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Iterative processes related to Riordan arrays: The reciprocation and the inversion of power series

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ABSTRACT

We point out how the Banach Fixed Point Theorem, together with the Picard successive approximation methods yielded by it, allows us to treat some mathematical methods in combinatorics. In particular we get, in this way, a proof of and an iterative algorithm for deriving the Lagrange Inversion Formula.

1. Introduction: beginning with a simple question

The results in this paper are consequences of special interpretations, as fixed point problems, of the two classical reversion processes in the realm of formal power series: *reciprocation*, i.e. the reversion for the Cauchy product, and *inversion*, i.e. the reversion for the composition of series.

The case of reciprocation was studied in [4,5]. To unify our approach, we also survey herein some of the previous results on this topic.

The aim of this paper is to show how the Picard successive approximation method yielded by the Banach Fixed Point Theorem and a mild generalization allow us to treat some mathematical methods in combinatorics, getting thus some associated algorithms—in particular:

(1) To construct all elements in the Riordan group as a consequence of the iterative process obtained for calculating the reciprocal of any power series admitting it. Also to present an algorithm and one pseudo-code description for it.
(2) To construct, approximatively, the inverse of any power series admitting one, in such a way that the Lagrange Inversion Formula can be first predicted and finally proved; we also describe the corresponding algorithm.

For completeness we are going to recall the metric fixed point theorems that we will use; see, for example, [1] for the first one and [10], page 212, for the second one.

The Banach Fixed Point Theorem (BFPT). Let (X, d) be a complete metric space and $f : X \to X$ contractive. Then f has a unique fixed point x_0 and $f^n(x) \to x_0$ for every $x \in X$.

The Generalized Banach Fixed Point Theorem (GBFPT). Let (X, d) be a complete metric space. Suppose $\{f_n\}_{n\in\mathbb{N}} : X \longrightarrow X$ is a sequence of contractive maps with the same contraction constant α and suppose that $\{f_n\} \longrightarrow f$, point to point. Then f is α-contractive and for any point $z \in X$ we have the sequence $\{f_n \circ \cdots \circ f_1(z)\} \xrightarrow[n\to\infty]{} x_0$, where x_0 is the unique fixed point of f.

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Our framework is the following. We consider $\mathbb{K}$ a field of characteristic zero and the ring of power series $\mathbb{K}[[x]]$ with coefficients in $\mathbb{K}$. If g is any series given by $g = \sum_{n\geq 0} g_n x^n$, we recall that the order of g, $\omega(g)$, is the smallest nonnegative integer number n such that $g_n \neq 0$ if any exists. Otherwise, that is if $g = 0$, we say that its order is ∞. It is well-known, and easy to prove, that the space $(\mathbb{K}[[x]], d)$ is a complete ultrametric space where the distance between f and g is given by $d(f, g) = \frac{1}{2^{\omega(f-g)}}$, $f, g \in \mathbb{K}[[x]]$; see [7] and also [4]. Here we understand that $\frac{1}{2^\infty} = 0$. Moreover the distance between f and g is less than or equal to $\frac{1}{2^{n+1}}$, i.e. $d(f, g) \leq \frac{1}{2^{n+1}}$, if and only if their n-degree Taylor polynomials are equal, $T_n(f) = T_n(g)$. Finally the sum and product of series are continuous if we consider the corresponding product topology in $\mathbb{K}[[x]] \times \mathbb{K}[[x]]$. See for example [7,4,5] on these topics.

In this paper $\mathbb{N}$ represents the set of natural numbers including 0. Throughout the paper we represent by $\frac{1}{f}$ the product inverse of f and by f^{-1} the compositional inverse, when they exist.

This work is motivated by the following question:

Question 1. Can we sum the arithmetic–geometric series $\sum_{k=1}^{\infty} kx^{k-1}$ using the Banach Fixed Point Theorem?

We can sum the geometric series using **BFPT**. A visual proof of this can be found in [13]. Herein we recall an analytic proof. The peculiar name of the following function will be justified later on. We consider

$$\begin{array}{rccc} h_{m,1} : & (\mathbb{K}[[x]], d) & \to & (\mathbb{K}[[x]], d) \\ & t & \mapsto & xt + 1. \end{array}$$

We iterate at $t = 0$ and we obtain

$$h_{m,1}(0) = 1$$
$$h_{m,1}^2(0) = x + 1$$
$$h_{m,1}^3(0) = x^2 + x + 1$$
$$h_{m,1}^4(0) = x^3 + x^2 + x + 1,$$

that is,

$$h_{m,1}^{n+1}(0) = \sum_{k=0}^{n} x^k.$$

As the fixed point of $h_{m,1}$ is the solution of $xt+1 = t$, we have $t = \frac{1}{1-x}$. Since $h_{m,1}$ is contractive, in fact $d(h_{m,1}(t_1), h_{m,1}(t_2)) \leq \frac{1}{2} d(t_1, t_2)$, then from **BFPT** we deduce

$$h_{m,1}^{n+1}(0) = \sum_{k=0}^{n} x^k \xrightarrow[n \to \infty]{} \frac{1}{1-x}$$

which is the unique fixed point of the function; in this case, $h_{m,1}(t) \equiv h_1(t) = xt + 1$.

Now it is natural to ask Question 1.

We organize the paper in the following way:

In Section 2 we apply **BFPT** to a suitable function related to Question 1. We do not answer the question in this way but we find an interesting arithmetical triangle. Later, and using **GBFPT**, we answer the question. Actually, we construct the whole Pascal triangle by this method; see [4,5].

In Section 3 we generalize the method above to construct the Pascal triangle, finding thus a way to construct arithmetical triangles $T(f \mid g)$ for any pair of series f and g with non-null independent terms. Using the usual product of matrices, we identify the well-known Riordan group; see [4].

In the procedure described above, we obtain a new parametrization of the elements in the Riordan group and thus a new notation different from the usual ones. In Section 4, we try to justify the use of our notation as an alternative to the usual notation. Recall that a Riordan array $T(f \mid g)$ with $g(0) \neq 0$ is an infinite lower triangular matrix such that the generating function of the j-th column is $\frac{x^j f}{g^{j+1}}$, beginning at $j = 0$. Equivalently, the action yielded in $\mathbb{K}[[x]]$ is

$$T(f|g)(s) = \frac{f}{g} s\left(\frac{x}{g}\right) \quad \text{which represents the power series } \frac{f(x)}{g(x)} s\left(\frac{x}{g(x)}\right).$$

The elements of the Riordan group are the Riordan arrays, with $f(0) \neq 0$, and the operation is the usual product of matrices.

Our method of construction and our notation allow us to explain easily a way to add and delete columns suitably in a Riordan array to get another one. For triangles of a concrete kind, those denoted by $T(1 \mid a + bx)$, we can calculate the inverse by just adding adequately new columns to those triangles. In fact, this is an *elementary operations* method. We end this section by giving expressions involving the so called A and Z sequences of a Riordan array. These expressions are given in terms of our notation and they are related to the inversion in the Riordan group.

In Section 5 we give the main new results of this paper. We display an algorithm for constructing the inverse of a series and we show the relation with the Lagrange Inversion Formula. In particular we prove that the Banach Fixed Point Theorem gives rise to the Lagrange Inversion Formula.

2. Two answers: a curious triangle and an iterative method

To answer the previous question, in this section we are going to recall briefly some examples and tools widely studied in [4,5].

It can be easily shown—see page 2257 in [5]—that there is no first-degree polynomial with coefficients in $\mathbb{K}[[x]]$, $f(t) = g(x)t + h(x)$, and no point, x_0, such that its $(n+1)$-st iteration coincides with the partial sum of the arithmetic–geometric series, that is $\nexists f, x_0/\forall n, f^{n+1}(x_0) = \sum_{k=0}^{n}(k+1)x^k$.

In view of this, we are going to iterate a polynomial whose fixed point is the sum of the arithmetic–geometric series, that is $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$. Since the equality $t = \frac{1}{(1-x)^2}$ can be converted into $t = 1 + (2x - x^2)t$, we consider the polynomial $f(t) = 1 + (2x - x^2)t$, with coefficients in $\mathbb{K}[[x]]$, and we initiate the iteration process at $t = 0$:

$$
\begin{aligned}
&f(0) = 1\\
&f^2(0) = 1 + 2x - \mathbf{x^2}\\
&f^3(0) = 1 + 2x + 3x^2 - \mathbf{4x^3} + \mathbf{x^4}\\
&f^4(0) = 1 + 2x + 3x^2 + 4x^3 - \mathbf{11x^4} + \mathbf{6x^5} - \mathbf{x^6}\\
&f^5(0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 - \mathbf{26x^5} + \mathbf{23x^6} - \mathbf{8x^7} + \mathbf{x^8}\\
&f^6(0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 - \mathbf{57x^6} + \mathbf{72x^7} - \mathbf{39x^8} + \mathbf{10x^9} - \mathbf{x^{10}}.
\end{aligned}
$$

We know that the sequence of iterations converges to the sum of the arithmetic–geometric series, because f is contractive in $(\mathbb{K}[[x]], d)$, but we can observe that in each iteration the partial sum of such series appears, plus a remainder. We want to control the difference from the partial sum. To do this, we display the coefficients of the remainder as a matrix, that is,

$$
\begin{pmatrix}
-1 & & & & & & \\
-4 & 1 & & & & & \\
-11 & 6 & -1 & & & & \\
-26 & 23 & -8 & 1 & & & \\
-57 & 72 & -39 & 10 & -1 & & \\
-120 & 201 & -150 & 59 & -12 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The main properties of this matrix are described below. For their proofs and a more exhaustive development of this triangle see [5].

(1) The rule of construction is similar to that for the Pascal triangle: each element is twice the above element minus the element above to the left, that is, $a_{n,k} = 2a_{n-1,k} - a_{n-1,k-1}$.
(2) The elements in the first column are Eulerian numbers except for the sign.
(3) The sum of the elements in any row are triangular numbers with negative sign.
(4) For every element, the sum of all elements in its row to the right and all elements above in its column is zero. That is, $\sum_{k=1}^{i-1} a_{kj} + \sum_{k=j+1}^{n} a_{ik} = 0$.
(5) The general term is $a_{n,j} = n + j - 1 + \sum_{k=1}^{j-1}(-1)^k \binom{n+j-1-k}{n+j-2k} 2^{n+j-2k}$.

The above approach, using **BFPT**, does not give us an exact answer in the sense that in the n-th iteration there appears the partial sum of the series plus a remainder. To find an adequate answer we consider the **GBFPT**. For computability facts we consider the sequence of functions, with polynomial coefficients, given by $h_{0,2}(t) = xt$, $h_{1,2}(t) = xt + x$, $h_{2,2}(t) = xt + x + x^2$, $h_{3,2}(t) = xt + x + x^2 + x^3$,

$$h_{m,2}(t) = xt + x\sum_{k=0}^{m-1} x^k.$$

Each function $h_{m,2}$ is $\frac{1}{2}$-contractive, so $\{h_{m,2}\}$ is an equicontractive sequence of one-degree polynomials that converges to $h_2(t) = xt + \frac{x}{1-x}$, i.e.,

$$\{h_{m,2}\} \longrightarrow h_2(t) = xt + \frac{x}{1-x}.$$

It is easy to see that the following iterations yielded by **GBFPT** are just the corresponding partial sums of the arithmetic–geometric series:

$$
\begin{aligned}
&h_{0,2}(0) = 0\\
&h_{1,2}(h_{0,2}(0)) = x\\
&h_{2,2}(h_{1,2}(h_{0,2}(0))) = x + 2x^2\\
&h_{3,2}(h_{2,2}(h_{1,2}(h_{0,2}(0)))) = x + 2x^2 + 3x^3.
\end{aligned}
$$

Now using again **GBFPT** we obtain that, since $xt + \frac{x}{1-x} = t \Rightarrow t = \frac{x}{(1-x)^2}$, then

$$(h_{m,2} \circ \cdots \circ h_{0,2})(0) \longrightarrow \frac{x}{(1-x)^2}.$$

From now on we call such iterations the *crossed* iterations of the corresponding sequence of functions. These crossed iterations at zero converge to the unique fixed point of h_2, that is, the sum of the arithmetic–geometric series. So the answer to our question is yes, if we are allowed to use the generalized version of the **BFPT**.

Recall that the Pascal triangle is given by

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
1 & 1 & & & & & & & \\
1 & 2 & 1 & & & & & & \\
1 & 3 & 3 & 1 & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \binom{n}{5} & \binom{n}{6} & \cdots & \binom{n}{n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{1-x} & \frac{x}{(1-x)^2} & \frac{x^2}{(1-x)^3} & \frac{x^3}{(1-x)^4} & \frac{x^4}{(1-x)^5} & \frac{x^5}{(1-x)^6} & \frac{x^6}{(1-x)^7} & \cdots & \frac{x^{n-1}}{(1-x)^n}
\end{array}$$

We have just constructed the first two columns of the Pascal triangle using **BFPT**. In fact we needed only **BFPT** to construct the first one and **GBFPT** to get the second one. The main observation is that we can follow this iterative procedure to construct all columns. For example we can repeat the process to construct the third column. To achieve this goal, we interpret the above equicontractive sequence $h_{m,2}$ in the following way:

$$h_{m,2}(t) = xt + x\sum_{k=0}^{m-1} x^k = xt + xT_{m-1,1}$$

where $T_{m-1,1}$ is the $m-1$-degree Taylor polynomial of the first column which is a geometric series. So in a similar way we consider the following equicontractive sequence:

$$h_{m,3}(t) = xt + x\sum_{k=0}^{m-1} kx^k = xt + xT_{m-1,2}$$

where $T_{m-1,2}$ is the Taylor polynomial of the second column which is an arithmetic–geometric series. So, as one can easily prove, the crossed iterations for this sequence coincide with the partial sums of the third column:

$$\begin{aligned}
&h_{0,3}(0) = 0\\
&h_{1,3}(h_{0,3}(0)) = 0,\\
&h_{2,3}(h_{1,3}(h_{0,3}(0))) = x^2,\\
&h_{3,3}(h_{2,3}(h_{1,3}(h_{0,3}(0)))) = x^2 + 3x^3,\\
&h_{4,3}(h_{3,3}(h_{2,3}(h_{1,3}(h_{0,3}(0))))) = x^2 + 3x^3 + 6x^4.
\end{aligned}$$

Using **GBFPT** once more, we obtain that these crossed iterations converge to the unique fixed point of the limit function $h_3(t) = xt + x\frac{x}{(1-x)^2}$. Since

$$h_3(t) = xt + x\frac{x}{(1-x)^2} \Rightarrow t = \frac{x^2}{(1-x)^3}$$

then

$$(h_{m,3} \cdots h_{0,3})(0) = \sum_{k=0}^{m} \binom{k}{2} x^k \longrightarrow \frac{x^2}{(1-x)^3}.$$

Actually, as we said before, we can construct every column of the Pascal triangle using this process:

Proposition 2. *For $n \geq 2$, the n-th column in Pascal's triangle is obtained from the $(n-1)$-st column by applying the crossed iterations in* ***GBFPT*** *to the sequence $\{h_{m,n}\}_{m\in\mathbb{N}}$ where*

$$h_{m,n}(t) = xt + xT_{m-1,n-1}$$

with $T_{m-1,n-1}$ being the $(m-1)$-st Taylor polynomial of the $(n-1)$-st column.

3. The group of all arithmetical triangles $T(f \mid g)$

Now we generalize the previous iterative method for any pair of series $f = \sum_{n\geq 0} f_n x^n$ and $g = \sum_{n\geq 0} g_n x^n$ such that $f_0 \neq 0$ and $g_0 \neq 0$ in order to construct a general arithmetical triangle $T(f \mid g)$. See [4] for a more detailed description. In this notation the Pascal triangle is $T(1 \mid 1-x)$. In the following description the series f plays the role of the series 1 and the series g plays the role of $1-x$.

$$\begin{array}{c|cccc}
f_0 & & & & \\
f_1 & d_{0,0} & & & \\
f_2 & d_{1,0} & d_{0,1} & & \\
f_3 & d_{2,0} & d_{2,1} & d_{2,2} & \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
f & \frac{f}{g} & \frac{xf}{g^2} & \frac{x^2 f}{g^3} & \cdots
\end{array}$$

In [4] we interpreted the calculation of $\frac{f}{g}$ as a fixed point problem. Consider the sequence

$$h_{m,1}(t) = T_m\left(\frac{g_0 - g}{g_0}\right) t + T_m\left(\frac{f}{g_0}\right)$$

where $T_m(f)$ is the m-degree Taylor polynomial of f. Observe that the sequence of crossed iterations has as its limit the unique fixed point of $h_1(t) = \frac{g_0-g}{g_0} t + \frac{f}{g_0}$, that is, $\frac{f}{g}$. It is the first column of $T(f \mid g)$.

To construct the successive columns we consider the equicontractive sequences

$$h_{m,n}(t) = T_m\left(\frac{g_0 - g}{g_0}\right) t + x T_{m-1}\left(\frac{x^{n-2} f}{g_0 g^{n-1}}\right).$$

Their corresponding limits are $h_n(t) = (\frac{g_0-g}{g_0})t + x(\frac{x^{n-2}f}{g_0 g^{n-1}})$, whose corresponding unique fixed points are

$$t_n = \frac{x^{n-1} f}{g^n}.$$

The series t_n is just the n-th column of our $T(f \mid g)$.

Theorem 3. *Let $f = \sum_{n\geq 0} f_n x^n$ and $g = \sum_{n\geq 0} g_n x^n$ be two series with $g_0 \neq 0$. Then the Riordan matrix $T(f \mid g) = (d_{n,k})$ can be constructed in the following way:*
if $k = 0$, then

$$d_{0,0} = \frac{f_0}{g_0}, \qquad d_{n,0} = -\frac{g_1}{g_0} d_{n-1,0} - \frac{g_2}{g_0} d_{n-2,0} \cdots - \frac{g_n}{g_0} d_{0,0} + \frac{f_n}{g_0},$$

and if $k > 0$, then

$$d_{n,k} = -\frac{g_1}{g_0} d_{n-1,k} - \frac{g_2}{g_0} d_{n-2,k} \cdots - \frac{g_{n-k}}{g_0} d_{k,k} + \frac{d_{n-1,k-1}}{g_0}.$$

This theorem gives us the following algorithm for constructing by columns any Riordan matrix:

Algorithm 4. Given $f = \sum_{n\geq 0} f_n x^n$ and $g = \sum_{n\geq 0} g_n x^n$ with $g_0 \neq 0$:
Step 1. Calculate the first column $d_{n,0}$.

$$d_{0,0} = \frac{f_0}{g_0}, \qquad d_{n,0} = -\frac{g_1}{g_0} d_{n-1,0} - \frac{g_2}{g_0} d_{n-2,0} \cdots - \frac{g_n}{g_0} d_{0,0} + \frac{f_n}{g_0}.$$

Step k. Calculate the k-th column, $d_{n,k}$, using the $k-1$-st column.

$$d_{n,k} = -\frac{g_1}{g_0} d_{n-1,k} - \frac{g_2}{g_0} d_{n-2,k} \cdots - \frac{g_{n-k}}{g_0} d_{k,k} + \frac{d_{n-1,k-1}}{g_0}.$$

We can write this algorithm in an informal pseudo-code:

```
READ (f,g,n)
SET (d,aux)
CALCULATE d[0,0]=f[0]/g[0]
% We calculate the first column
FOR i=1 to n
```

```
        FOR k=1 to n
            CALCULATE aux(k,i)=g[i-k]*d[k,0]
        END
        CALCULATE d(i,0)=1/g[0]*(f[i]-SUM(aux(:,i)))
END
% We calculate the remaining columns
FOR j=1 to n
        FOR i=1 to n
                FOR k=1 to i
                        CALCULATE aux(k,i)=g[i-k]*d[k,j]
                END
                CALCULATE d(i,j)=1/g[0]*(d(i-1,j-1)-SUM(aux(:,i))
        END
END
PRINT(f,g,d)
```

So to construct the arithmetical triangle $T(f \mid g)$ it is enough to know the ordered pair of series f and g, i.e. the data, and the algorithm for dividing two series. Every column is constructed by the same rule as that in $\frac{f}{g}$ but the coefficients of $\frac{f}{g}$ are replaced with the coefficients of the previous column. Except for the first column, here we need an auxiliary column: the coefficients of f.

We can consider the matrix $T(f \mid g)$, like in linear algebra, as the matrix associated with a $\mathbb{K}$-linear continuous function (see [4]):

$$\begin{array}{rcl} T(f \mid g) : (\mathbb{K}[[x]], d) & \to & (\mathbb{K}[[x]], d) \\ h & \mapsto & T(f \mid g)(h) = \dfrac{f}{g} h\left(\dfrac{x}{g}\right). \end{array}$$

Using the classical definition of composition of maps and the behavior of the associated matrix, we can easily find the formulas for the product and the inverse for these triangles:

$$T(f_1 \mid g_1)T(f_2 \mid g_2) = T\left(f_1 f_2\left(\frac{x}{g_1}\right)\middle| g_1 g_2\left(\frac{x}{g_1}\right)\right)$$

$$(T(f \mid g))^{-1} = T\left(\frac{1}{f\left(\omega^{-1}\right)}\middle| \frac{1}{g\left(\omega^{-1}\right)}\right), \qquad \omega = \left(\frac{x}{g}\right), \qquad \omega \circ \omega^{-1} = \omega^{-1} \circ \omega = x.$$

So if we consider the set of the all arithmetical triangles with $f_0 \neq 0$ and $g_0 \neq 0$ and the usual product of matrices, we obtain a group. Actually this group is the well-known Riordan group.

4. On the $T(f \mid g)$ notation

We have received some criticism as regards our notation. Some might think that our notation is, in some sense, cumbersome. Of course it depends strongly on the way you approach or you run into this group. In this section we are going to give some reasons for our notation being highly appropriate. The basic formula relating our notation to the classical notation is

$$(d(x), h(x)) = T\left(\frac{xd}{h}\middle|\frac{x}{h}\right) = \left(\frac{f(x)}{g(x)}, \frac{x}{g}\right) = T(f \mid g) = (d_{i,j})_{i,j \geq 0}.$$

The fundamental equality with our notation is

$$T(f|g) = T(f|1)T(1|g). \tag{1}$$

This equality in the other notation is

$$\left(\frac{f(x)}{g(x)}, \frac{x}{g(x)}\right) = (f(x), x)\left(\frac{1}{g(x)}, \frac{x}{g(x)}\right).$$

By (1), every element of the Riordan group [9] can be expressed by means of the product of a lower triangular Toepliz matrix whose columns are the coefficients of series f, shifted conveniently, the matrix $T(f \mid 1)$, and a renewal array, the matrix $T(1 \mid g)$ described by Rogers in [8]. Matrices of this last kind are really similar to the Jabotinsky matrices; see [3]. We wish to point out that the structure of every element of the Riordan group is *essentially* in the structure of the matrix $T(1 \mid g)$. For example, knowing a closed formula for the general term of $T(1 \mid g)$ gives us at once a closed formula for the general term in $T(f \mid g)$. This matrix $T(1 \mid g)$, for us, is intrinsically related to the calculation of $\frac{1}{g}$, which is its first column.

A comparative table of the notations is given here:

Name	$(d(t), th(t))$	$T(f \mid g)$
Identity	$(1, t)$	$T(1 \mid 1)$
Pascal	$\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$	$T(1 \mid 1-t)$
Appel subgroup element	$(d(t), t)$	$T(d \mid 1)$
Associated subgroup element	$(1, th(t))$	$T\left(\frac{1}{h} \mid \frac{1}{h}\right)$
Bell subgroup element	$(d(t), td(t))$	$T\left(1 \mid \frac{1}{d}\right)$

From our viewpoint, a curious and emblematic property of our notation is the way of giving, by means of the parameters, the natural powers of the Pascal triangle:

Proposition 5. *For every $n \in \mathbb{N}$, we have $T^n(1 \mid 1-x) = T(1 \mid 1-nx)$.*

Proof. Let us proceed by induction.

For $n = 2$: $T^2(1 \mid 1-x) = T(1 \mid 1-x)T(1 \mid 1-x)$, $\omega = \frac{x}{1-x}$. So

$$T^2(1 \mid 1-x) = T\left(1 \mid (1-x)\left(1 - \frac{x}{1-x}\right)\right) = T(1 \mid 1-2x).$$

Suppose that $T^{n-1}(1 \mid 1-x) = T(1 \mid 1-(n-1)x)$; then

$$\begin{aligned} T^n(1 \mid 1-x) &= T(1 \mid 1-x)T^{n-1}(1 \mid 1-x) = T(1 \mid 1-x)T(1 \mid 1-(n-1)x) \\ &= T\left(1 \mid (1-x)\left(1-(n-1)\frac{x}{1-x}\right)\right) = T(1 \mid 1-nx). \quad \square \end{aligned}$$

Another reason why for us our notation is natural is related to the way we began to study these topics. One of the first things we did was to find our curious triangle described in Section 2. From our notation, the description as a Riordan array is

$$\begin{pmatrix} -1 \\ -4 & 1 \\ -11 & 6 & -1 \\ -26 & 23 & -8 & 1 \\ -57 & 72 & -39 & 10 & -1 \\ -120 & 201 & -150 & 59 & -12 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = T\left(\frac{1}{(1-x)^2} \middle| 2x-1\right).$$

This notation resembles both the problem we were treating and the algorithm of construction.

Another thing we can describe easily with our notation is the fact that, with our construction method by columns, we can add new columns to the left for every element of the Riordan group to obtain again a Riordan array intrinsically related to the initial one, for example,

$$\begin{pmatrix} 1 \\ 2 & -1 \\ 3 & -4 & 1 \\ 4 & -11 & 6 & -1 \\ 5 & -26 & 23 & -8 & 1 \\ 6 & -57 & 72 & -39 & 10 & -1 \\ 7 & -120 & 201 & -150 & 59 & -12 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = T\left(\frac{2x-1}{(1-x)^2} \middle| 2x-1\right).$$

Note that we added a new column to the left and look at the way the parameters changed in our notation.

In general we can construct a closely related family of new Riordan matrices. For example by the definition of the Riordan array we get

$$T(fg \mid g) = \begin{pmatrix} f_0 \\ f_1 & d_{0,0} \\ f_2 & d_{1,0} & d_{1,1} \\ f_3 & d_{2,0} & d_{2,1} & d_{2,2} \\ f_4 & d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$T\left(\frac{f}{g}\middle| g\right) = \begin{pmatrix} d_{1,1} & & & & & \\ d_{2,1} & d_{2,2} & & & & \\ d_{3,1} & d_{3,2} & d_{3,3} & & & \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & & \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $f = \sum_{n\geq 0} f_n x^n$ and $T(f \mid g) = (d_{n,k})_{n,k\in\mathbb{N}}$.

In the same way we observe that $T(\frac{f}{g^m} \mid g)$ for $m \in \mathbb{N}$ is the matrix obtained from $T(f \mid g)$ by deleting the first m rows and m columns. Moreover $T(fg^m \mid g)$ is the unique Riordan matrix with the property that by deleting the first m rows and m columns from it, we obtain $T(f \mid g)$. In fact one can easily prove the following:

Proposition 6. *Let $T(f \mid g) = (d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan matrix and $m \in \mathbb{Z}$. Then $T(fg^m \mid g) = (\tilde{d}_{n,k})_{n,k\in\mathbb{N}}$ with $\tilde{d}_{n,k} = [x^{n-k}]fg^{m-k-1}$, where $[x^j]S$ stands for the j-th coefficient of the formal power series S.*

We can always embed any $T(f|g)$ in a bi-infinite lower triangular matrix.

For our example we have

$$\begin{pmatrix} & & & & & & \vdots & & & & & \\ & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & -23 & -6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 26 & 11 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \cdots & -5 & -4 & -3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & -4 & -3 & -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ & -3 & -2 & -1 & 0 & 1 & 2 & -1 & 0 & 0 & 0 & \\ & -2 & -1 & 0 & 1 & 2 & 3 & -4 & 1 & 0 & 0 & \\ & -1 & 0 & 1 & 2 & 3 & 4 & -11 & 6 & -1 & 0 & \\ & 0 & 1 & 2 & 3 & 4 & 5 & -26 & 23 & -8 & 1 & \\ & & & & & & \vdots & & & & & \end{pmatrix}.$$

This construction is very nice in the case of Riordan matrices of the kind $T(1 \mid a+bx)$, treated as a change of variables in [5]. For these matrices we have

$$\begin{pmatrix} & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{a}\left(\frac{a}{1-bx}\right)^4 & \leftarrow & a^3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{a}\left(\frac{a}{1-bx}\right)^3 & \leftarrow & 3a^2b & a^2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{a}\left(\frac{a}{1-bx}\right)^2 & \leftarrow & 3ab^2 & 2ab & a & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{a}\frac{a}{1-bx} & \leftarrow & b^3 & b^2 & b & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{a} & \leftarrow & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & \cdots \\ \frac{1}{a}\left(\frac{1-bx}{a}\right) & \leftarrow & 0 & 0 & 0 & 0 & -\frac{b}{a^2} & \frac{1}{a^2} & 0 & \cdots \\ \frac{1}{a}\left(\frac{1-bx}{a}\right)^2 & \leftarrow & 0 & 0 & 0 & 0 & \frac{b^2}{a^3} & -\frac{2b}{a^3} & \frac{1}{a^3} & \cdots \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \vdots \\ & & (a+bx)^3 & (a+bx)^2 & (a+bx) & 1 & \frac{1}{a+bx} & \frac{1}{(a+bx)^2} & \frac{1}{(a+bx)^3} & \cdots \end{pmatrix}.$$

Note that, as for the Pascal triangle, we can see the above matrix in the following way:

$$\begin{pmatrix} \frac{1}{a}T^{-1}(1 \mid a+bx)\blacklozenge & 0 \\ 0 & T(1 \mid a+bx) \end{pmatrix}$$

where $\frac{1}{a}T^{-1}(1 \mid a+bx)\blacklozenge$ is $\frac{1}{a}T^{-1}(1 \mid a+bx)$ placed in the same way as the inverse of the Pascal triangle is placed in the Hexagon of Pascal on page 194 in [2]. Note that $T^{-1}(1 \mid a+bx) = T(1 \mid \frac{1-bx}{a})$. This gives us a method for calculating $T^{-1}(1 \mid a+bx)$ by means of elementary operations.

It is usual in Riordan array theory to make the following statement:

$D = (d_{n,k})$ is a Riordan array if and only if there exist two sequences $A = (a_n)_{n\in\mathbb{N}}$ and $Z = (z_n)_{n\in\mathbb{N}}$, called the A and Z sequences of D, such that

$$d_{n+1,k+1} = \sum_{j=0}^{n-k} a_j d_{n,j+k} \quad \text{and} \quad d_{n+1,0} = \sum_{j=0}^{n} z_j d_{n,j}.$$

See [6,8,11].

Our construction algorithm does not need the A and Z sequences of a Riordan array, but, in our notation, they appear in the expression for the inverse of the Riordan array $T(f \mid g)$ giving us an especially aesthetically pleasing formula:

Proposition 7. *Let $f = \sum_{n\geq 0} f_n x^n$ and $g = \sum_{n\geq 0} g_n x^n$ be two formal power series with $f_0 \neq 0$ and $g_0 \neq 0$. Suppose that A and Z represent the A sequence and the Z sequence, respectively, of $T(f \mid g)$. Then*

(i) $T^{-1}(1 \mid g) = T(1 \mid A)$
(ii) $T^{-1}(f \mid g) = T\left(\frac{g_0}{f_0}(A - xZ)|A\right)$.

Proof. (i) From Theorem 1.3 of [11], the A sequence is the unique series with $A(0) \neq 0$ such that $\frac{1}{g} = A(\frac{x}{g})$. So $\frac{x}{g} = xA(\frac{x}{g})$. If $\omega = \frac{x}{g}$ then $\omega = xA(\omega)$. On the other hand as $\omega^{-1} \circ \omega = x$ and $\omega = \frac{x}{g}$ then $x = \omega g$; composing with ω^{-1} we get $\omega^{-1} = xg(\omega^{-1})$ and $\frac{\omega^{-1}}{x} = g(\omega^{-1})$. So composing with ω^{-1} but now in $\omega = xA(\omega)$ we get $x = \omega^{-1}A(x)$; then $1 = \frac{\omega^{-1}}{x}A(x)$ and so $1 = g(\omega^{-1})A(x)$; then $A(x) = \frac{1}{g(\omega^{-1})}$. This is since $T^{-1}(1 \mid g) = T(1 \mid \frac{1}{g(\omega^{-1})}) = T(1 \mid A)$.

(ii) From Theorem 2.3 in [6] we obtain that Z is determined by the equality $\omega^{-1}Z = 1 - \frac{f_0 g(\omega^{-1})}{g_0 f(\omega^{-1})}$. From here we get $\frac{1}{f(\omega^{-1})} = \frac{g_0}{f_0}(A - xZ)$. So

$$T^{-1}(f \mid g) = T\left(\frac{1}{f(\omega^{-1})}\middle| \frac{1}{g(\omega^{-1})}\right) = T\left(\frac{g_0}{f_0}(A - xZ)|A\right). \quad \square$$

Corollary 8.

$$T^{-1}(f \mid g) = T(1 \mid A)T\left(\frac{1}{f}\middle| 1\right).$$

5. The Lagrange Inversion Formula obtained via the Banach Fixed Point Theorem

In the previous section we showed that to calculate the inverse of $T(1 \mid g)$ we need, in particular, to calculate ω^{-1}, where $\omega = \frac{x}{g}$, and then $\omega^{-1} = xg(\omega^{-1})$. So we consider the function $F : x\mathbb{K}[[x]] \to x\mathbb{K}[[x]]$ defined by $F(y) = xg(y)$. Here $x\mathbb{K}[[x]]$ represents the series with null independent term. This function is $\frac{1}{2}$-contractive since

$$d(F(y_1), F(y_2)) = \frac{1}{2^{\omega(xg(y_1)-xg(y_2))}} = \frac{1}{2^{\omega(g(y_1)-g(y_2))+1}} \leq \frac{1}{2}d(y_1, y_2).$$

The domain, $x\mathbb{K}[[x]]$, of F is the closed ball, in $(\mathbb{K}[[x]], d)$, whose center is the series $y = 0$ and the ratio is $\frac{1}{2}$. Consequently our domain is also complete with the relative metric. So the unique fixed point of F is $\omega^{-1} = (\frac{x}{g})^{-1}$ and **BFPT** can be applied.

The **BFPT** gives us a theoretical iterative process for calculating ω^{-1}. To convert this method into an effective approximation process we first note that the relation $d(S_1, S_2) \leq \frac{1}{2^{m+1}}$ means that the m-degree Taylor polynomials of the two series are equal, that is $T_m(S_1) = T_m(S_2)$. Then we obtain the following algorithm:

Suppose $g = \sum_{n\geq 0} g_n x^n$. We begin to iterate the function $F(y) = xg(y)$ at $y = 0$. So $F(0) = xg(0) = g_0 x$ since $d(F(0), F(\omega^{-1})) = d(F(0), \omega^{-1}) \leq \frac{1}{4}$. Consequently $T_1(\omega^{-1}) = g_0 x$. Using again the $\frac{1}{2}$-contractivity of F we get $T_2(F(g_0 x)) = T_2(\omega^{-1})$. Since $F(g_0 x) = g_0 x + g_0 g_1 x^2 + \cdots$ we obtain $T_2(\omega^{-1}) = g_0 x + g_0 g_1 x^2$. Similar arguments allow us to prove that $T_3(F(g_0 x + g_0 g_1 x^2)) = T_3(\omega^{-1})$. The above construction can be summarized as follows, with the notation as above:

Proposition 9.

$$T_m(\omega^{-1}) = T_m(F(T_{m-1}(F(\cdots(F(T_1(F(0))))\cdots)))).$$

Following this process we get

$$T_1(\omega^{-1}) = g_0x$$
$$T_2(\omega^{-1}) = g_0x + g_0g_1x^2$$
$$T_3(\omega^{-1}) = g_0x + g_0g_1x^2 + (g_0g_1^2 + g_0^2g_2)x^3$$
$$T_4(\omega^{-1}) = g_0x + g_0g_1x^2 + (g_0g_1^2 + g_0^2g_2)x^3 + (g_0g_1^3 + 3g_0^2g_1g_2 + g_0^3g_3)x^4$$
$$T_5(\omega^{-1}) = g_0x + g_0g_1x^2 + (g_0g_1^2 + g_0^2g_2)x^3 + (g_0g_1^3 + 3g_0^2g_1g_2 + g_0^3g_3)x^4 + (g_0g_1^4 + 6g_0^2g_1^2g_2 + 2g_0^3g_2^2 + 4g_0^3g_1g_3 + g_0^4g_4)x^5.$$

If we recall the Cauchy powers of the series g:

$$g(x) = \mathbf{g_0} + g_1x + g_2x^2 + g_3x^3 + g_4x^4 + \cdots$$
$$g^2(x) = g_0^2 + \mathbf{2g_0g_1}x + (2g_0g_2 + g_1^2)x^2 + (2g_0g_3 + 2g_1g_2)x^3 \cdots$$
$$g^3(x) = g_0^3 + 3g_0^2g_1x + \mathbf{3(g_0g_1^2 + g_0^2g_2)}x^2 + (6g_0g_1g_2 + 3g_0^2g_3 + g_1^3)x^3 + \cdots$$
$$g^4(x) = g_0^4 + 4g_0^3g_1x + (4g_0^3g_2 + 6g_0^2g_1^2)x^2 + \mathbf{4(g_0g_1^3 + 3g_0^2g_1g_2 + g_0^3g_3)}x^3 + \cdots$$
$$g^5(x) = g_0^5 + 5g_0^4g_1x + (5g_0^4g_2 + 10g_0^3g_1^2)x^2 + (20g_0^3g_1g_2 + 10g_0^2g_1^3 + 5g_0^4g_3)x^3 + \mathbf{5(g_0g_1^4 + 6g_0^2g_1^2g_2 + 2g_0^3g_2^2 + 4g_0^3g_1g_3 + g_0^4g_4)}x^4 + \cdots,$$

then comparing appropriately the coefficients of ω^{-1} and the powers of g we obtain the following relationships:

$$[x]\omega^{-1} = [x^0]g$$
$$[x^2]\omega^{-1} = \frac{1}{2}[x^1]g^2$$
$$[x^3]\omega^{-1} = \frac{1}{3}[x^2]g^3$$
$$[x^4]\omega^{-1} = \frac{1}{4}[x^3]g^4$$
$$[x^5]\omega^{-1} = \frac{1}{5}[x^4]g^5.$$

These equalities allow us to predict and motivate the classical Lagrange Inversion Formula (see [12], page 36):

$$[x^{n+1}]\omega^{-1} = \frac{1}{n+1}[x^n]g^{n+1}, \quad \text{with } \omega = \frac{x}{g}.$$

From now on we write $T_j \equiv T_j(\omega^{-1})$. To show how this process works, note that

$$\begin{aligned} F(T_n) &= x(g_0 + g_1T_n + g_2T_n^2 + \cdots + g_nT_n^n + \cdots) \\ &= T_n + (g_1[x^n]T_n + g_2[x^n]T_n^2 + \cdots + g_n[x^n]T_n^n)x^{n+1} + S_{n+2} \quad \text{with } S_{n+2} \in x^{n+2}\mathbb{K}[[x]]. \end{aligned}$$

So

$$[x^{n+1}]F(T_n) = \sum_{k=1}^{n}[x^k]g[x^n](\omega^{-1})^k.$$

This is because $[x^n](\omega^{-1})^k = [x^n](T_n)^k$ for any $k \le n$. Suppose now that we know

$$n[x^n](\omega^{-1})^k = k[x^{n-k}]g^n \quad \text{for } k \le n$$

then

$$[x^{n+1}]F(T_n) = [x^{n+1}]\omega^{-1} = \frac{1}{n}\sum_{k=1}^{n}k[x^k]g[x^{n-k}]g^n = \frac{1}{n}[x^{n-1}]g'g^n = \frac{1}{n+1}[x^n]g^{n+1}.$$

Note that in the above development we need to know $n[x^n](\omega^{-1})^k = k[x^{n-k}]g^n$ for $k \le n$. In fact we can give a proof of all of the above using essentially the fact that ω^{-1} is the fixed point of the contractive function F.

Theorem 10 (*Lagrange Inversion Obtained via the Banach Fixed Point Theorem*). *Let $\mathbb{K}$ be a field of characteristic zero. Suppose that ω is a formal power series in $\mathbb{K}[[x]]$ with $\omega(0) = 0$ and $\omega'(0) \neq 0$. Then*

$$n[x^n](\omega^{-1})^k = k[x^{n-k}]\left(\frac{x}{\omega}\right)^n \quad \textit{for } n, k \in \mathbb{N}.$$

Proof. Let $g = \frac{x}{\omega}$. So $[x^0]g \neq 0$. As proved before, ω^{-1} is the unique fixed point of the $\frac{1}{2}$-contractive function $F : x\mathbb{K}[[x]] \to x\mathbb{K}[[x]]$ defined by $F(y) = xg(y)$. Iterating at $y = 0$ we get

$$[x^1]\omega^{-1} = [x^1]F(0) = [x^0]g.$$

If $k > 1$, note that $[x^1](\omega^{-1})^k = 0$ and $[x^{1-k}]g = 0$, and then

$$[x^1]\omega^{-1} = k[x^{1-k}]g.$$

Let us proceed by induction on n. Suppose that

$$j[x^j](\omega^{-1})^k = k[x^{j-k}]g^j \quad \text{for } j \le n,\ k \ge 1.$$

In fact we are supposing only $[x^j](\omega^{-1})^k = k[x^{j-k}]g^j$ for $0 \le k \le j \le n$, because if $j < k$, then $[x^{j-k}]g^j = 0 = [x^j](\omega^{-1})^k$. Then the equality holds trivially.

Since $\omega^{-1} = xg(\omega^{-1})$, then for any k, $(\omega^{-1})^k = x^k g^k(\omega^{-1})$. Consequently

$$[x^{n+1}](\omega^{-1})^k = [x^{n+1}]x^k g^k(\omega^{-1}) = [x^{n+1-k}]g^k(\omega^{-1}) = \sum_{j=0}^{n+1-k}[x^j]g^k[x^{n+1-k}](\omega^{-1})^j$$

and by the induction hypothesis

$$[x^{n+1}](\omega^{-1})^k = \frac{1}{n+1-k}\sum_{j=0}^{n+1-k} j[x^j]g^k[x^{n+1-k-j}]g^{n+1-k}.$$

Let us define $h = g^k$;

$$\begin{aligned}[x^{n+1}](\omega^{-1})^k &= \frac{1}{n+1-k}\sum_{j=0}^{n+1-k} j[x^j]h[x^{n+1-k-j}]h^{\frac{n+1-k}{k}} = \frac{1}{n+1-k}\sum_{j=1}^{n+1-k}[x^{j-1}]h'[x^{n+1-k-j}]h^{\frac{n+1-k}{k}}\\ &= \frac{1}{n+1-k}\sum_{j=0}^{n-k}[x^j]h'[x^{n-k-j}]h^{\frac{n+1-k}{k}} = \frac{1}{n+1-k}[x^{n-k}](h'h^{\frac{n+1-k}{k}})\\ &= \frac{1}{n+1-k}[x^{n-k}]\left(\frac{k}{n+1}h^{\frac{n+1-k}{k}}\right)' = \frac{k}{n+1}[x^{n+1-k}]h^{\frac{n+1}{k}} = \frac{k}{n+1}[x^{n+1-k}]g^{n+1}. \quad \square\end{aligned}$$

The development above gives us the following algorithm for calculating the first n coefficients of the compositional inverse of $\omega = \frac{x}{g}$, because the m-degree Taylor polynomial of ω^{-1} coincides with the m-degree Taylor polynomial of $F(T_{m-1})$ as we proved in Proposition 9.

Algorithm 11. Given $g = \sum_{j\ge 0} g_j x^j$ with $g_0 \neq 0$, and given $F(y) = xg(y)$ with $y \in x\mathbb{K}[[x]]$:

Step 1. (Initial.) $T_1 = g_0 x$.

Step i (2 to n). Calculate the Taylor polynomial of order i of $F(T_{i-1})$.

We can write this algorithm in an informal pseudo-code:

```
READ (g,F,n)
SET T
FOR i=2 to n
        CALCULATE T[i]=TAYLOR(F(T[i-1]))
END
PRINT T
```

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