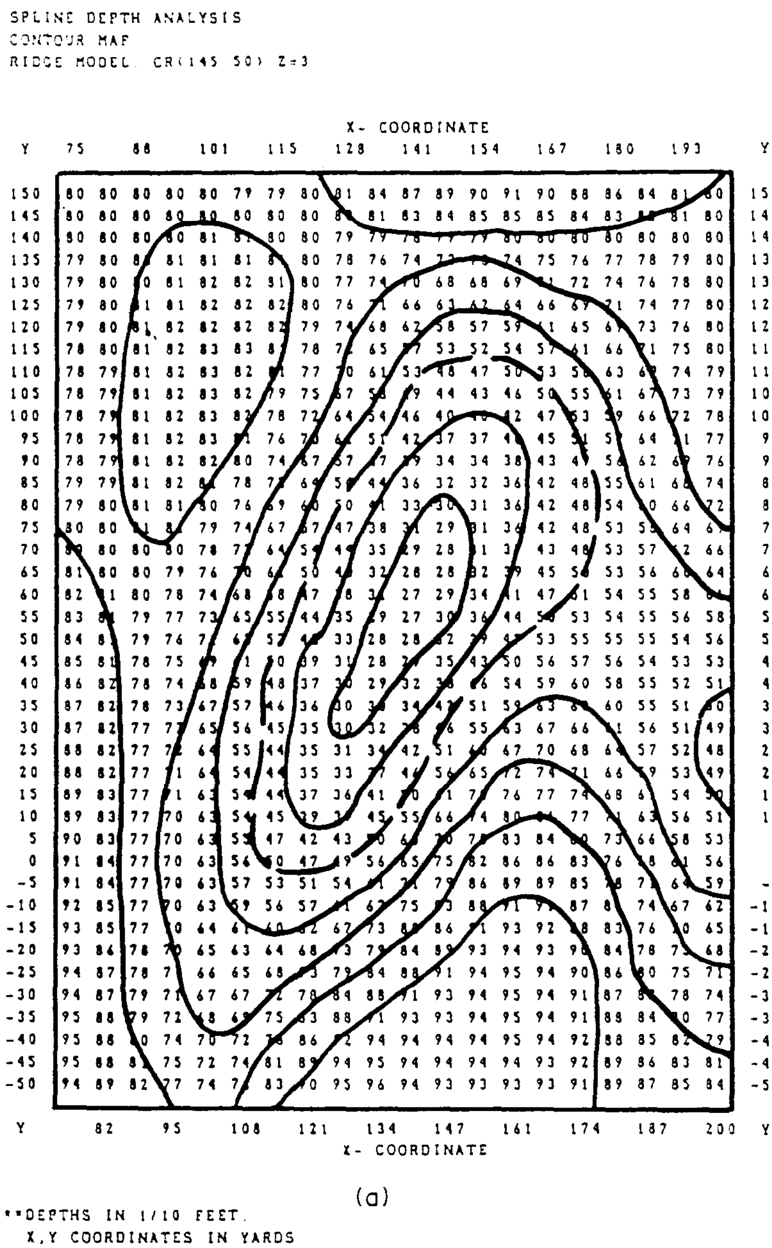
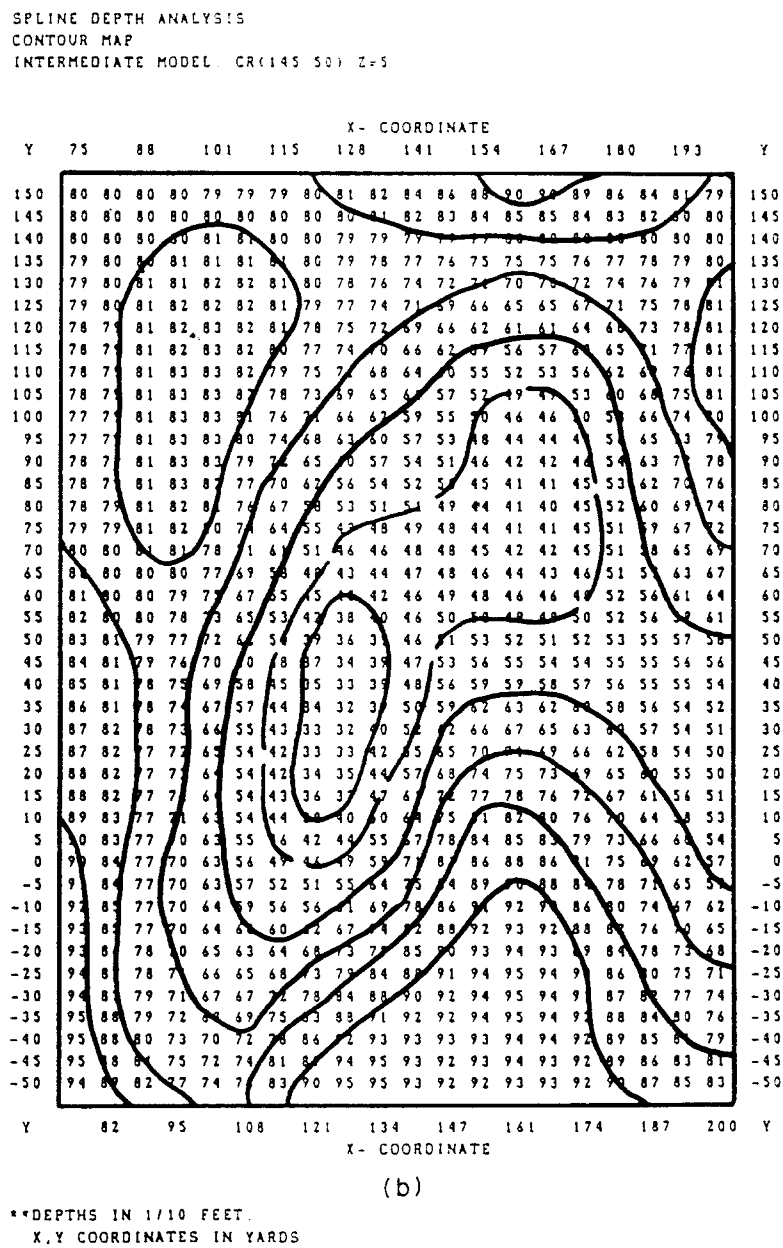


Fig. 2. (a) Spline depth analysis contour map. Depths in $\frac{1}{10}$ ft; X, Y coordinates in yards. Ridge Model: CR(145, 50) $Z = 3$; (b) Intermediate Model: CR(145, 50) $Z = 5$; (c) Hill Model: CR(145, 50) $Z = 7$.

Thus, for the analysis of the main critical point variations, they are assigned values: for CR(125, 115), z = 7.5 and for CR(185, −20), z = 8.

ANALYSIS

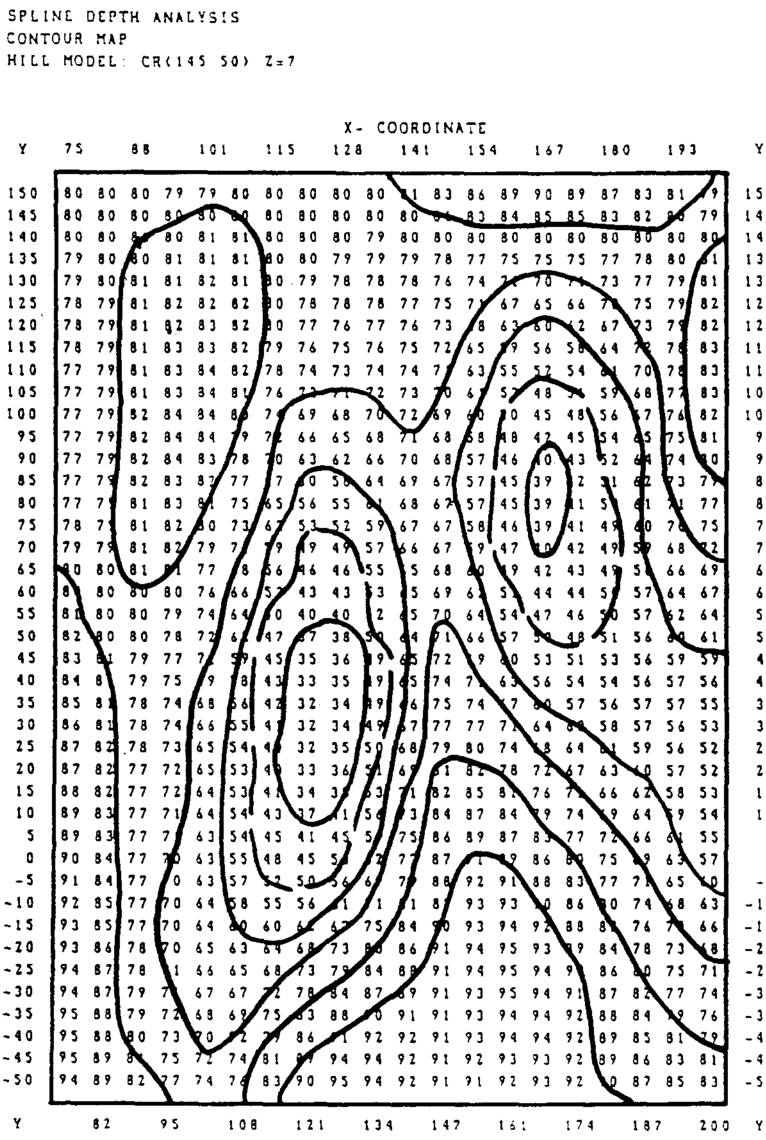
Thirteen bicubic spline analyses were run. The results are summarized in Table 2. Most runs were performed with the aim of determining the effects from the variation of the depths of the mild critical points. These effects were not the main purpose of the analysis but were used to determine sensible values of the depths of the milder critical points. The



(b)

Fig. 2. (continued)

primary analysis was the variation of the depth of the main critical point. Figures 2(a), (b), (c) are contour maps that represent the topological features of the sea bed. Fig. 2(a) is the ridge model (i.e. $z = 3$) in which the main feature is a large central ridge. The area within the dashed contour line represents the area impassable to the boat. This area represents about 17.8% of the total area in consideration. Figure 2(c) is the hill model with $z = 7$. It is observed that there are two distinct impassable shallow regions with a small passable ridge between them. Figure 3 is a plot of the percent of the area impassable to the ship as a function of depth of the main critical point CR(145, 50). Note that the contour map for the ridge model shows a second shallow region at (200, 30). This result was totally unexpected and further analysis [variation of critical point CR(185, −20) and



(c)

Fig. 2. (continued)

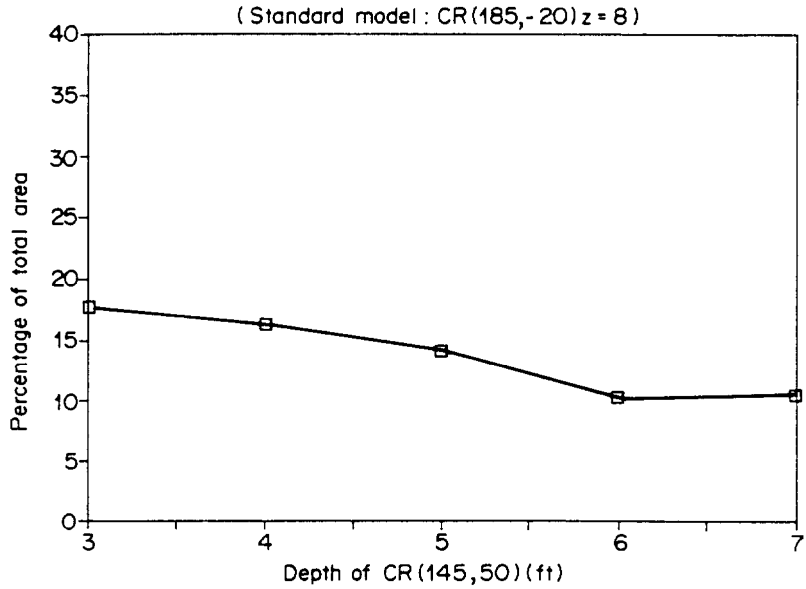


Fig. 3. Impassable area vs. depth of CR(145, 50).

other points in column 6] showed it to be stable to variations. Thus this must be a real result.

CONCLUSION

The single largest source of error in the analysis is the lack of sufficient data. If a larger set of depth measurements are systematically made, then we would feel more confident telling the captain of the boat to use our contour maps. As it stands, a decision as to which model is correct, the ridge or hill model, could be obtained by making a depth measurement at the critical point CR(145, 50). Further analysis with larger sets of data would determine whether the overall assumptions made are valid. We were in the process of running the spline program on data taken from real contour maps and making further comparisons before time ran out.

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REFERENCES

1. W. H. Press *et al.*, *Numerical Recipes*. Cambridge U. P., Cambridge.
2. R. L. Burden, J. D. Faires and A. C. Reynolds, *Numerical Analysis*, 2nd Edn. Prindle, Weber & Schmidt, Boston, MA (1981).
3. J. F. A. Sleath, *Sea Bed Mechanics* (1984).

APPENDIX

Program listing

```
Source Line          IBM Personal Computer BASIC Compiler V1 00

REM TWO DIMENISIONAL SPLINE WITH NATURAL BOUNDARY CONDITIONS

REM GXY-INPUTTED GRID
REM ZXY- INPUTTED DEPTH
REM ZA- SPLINE CALCULATED DEPTH
REM Z2- CALCULATED SECOND DERIVATIVE
REM RT,Z,U- TEMP VARIABLE USED FOR CALC OF SPLINE
REM NGX- NUMBER OF DEPTH MEASUREMENTS ALONG X-COLUMNS
REM ZXT,YT- TEMPORARY CALCULATED DEPTH AND Y POSITION ALONG X-C
OLUMNS
REM XT- X POSITION ALONG Y-COLUMNS
REM NX,NY- NUMBER OF DEPTHS CALC ALONG X AND Y AXIS
REM XCOL- CENTER POSITION OF X-COLUMNS /85/105/125/145/165/185
    DEFINT I-N
    DEFDBL A-H, O-Z
    DIM GXY(6,15),ZXY(6,15),ZA(26,44),RT(44),Z(44),Z2(44),ZXT(6
,44)
    DIM YT(44),XT(26),NGY%(6),U(44),XCOL(6)

REM RUN PROGRAM BLOCK
    WIDTH "LPT1:",255
    GOSUB 100
REM CALCULATE SPLINE FOR COLUMNS 1-6 (X)
    INCY!= 200/(NY-1)
    INCX!= 125/(NX-1)
    FOR JL=1 TO 6
    N=NGY%(JL)
    FOR JJ=1 TO N
    RT(JJ)=GXY(JL,JJ): Z(JJ)= ZXY(JL,JJ): NEXT JJ
    GOSUB 120   'CALC SECOND DERIVATIVE ROUTINE Z2
    RT= -50
    FOR JJ=1 TO NY
     GOSUB 145  'SPLINE CALC ROUTINE
     ZXT(JL,JJ)=Z
     YT(JJ)= RT
     RT=RT + INCY!
    NEXT JJ
    NEXT JL

REM CALCULATE SPLINE ALONG Y=CONST
    N=6
    FOR JL=1 TO NY
    FOR JJ=1 TO N
    RT(JJ)= XCOL(JJ): Z(JJ)= ZXT(JJ,JL)
    NEXT JJ
    GOSUB 120  'CALC SECOND DERIVATIVES ROUTINE Z2
    RT= 75
    FOR JJ=1 TO NX
     GOSUB 145   'SPLINE CALC ROUTINE
     ZA(JJ,JL)=Z
     XT(JJ)= RT
     RT=RT+INCX!
    NEXT JJ
    NEXT JL
REM OUTPUT ROUTINE
    LPRINT CHR$ (27);"Q,4,$",
    LPRINT "SPLINE DEPTH ANALYSIS"
    LPRINT "CONTOUR MAP"
    LPRINT A$ LPRINT LPRINT
    LPRINT "                              X- COORDINATE"
    LPRINT "       75    88    101   115   128   141   154   16
7   180   193"
    LPRINT "      ---------------------------------------------
------------------"
    LPRINT " Y  :
                 :"
    FOR K= NY TO 1 STEP -1
    LPRINT CHR$(27);"Q,4,$";
    LPRINT USING "### : ";YT(K);
    FOR I=1 TO NX
    IF ZA(I,K)(=0 THEN LPRINT CHR$(27),"Q,2,$", GOSUB 80:GOSUB
90:LPRINT CHR$(27);"Q,4,$";:COSUB 95 GOTO 96
    IF ZA(I,K)(=3 04 THEN LPRINT CHR$(27),"Q,1,$";:GOSUB 80 GOT
O 96
    IF ZA(I,K)(=4 04 THEN LPRINT CHR$(27),"Q,1,$"; GOSUB 80 GOS
UB 90 LPRINT CHR$(27);"Q,2,$";:GOSUB 95 GOTO 96
    IF ZA(I,K)(=5 04 THEN LPRINT CHR$(27),"Q,2,$";:GOSUB 80 GOT
O 96
    IF ZA(I,K)(=6.04 THEN LPRINT CHR$(27),"Q,1,$",:GOSUB 80 GOS
UB 90 LPRINT CHR$(27);"Q,3,$";:GOSUB 95 GOTO 96
    IF ZA(I,K)(=7.04 THEN LPRINT CHR$(27),"Q,3,$";:GOSUB 80 GOT
O 96
    IF ZA(I,K)(=8.04 THEN LPRINT CHR$(27),"Q,2,$";:GOSUB 80 GOS
UB 90:LPRINT CHR$(27);"Q,3,$"; GOSUB 95 GOTO 96
    IF ZA(I,K)(=100  THEN LPRINT CHR$(27),"Q,4,$";:GOSUB 80 GOT
O 96
80  WIDTH "LPT1:",255
    LPRINT USING" ##";ZA(I,K)*10;
    RETURN
90  WIDTH "LPT1:",255
    LPRINT CHR$(8),:LPRINT CHR$(8),
    WIDTH "LPT1:",255
    RETURN
95  WIDTH "LPT1:",255
    LPRINT USING"##",ZA(I,K)*10,
    RETURN
96  WIDTH "LPT1:",255
    NEXT I
    LPRINT CHR$(27);"Q,4,$";
    LPRINT USING "  :###" ;YT(K)
    NEXT K
    LPRINT " Y  :
                 :"
    LPRINT "      ---------------------------------------------
------------------"
    LPRINT "          82    95    108   121   134   147   161
 174   187   200"
    LPRINT:LPRINT
    LPRINT"**DEPTHS IN 1/10 FEET "
    LPRINT"  X,Y COORDINATES IN YARDS."
    LPRINT CHR$(12)
    STOP

REM INPUT VARIABLES ROUTINE
100 PRINT "2D NATURAL SPLINE PROGRAM"
    INPUT "TITLE  ";A$
    INPUT "NUMBER OF PONTS ALONG X AXIS 1-20: "; NX
    INPUT "NUMBER OF PONTS ALONG Y AXIS 1-40: "; NY
    PRINT "INPUT CENTER POSITION FOR X-COLUMNS"
    FOR J=1 TO 6
    PRINT J;:INPUT"";XCOL(J) NEXT J
    FOR I=1 TO 6
    PRINT:PRINT "COLUMN "I
    INPUT "NUMBER OF DEPTH MEASUREMENTS ALONG X COLUMN  ",NGY%(
I)
    PRINT "INPUT Y POSITION, DEPTH"
    FOR K= 1 TO NGY%(I)
    PRINT K;:INPUT"",GXY(I,K),ZXY(I,K)
    NEXT K
    NEXT I
   LPRINT:LPRINT
110 RETURN

REM SECOND DERIVATIVE CALCULATION (Z2) ROUTINE, NATURAL BOUNDAR
Y CONDITIONS
120 Z2(1)=0: U(1)=0
130 FOR I= 2 TO N-1
     SIG= (RT(I)-RT(I-1))/(RT(I+1)-RT(I-1))
     P= SIG*Z2(I-1)+2
     Z2(I)= (SIG-1)/P
     U(I)= (6*((Z(I+1)-Z(I))/(RT(I+1)-RT(I))-(Z(I)-Z(I-1))/(RT(
I)-RT(I-1)))/(RT(I+1)-RT(I-1))-SIG*U(I-1))/P
    NEXT I
140 Z2(N)= 0
    FOR K= N-1 TO 1 STEP -1
     Z2(K)= Z2(K)*Z2(K+1)+U(K)
    NEXT K
    RETURN

REM SPLINE CALCULATION ROUTINE (GIVEN RT,Z)
145 KLO= 1
    KHI= N
150 IF (KHI-KLO)) 1 THEN K= (KHI+KLO)/2 ELSE 160
    IF RT(K)) RT THEN KHI= K ELSE KLO= K
    GOTO 150
160 H= RT(KHI)-RT(KLO)
    IF H=0 THEN PRINT "Error  Bad input"  STOP
    A= (RT(KHI)-RT)/H
    B= (RT-RT(KLO))/H
    Z= A*Z(KLO)+B*Z(KHI)+((A^3-A)*Z2(KLO)+(B^3-B)*Z2(KHI))*(H^2
)/6
    RETURN
END
```