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Controllability of nonlinear neutral evolution integrodifferential systems

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Abstract

Sufficient conditions for controllability of nonlinear neutral evolution integrodifferential systems in a Banach space are established. The results are obtained by using the resolvent operators and the Schaefer fixed-point theorem. An application to partial integrodifferential equation is given.

Keywords: Controllability; Mild solution; Resolvent operator; Neutral evolution integrodifferential system

1. Introduction

The problem of controllability of nonlinear systems represented by ordinary differential equations in infinite-dimensional spaces has been extensively studied by several authors [2,3,13,14]. Zhang [16] studied the local exact controllability of semilinear evolution systems by means of the contraction mapping principle. Klamka [9] considered the dynamical control systems described by nonlinear abstract differential equations and derived the sufficient conditions for controllability of nonlinear systems by using the Schauder fixed point theorem. Bian [5] investigated the approximate controllability for a class of semilinear systems. Bal-

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achandran et al. [1] established the local null controllability of nonlinear functional differential systems in Banach spaces by using the fractional power operators and the Schauder fixed point theorem. Controllability of nonlinear Volterra integrodifferential systems in abstract spaces has been studied by Naito [11]. Recently Balachandran et al. [4] derived a set of sufficient conditions for the controllability of neutral functional integrodifferential systems in Banach spaces by using the semigroup theory. The purpose of this paper is to study the controllability of nonlinear neutral evolution integrodifferential systems in Banach spaces by using the resolvent operators and the Schaefer fixed-point theorem. The nonlinear neutral evolution integrodifferential systems with resolvent operators considered here serves as an abstract formulation of partial integrodifferential equations which arises in various applications such as viscoelasticity, heat equations and many other physical phenomena [8,10,12].

2. Preliminaries

Consider the nonlinear neutral evolution integrodifferential system of the form

$$\begin{aligned} &\frac{d}{dt}\big[x(t)+g(t,x_t)\big] \\ &\quad = A(t)x(t)+\int_0^t B(t,s)x(s)\,ds \\ &\qquad + (Gu)(t)+f\left(t,x_t,\int_0^t h(t,s,x_s)\,ds\right), \quad t\in J=[0,b], \\ &x_0=\phi, \quad \text{on } [-r,0], \end{aligned} \tag{1}$$

where the state $x(\cdot)$ takes values in a Banach space X with the norm $\|\cdot\|$, and the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Here $A(t)$ and $B(t,s)$ are closed linear operators on X with dense domain $D(A)$ which is independent of t, G is a bounded linear operator from U into X, $h: J\times J\times C\to X$, $f: J\times C\times X\to X$ and $g: J\times C\to X$, are continuous functions. Here $C=C([-r,0],X)$ is the Banach space of all continuous functions $\phi:[-r,0]\to X$ endowed with the norm $\|\phi\|=\sup\{|\phi(\theta)|:\ -r\leqslant\theta\leqslant 0\}$. Also for $x\in C([-r,b],X)$ we have $x_t\in C$ for $t\in[0,b]$, $x_t(\theta)=x(t+\theta)$ for $\theta\in[-r,0]$.

We shall make the following assumptions [7]:

(I) $A(t)$ generates a strongly continuous semigroup of evolution operators in the Banach space X.

(II) Suppose Y is the Banach space formed from $D(A)$ with the graph norm. $A(t)$ and $B(t,s)$ are closed operators it follows that $A(t)$ and $B(t,s)$ are in the set of bounded operators from Y to X, $B(Y,X)$, for $0 \leqslant t \leqslant b$ and $0 \leqslant s \leqslant t \leqslant b$, respectively. Further $A(t)$ and $B(t,s)$ are continuous on $0 \leqslant t \leqslant b$ and $0 \leqslant s \leqslant t \leqslant b$, respectively, into $B(Y,X)$.

Definition 2.1. A resolvent operator for (1) is a bounded operator valued function $R(t,s) \in B(X)$, $0 \leqslant s \leqslant t \leqslant b$, the space of bounded linear operators on X, having the following properties:

(a) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$, $0 \leqslant s \leqslant b$, $\|R(t,s\| \leqslant Me^{\beta(t-s)}$ for some constants M and β.
(b) $R(t,s)Y \subset Y$, $R(t,s)$ is strongly continuous in s and t on Y.
(c) For each $x \in D(A)$, $R(t,s)x$ is strongly continuously differentiable in t and s and

$$\frac{\partial R}{\partial t}(t,s)x = A(t)R(t,s)x + \int_s^t B(t,r)R(r,s)x\,dr,$$

$$\frac{\partial R}{\partial s}(t,s)x = -R(t,s)A(s)x - \int_s^t R(t,r)B(r,s)x\,dr$$

with $\frac{\partial R}{\partial t}(t,s)x$ and $\frac{\partial R}{\partial s}(t,s)x$ strongly continuous on $0 \leqslant s \leqslant t \leqslant b$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space. It will not, however, be an evolution operator because it will not satisfy an evolution or semigroup property. Because a number of results follow directly from the definition of the resolvent operator.

Definition 2.2. A solution $x \in C([-r,b],X)$ is a mild solution of the problem (1) if the following holds: $x_0 = \phi$ on $[-r,0]$ and $s \in [0,t)$, the function $A(s)R(t,s)g(s,x(s))$, is integrable and the integral equation

$$\begin{aligned} x(t) = {} & R(t,0)\big[\phi(0) + g(0,\phi)\big] - g(t,x_t) - \int_0^t R(t,s)A(s)g(s,x_s)\,ds \\ & - \int_0^t R(t,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds \\ & + \int_0^t R(t,s)\left[(Gu)(s) + f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\,d\tau\right)\right]ds \end{aligned} \tag{2}$$

is satisfied.

Schaefer's theorem [15]. *Let E be a normed linear space. Let $F: E \to E$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\zeta(F) = \{x \in E;\ x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Definition 2.3. The system (1) is said to be controllable on the interval J if for every continuous initial function $\phi \in C$, $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(t)$ of (1) satisfies $x(b) = x_1$.

Further we assume the following hypotheses:

(i) The resolvent operator $R(t,s)$ is compact and there exist constants $M_i > 0$, $i = 1, 2, 3$, such that $|R(t,s)| \leqslant M_1$, $|R(t,s)A(s)| \leqslant M_2$ and $|B(t,s)| \leqslant M_3$.

(ii) The linear operator $W: L^2(J,U) \to X$ defined by

$$Wu = \int_0^b R(b,s)Gu(s)\,ds$$

has an induced inverse operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)/\ker W$ and there exist positive constants M_4, M_5 such that $|G| \leqslant M_4$ and $|\widetilde{W}^{-1}| \leqslant M_5$ (see [6]).

(iii) The function $g: J \times C \to X$ is completely continuous and for any bounded set D in $C([-r,b], X)$ the set $\{t \to g(t, x_t):\ x \in D\}$ is equicontinuous in $C([0,b], X)$ and there exists a constant $L > 0$ such that

$$\big|g(t,\phi)\big| \leqslant L, \quad t \in J,\ \phi \in C.$$

(iv) For each $t, s \in J \times J$, the function $h(t,s,\cdot): C \to X$ is continuous and for each $x \in C$ the function $h(\cdot,\cdot,x): J \times J \to X$ is strongly measurable.

(v) For each $t \in J$ the function $f(t,\cdot,\cdot): C \times X \to X$ is continuous and for each $(x,y) \in C \times X$ the function $f(\cdot,x,y): J \to X$ is strongly measurable.

(vi) For every positive integer k there exists $\alpha_k \in L^1(0,b)$ such that for a.e. $t \in J$

$$\sup_{\|x\|,|y| \leqslant k} \big|f(t,x,y)\big| \leqslant \alpha_k(t).$$

(vii) There exists an integrable function $m: J \times J \to [0,\infty)$ such that

$$\big|h(t,s,x)\big| \leqslant m(t,s)\Omega\big(\|x\|\big), \quad t,s \in J,\ x \in C,$$

where $\Omega: [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(viii) There exists an integrable function $p: J \to [0,\infty)$ such that

$$\big|f(t,x,y)\big| \leqslant p(t)\Omega_0\big(\|x\| + |y|\big), \quad t \in J,\ x \in C,\ y \in X,$$

where $\Omega_0 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(ix) $\int_0^b \hat{m}(s)\,ds < \int_c^\infty \frac{ds}{\Omega(s)+\Omega_0(s)}$, where $c = M_1(\|\phi\| + L) + L + M_2Lb + M_1M_3Lb^2 + M_1Nb$, $\hat{m}(t) = \{M_1p(t), m(t,t)\}$ and

$$N = M_4M_5\Bigg[|x_1| + M_1\big(\|\phi\| + L\big) + L + M_2Lb + M_1M_3Lb^2 + M_1\int_0^b p(s)\Omega_0\left(\|x_s\| + \int_0^s m(s,\tau)\Omega\big(\|x_\tau\|\big)\,d\tau\right)ds\Bigg].$$

Then the system (1) has a mild solution of the following form

$$x(t) = R(t,0)\big[\phi(0) + g(0,\phi)\big] - g(t,x_t) - \int_0^t R(t,s)A(s)g(s,x_s)\,ds - \int_0^t R(t,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds + \int_0^t R(t,s)\left[(Gu)(s) + f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right)\right]ds.$$

3. Controllability result

Theorem. *If the hypotheses* (i)–(ix) *are satisfied, then the system* (1) *is controllable on J.*

Proof. Consider the space $C_b = C([-r,b],X)$, with the norm

$$\|x\|_1 = \sup\big\{\big|x(t)\big| : -r \leqslant t \leqslant b\big\}.$$

Using the hypothesis (ii) for an arbitrary function $x(\cdot)$, define the control

$$u(t) = \widetilde{W}^{-1}\Bigg[x_1 - R(b,0)\big[\phi(0) + g(0,\phi)\big] + g(b,x_b) + \int_0^b R(b,s)A(s)g(s,x_s)\,ds + \int_0^b R(b,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds$$

$$- \int_0^b R(b,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) ds\Bigg](t).$$

Define $C_b^0 = \{x \in C_b\colon\ x_0 = \phi \text{ on } [-r, 0]\}$ and we now show that when using the control $u(t)$, the operator $F : C_b^0 \to C_b^0$, defined by

$$\begin{aligned}
(Fx)(t) &= R(t,0)\big[\phi(0) + g(0,\phi)\big] - g(t,x_t) - \int_0^t R(t,s)A(s)g(s,x_s)\, ds \\
&\quad - \int_0^t R(t,s) \int_0^s B(s,\tau)g(\tau,x_\tau)\, d\tau\, ds \\
&\quad + \int_0^t R(t,\eta) G \widetilde{W}^{-1}\Bigg[x_1 - R(b,0)\big(\phi(0) + g(0,\phi)\big) + g(b,x_b) \\
&\quad + \int_0^b R(b,s)A(s)g(s,x_s)\, ds \\
&\quad + \int_0^b R(b,s) \int_0^s B(s,\tau)g(\tau,x_\tau)\, d\tau\, ds \\
&\quad - \int_0^b R(b,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) ds\Bigg](\eta)\, d\eta \\
&\quad + \int_0^t R(t,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) ds
\end{aligned}$$

has a fixed point. This fixed point is then a solution of Eq. (1).

Clearly $x(b) = x_1$ which means that the control u steers the system (1) from the initial function ϕ to x_1 in time b, provided we can obtain a fixed point of the nonlinear operator F.

In order to study the controllability problem of (1), we introduce a parameter $\lambda \in (0,1)$ and consider the following system

$$\begin{aligned}
&\frac{d}{dt}\big[x(t) + \lambda g(t,x_t)\big] \\
&\quad = A(t)x(t) + \lambda \int_0^t B(t,s)x(s)\, ds + \lambda (Gu)(t)
\end{aligned}$$

$$+\lambda f\left(t, x_t, \int_0^t h(t,s,x_s)\,ds\right), \quad t \in J = [0,b],$$
$$x_0 = \lambda\phi, \quad \text{on } [-r,0]. \tag{3}$$

First we obtain a priori bounds for the mild solution of Eq. (3). Then from

$$\begin{aligned} x(t) = {} & \lambda R(t,0)\big[\phi(0)+g(0,\phi)\big] - \lambda g(t,x_t) - \lambda \int_0^t R(t,s)A(s)g(s,x_s)\,ds \\ & - \lambda \int_0^t R(t,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds \\ & + \lambda \int_0^t R(t,\eta)G\widetilde{W}^{-1}\Bigg[x_1 - R(b,0)\big(\phi(0)+g(0,\phi)\big) + g(b,x_b) \\ & + \int_0^b R(b,s)A(s)g(s,x_s)\,ds + \int_0^b R(b,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds \\ & - \int_0^b R(b,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right) ds\Bigg](\eta)\,d\eta \\ & + \lambda \int_0^t R(t,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right) ds, \end{aligned}$$

we have

$$\begin{aligned} \big|x(t)\big| \leqslant {} & M_1\big(\|\phi\|+L\big) + L + M_2Lb + M_1M_3Lb^2 \\ & + \int_0^t \big|R(t,\eta)\big| M_4M_5 \\ & \times \Bigg[|x_1| + M_1\big(\|\phi\|+L\big) + L + M_2Lb + M_1M_3Lb^2 \\ & + M_1\int_0^b p(s)\Omega_0\left(\|x_s\| + \int_0^s m(s,\tau)\Omega\big(\|x_\tau\|\big)\,d\tau\right) ds\Bigg] d\eta \\ & + M_1\int_0^t p(s)\Omega_0\left(\|x_s\| + \int_0^s m(s,\tau)\Omega\big(\|x_\tau\|\big)\,d\tau\right] ds. \end{aligned}$$

Consider the function μ defined by

$$\mu(t) = \sup\{|x(s)|:\ -r \leqslant s \leqslant t\}, \quad 0 \leqslant t \leqslant b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, b]$ by the previous inequality we have

$$\mu(t) \leqslant M_1(\|\phi\| + L) + L + M_2Lb + M_1M_3Lb^2 + M_1Nb + M_1\int_0^{t^*} p(s)\Omega_0\left[\mu(s) + \int_0^s m(s,\tau)\Omega(\mu(\tau))\,d\tau\right]ds,$$

$$\mu(t) \leqslant M_1(\|\phi\| + L) + L + M_2Lb + M_1M_3Lb^2 + M_1Nb + M_1\int_0^{t} p(s)\Omega_0\left[\mu(s) + \int_0^s m(s,\tau)\Omega(\mu(\tau))\,d\tau\right]ds.$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \geqslant 1$.

Denoting by $v(t)$ the right-hand side of the above inequality, we have

$$c = v(0) = M_1(\|\phi\| + L) + L + M_2Lb + M_1M_3Lb^2 + M_1Nb,$$
$$\mu(t) \leqslant v(t),\ 0 \leqslant t \leqslant b$$

and

$$v'(t) = M_1p(t)\Omega_0\left[\mu(t) + \int_0^t m(t,\tau)\Omega(\mu(\tau))\,d\tau\right] \leqslant M_1p(t)\Omega_0\left[v(t) + \int_0^t m(t,\tau)\Omega(v(\tau))\,d\tau\right].$$

Let $w(t) = v(t) + \int_0^t m(t,\tau)\Omega(v(\tau))\,d\tau$. Then $w(0) = v(0)$, $v(t) \leqslant w(t)$, and

$$\begin{aligned} w'(t) &= v'(t) + m(t,t)\Omega(v(t)) \\ &\leqslant M_1p(t)\Omega_0(w(t)) + m(t,t)\Omega(w(t)) \\ &\leqslant \hat{m}(t)[\Omega_0(w(t)) + \Omega(w(t))]. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leqslant \int_0^b \hat{m}(s)\,ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J.$$

This inequality implies that $w(t) < \infty$. So there is a constant K such that $v(t) \leqslant K$, $t \in J$ and hence $\mu(t) \leqslant K$, $t \in [0, b]$. Since $\|x_t\| \leqslant \mu(t)$, we have

$$\|x\|_1 = \sup\{|x(t)|:\ -r \leqslant t \leqslant b\} \leqslant K,$$

where K depends only on b and on the functions m, Ω, and Ω_0.

Next we must prove that the operator F is a completely continuous operator.

Let $B_k = \{x \in C_b^0\colon\ \|x\|_1 \leqslant k\}$ for some $k \geqslant 1$. We first show that the set $\{Fx\colon\ x \in B_k\}$ is equicontinuous. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leqslant b$,

$$\begin{aligned}
&\bigl|(Fx)(t_1) - (Fx)(t_2)\bigr| \\
&\quad\leqslant \bigl|R(t_1,0) - R(t_2,0)\bigr|\bigl|\phi(0) + g(0,\phi)\bigr| + \bigl|g(t_1,x_{t_1}) - g(t_2,x_{t_2})\bigr| \\
&\qquad + \left|\int_0^{t_1}\bigl[R(t_1,s) - R(t_2,s)\bigr]A(s)g(s,x_s)\,ds\right| \\
&\qquad + \left|\int_{t_1}^{t_2} R(t_2,s)A(s)g(s,x_s)\,ds\right| \\
&\qquad + \left|\int_0^{t_1}\bigl[R(t_1,s) - R(t_2,s)\bigr]\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds\right| \\
&\qquad + \left|\int_{t_1}^{t_2} R(t_2,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds\right| \\
&\qquad + \left|\int_0^{t_1}\bigl[R(t_1,\eta) - R(t_2,\eta)\bigr]G\widetilde{W}^{-1}\Bigg[x_1 - R(b,0)\bigl(\phi(0) + g(0,\phi)\bigr)\right. \\
&\qquad + g(b,x_b) + \int_0^b R(b,s)A(s)g(s,x_s)\,ds \\
&\qquad + \int_0^b R(b,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds \\
&\qquad \left. - \int_0^b R(b,s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right)ds\Bigg](\eta)\,d\eta\right| \\
&\qquad + \left|\int_{t_1}^{t_2} R(t_2,\eta)G\widetilde{W}^{-1}\Bigg[x_1 - R(b,0)\bigl(\phi(0) + g(0,\phi)\bigr) + g(b,x_b)\right. \\
&\qquad + \int_0^b R(b,s)A(s)g(s,x_s)\,ds + \int_0^b R(b,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^b R(b,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right) ds\Bigg](\eta)\,d\eta\Bigg| \\
&+ \left|\int_0^{t_1} \big[R(t_1,s) - R(t_2,s)\big] f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right) ds\right| \\
&+ \left|\int_{t_1}^{t_2} R(t_2,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right) ds\right| \\
\leqslant{} & \big|R(t_1,0) - R(t_2,0)\big|\big|\phi(0) + g(0,\phi)\big| + \big|g(t_1,x_{t_1}) - g(t_2,x_{t_2})\big| \\
&+ L\int_0^{t_1} \big|\big[R(t_1,s) - R(t_2,s)\big]A(s)\big|\,ds + L\int_{t_1}^{t_2} \big|R(t_2,s)A(s)\big|\,ds \\
&+ L\int_0^{t_1} \big|R(t_1,s) - R(t_2,s)\big| \int_0^s \big|B(s,\tau)\big|\,d\tau\,ds \\
&+ L\int_{t_1}^{t_2} \big|R(t_2,s)\big| \int_0^s \big|B(s,\tau)\big|\,d\tau\,ds \\
&+ \int_0^{t_1} \big|R(t_1,\eta) - R(t_2,\eta)\big| M_4 M_5 \Bigg[|x_1| \\
&+ M_1\big(\|\phi\| + L\big) + L + M_2 L b + M_1 M_3 L b^2 + M_1 \int_0^b \alpha_k(s)\,ds\Bigg] d\eta \\
&+ \int_{t_1}^{t_2} \big|R(t_2,\eta)\big| M_4 M_5 \Bigg[|x_1| + M_1\big(\|\phi\| + L\big) + L + M_2 L b \\
&+ M_1 M_3 L b^2 + M_1 \int_0^b \alpha_k(s)\,ds\Bigg] d\eta \\
&+ \int_0^{t_1} \big|R(t_1,s) - R(t_2,s)\big| \alpha_k(s)\,ds + \int_{t_1}^{t_2} \big|R(t_2,s)\big| \alpha_k(s)\,ds.
\end{aligned}
$$

The right-hand side is independent of $x \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since g is completely continuous and the compactness of $R(t,s)$ for $t, s > 0$ implies the continuity in the uniform operator topology.

Thus the set $\{Fx\colon x \in B_k\}$ is equicontinuous.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family FB_k is uniformly bounded. Next we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela–Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \leqslant s \leqslant b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$ we define

$$\begin{aligned}(F_\epsilon x)(t) = {} & R(t,0)\big[\phi(0) + g(0,\phi)\big] - g(t-\epsilon, x_{t-\epsilon}) \\ & - \int_0^{t-\epsilon} R(t,s)A(s)g(s,x_s)\,ds \\ & - \int_0^{t-\epsilon} R(t,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds \\ & + \int_0^{t-\epsilon} R(t,\eta)G\widetilde{W}^{-1}\Bigg[x_1 - R(b,0)\big(\phi(0)+g(0,\phi)\big) \\ & + g(b,x_b) + \int_0^b R(b,s)A(s)g(s,x_s)\,ds \\ & + \int_0^b R(b,s)\int_0^s B(s,\tau)g(\tau,x_\tau)\,d\tau\,ds \\ & - \int_0^b R(b,s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right)ds\Bigg](\eta)\,d\eta \\ & + \int_0^{t-\epsilon} R(t,s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right)ds.\end{aligned}$$

Since $R(t,s)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t)\colon x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $x \in B_k$ we have

$$\begin{aligned}&\big|(Fx)(t) - (F_\epsilon x)(t)\big| \\ &\quad \leqslant \big|g(t,x_t) - g(t-\epsilon, x_{t-\epsilon})\big| + \int_{t-\epsilon}^t \big|R(t,s)A(s)g(s,x_s)\big|\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \int_{t-\epsilon}^{t} \left| R(t,s) \int_0^s B(s,\tau) g(\tau, x_\tau) \right| d\tau\, ds \\
&+ \int_{t-\epsilon}^{t} \left| R(t,\eta) G \widetilde{W}^{-1} \Bigg[x_1 - R(b,0)\big(\phi(0) + g(0,\phi)\big) + g(b, x_b) \right. \\
&+ \int_0^b R(b,s) A(s) g(s, x_s)\, ds + \int_0^b R(b,s) \int_0^s B(s,\tau) g(\tau, x_\tau)\, d\tau\, ds \\
&\left. - \int_0^b R(b,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) ds \Bigg](\eta) \right| d\eta \\
&+ \int_{t-\epsilon}^{t} \left| R(t,s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) \right| ds \\
\leqslant{}& \big| g(t,x_t) - g(t-\epsilon, x_{t-\epsilon}) \big| + L \int_{t-\epsilon}^{t} \big| R(t,s) A(s) \big|\, ds \\
&+ L \int_{t-\epsilon}^{t} \big| R(t,s) \big| \int_0^s \big| B(s,\tau) \big|\, d\tau\, ds \\
&+ \int_{t-\epsilon}^{t} \big| R(t,\eta) \big| M_4 M_5 \Bigg[|x_1| + M_1\big(\|\phi\| + L\big) + L + M_2 L b \\
&+ M_1 M_3 L b^2 + M_1 \int_0^b \alpha_k(s)\, ds \Bigg] d\eta + \int_{t-\epsilon}^{t} \big| R(t,s) \big| \alpha_k(s)\, ds.
\end{aligned}
$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t)\colon\ x \in B_k\}$. Hence the set $\{(Fx)(t)\colon\ x \in B_k\}$ is precompact in X.

It remains to show that $F\colon\ C_b^0 \to C_b^0$ is continuous. Let $\{x_n\}_0^\infty \subseteq C_b^0$ with $x_n \to x$ in C_b^0. Then there is an integer r such that $|x_n(t)| \leqslant r$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$.

By (v) $f(t, x_{n_t}, \int_0^s h(s,\tau,x_{n_\tau})\, d\tau) \to f(t, x_t, \int_0^s h(s,\tau,x_\tau)\, d\tau)$ for almost each $t \in J$ and since $|f(t, x_{n_t}, \int_0^s h(s,\tau,x_{n_\tau})\, d\tau) - f(t, x_t, \int_0^s h(s,\tau,x_\tau)\, d\tau)| \leqslant 2\alpha_r(t)$ and also g is completely continuous, we have by dominated convergence theorem

$$
\begin{aligned}
&\|Fx_n - Fx\| \\
&\quad = \sup_{t\in J}\Bigg|\big[g(t,x_{n_t}) - g(t,x_t)\big] + \int_0^t R(t,s)A(s)\big[g(s,x_{n_s}) - g(s,x_s)\big]\,ds \\
&\qquad + \int_0^t R(t,s)\int_0^s B(s,\tau)\big[g(\tau,x_{n_\tau}) - g(\tau,x_\tau)\big]\,d\tau\,ds \\
&\qquad + \int_0^t R(t,\eta)G\widetilde{W}^{-1}\Bigg[g(b,x_{n_b}) - g(b,x_b) \\
&\qquad + \int_0^b R(b,s)A(s)\big[g(s,x_{n_s} - g(s,x_s)\big]\,ds \\
&\qquad + \int_0^b R(b,s)\int_0^s B(s,\tau)\big[g(\tau,x_{n_\tau}) - g(\tau,x_\tau)\big]\,d\tau\,ds \\
&\qquad + \int_0^b R(b,s)\Bigg[f\Bigg(s,x_{n_s},\int_0^s h(s,\tau,x_{n_\tau})\,d\tau\Bigg) \\
&\qquad - f\Bigg(s,x_s,\int_0^s h(s,\tau,x_\tau)\,d\tau\Bigg)\Bigg]\,ds\Bigg](\eta)\,d\eta \\
&\qquad + \int_0^t R(t,s)\Bigg[f\Bigg(s,x_{n_s},\int_0^s h(s,\tau,x_{n_\tau})\,d\tau\Bigg) \\
&\qquad - f\Bigg(s,x_s,\int_0^s h(s,\tau,x_\tau)\,d\tau\Bigg)\Bigg]\,ds\Bigg| \\
&\quad \leqslant \big|g(t,x_{n_t}) - g(t,x_t)\big| + \int_0^b \big|R(t,s)A(s)\big|\big|g(s,x_{n_s}) - g(s,x_s)\big|\,ds \\
&\qquad + \int_0^b \big|R(t,s)\big|\int_0^s \big|B(s,\tau)\big|\big|g(\tau,x_{n_\tau}) - g(\tau,x_\tau)\big|\,d\tau\,ds \\
&\qquad + \int_0^b \big|R(t,\eta)\big|M_4M_5\Bigg[\big|g(b,x_{n_b}) - g(b,x_b)\big|
\end{aligned}
$$

$$
\begin{aligned}
&+ M_2 \int_0^b \left| g(s, x_{n_s}) - g(s, x_s) \right| ds \\
&+ M_1 M_3 \int_0^b \int_0^s \left| g(\tau, x_{n_\tau}) - g(\tau, x_\tau) \right| d\tau\, ds \\
&+ M_1 \int_0^b \left| f\left(s, x_{n_s}, \int_0^s h(s, \tau, x_{n_\tau})\, d\tau\right) \right. \\
&\left. - f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)\, d\tau\right) \right| ds \Bigg]\, d\eta \\
&+ \int_0^b \left| R(t, s) \right| \left| f\left(s, x_{n_s}, \int_0^s h(s, \tau, x_{n_\tau})\, d\tau\right) \right. \\
&\left. - f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)\, d\tau\right) \right| ds \\
\to{}& 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{x \in C_b^0\colon\ x = \lambda F x, \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently by Schaefer's theorem the operator F has a fixed point in C_b^0. This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Hence, the system (1) is controllable on J. □

4. Application

Consider the following partial integrodifferential equation of the form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[z(y, t) + \mu_1\big(t, z(y, t - r)\big)\right] \\
&\quad = a(t, y) \frac{\partial^2}{\partial y^2} z(y, t) + \int_0^t b(t, s) z(y, s)\, ds \\
&\qquad + \mu(t, y) + \mu_2\left(t, z(y, t - r), \int_0^t \mu_3\big(t, s, z(y, s - r)\big)\, ds\right), \\
&\qquad 0 \leqslant y \leqslant 1,\ t \in J, \qquad (4)
\end{aligned}
$$

$$z(0,t) = z(1,t) = 0, \quad t \geqslant 0,$$
$$z(t,y) = \phi(y,t), \quad -r \leqslant t \leqslant 0,$$

where ϕ and $a(t,y)$ are continuous and satisfy certain smoothness conditions and $b(t,s)$ is continuous such that $|b(t,s)| \leqslant k$.

Let $g(t,w_t)(y) = \mu_1(t,w(t-y))$, $h(t,s,w_s)(y) = \mu_3(t,s,w(s-y))$ and $f(t,w_t,v)(y) = \mu_2(t,w(t-y),v(y))$.

Take $X = L^2(J)$ and define $A(t): X \to X$ by $A(t)w = a(t,y)w''$ with domain $D(A) = \{w \in X\colon\ w, w'$ are absolutely continuous, $w'' \in X,\ w(0) = w(1) = 0\}$, generates an evolution system and $R(t,s)$ can be extracted from the evolution system [7,12] such that $|R(t,s)| \leqslant n_1$ and $|R(t,s)A(s)| \leqslant n_2$.

Let $Gu: J \to X$ be defined by

$$(Gu)(t)(y) = \mu(t,y), \quad y \in (0,1).$$

With the choice of $A(t)$, $B(t,s)$, g, h and f, (1) is the abstract formulation of (4).

Assume that the linear operator W is given by

$$(Wu)(y) = \int_0^b R(b,s)\mu(s,y)\,ds, \quad y \in (0,1),$$

has a bounded invertible operator $\widetilde{W}^{-1}$ in $L^2(J,U)/\ker W$.

Further the function $\mu_1 : J \times [0,1] \to [0,1]$ is completely continuous and there exists a constant $k_1 > 0$ such that

$$\bigl|\mu_1\bigl(t, w(t-y)\bigr)\bigr| \leqslant k_1.$$

Also, the functions $\mu_3 : J \times J \times [0,1] \to [0,1]$ and $\mu_2 : J \times [0,1] \times [0,1] \to [0,1]$ are measurable and there exist integrable functions $l : J \times J \to [0,\infty)$, $q : J \to [0,\infty)$ such that

$$\bigl|\mu_3(t,s,w)\bigr| \leqslant l(t,s)\Omega_1\bigl(\|w\|\bigr),$$

and

$$\bigl|\mu_2(t,v,w)\bigr| \leqslant q(t)\Omega_2\bigl(\|v\| + |w|\bigr),$$

where $\Omega_1, \Omega_2 : [0,\infty) \to (0,\infty)$ is continuous, nondecreasing and

$$\int_0^b \hat{n}(s)\,ds < \int_c^\infty \frac{ds}{\Omega_1(s) + \Omega_2(s)},$$

where $c = n_1(\|\phi\| + k_1) + k_1 + n_2k_1b + n_1kk_1b^2 + n_1Nb$. Here N depends on μ_1, μ_2 and μ_3. Further all the conditions stated in the above theorem are satisfied. Hence the system (4) is controllable on J.

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References

[1] K. Balachandran, J.P. Dauer, P. Balasubramaniam, Local null controllability of nonlinear functional differential systems in Banach spaces, J. Optim. Theory Appl. 88 (1996) 61–75.
[2] K. Balachandran, R. Sakthivel, Controllability of integrodifferential systems in Banach spaces, Appl. Math. Comput. 118 (2001) 63–71.
[3] K. Balachandran, R. Sakthivel, Controllability of semilinear functional integrodifferential systems in Banach spaces, Kybernetika 36 (2000) 465–476.
[4] K. Balachandran, R. Sakthivel, J.P. Dauer, Controllability of neutral functional integrodifferential systems in Banach spaces, Comput. Math. Appl. 39 (2000) 117–126.
[5] W. Bian, Approximate controllability for semilinear systems, Acta Math. Hungar. 81 (1998) 41–57.
[6] N. Carmichel, M.D. Quinn, Fixed point methods in nonlinear control, in: Lecture Notes in Control and Inform. Sci., Vol. 75, Springer-Verlag, Berlin, 1984, pp. 24–51.
[7] R. Grimmer, Resolvent operators for integral equations in a Banach space, Trans. Amer. Math. Soc. 273 (1982) 333–349.
[8] R. Grimmer, J.H. Liu, Integrated semigroups and integrodifferential equations, Semigroup Forum 48 (1994) 79–95.
[9] J. Klamka, Schauder's fixed point theorem in nonlinear controllability problems, Control Cybernet. 29 (2000) 153–165.
[10] Y. Lin, J.H. Liu, Semilinear integrodifferential equations with nonlocal Cauchy problem, Nonlinear Anal. 26 (1996) 1023–1033.
[11] K. Naito, On controllability for a nonlinear Volterra equation, Nonlinear Anal. 18 (1992) 99–108.
[12] J. Pruss, On resolvent operators for linear integrodifferential equations of Volterra type, J. Integral Equations 5 (1983) 211–236.
[13] J.Y. Park, S.N. Kang, Approximate controllability of neutral functional differential systems with unbounded delay, Internat. J. Math. Math. Sci. 26 (2001) 737–744.
[14] J.Y. Park, H.K. Han, Controllability of nonlinear functional integrodifferential systems in Banach spaces, Nihonkai Math. J. 8 (1997) 47–53.
[15] D.R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1974.
[16] X. Zhang, Exact controllability of semilinear evolution systems and its application, J. Optim. Theory Appl. 107 (2000) 415–432.