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ORIGINAL ARTICLE

Sufficiency and duality in nondifferentiable minimax fractional programming with (H_p, r)-invexity

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Abstract In the present paper, we discuss the optimality condition for an optimal solution to the problem and a dual model is formulated for a non differentiable minimax fractional programming problem. Weak, strong and strict converse duality results are concerned involving (H_p, r)-invexity.

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1. Introduction

Fractional programming is an fascinating and interesting topic for research that appeared in several types of optimization problems. These programming are widely used in different branches of engineering and sciences, for example it can be used in engineering and economics to minimize a ratio of functions between a given period of time and utilized resource in order to measure the efficiency or productivity of a system. In these types of problems the objective function is usually given as a ratio of functions in fractional programming form (see Stancu-Minasian [1]).

Optimization problems with minimax type functions are arise in the design of electronic circuits, however, minimax fractional problems appear in formulation of discrete and continuous rational approximation problem with respect to the Chebyshev norm [2], in continuous rational games [3], in multiobjective programming [4], in engineering design as well as in some portfolio solution problems discussed by Bajaona-Xandari and Martinez-Legaz [5].

Yadav and Mukherjee [6] formulated two dual models for primal problem and derived duality theorem for convex differentiable minimax fractional programming, a step forward Chandra and Kumar [7] improved the dual formulation of Yadav and Mukherjee and they provided two modified dual problems for minimax fractional programming and proved duality results. Lai et al. [8] proved necessary and sufficient optimality conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to established a parametric dual model and also discussed duality results. Many papers are appeared in this direction (see Yuan et al. [9],

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Ahmad [10,11], Lai et al. [8], Hu et al. [12] and Lai and Lee [13]).

In the course of generalization of convex functions, Avriel [14] first introduced the definition of r-convex functions and established some characterizations and relations between r-convexity and other generalization of convexity . Antczak [16] introduced the concept of a class of r-preinvex functions, which is a generalization of r-convex function and preinvex function, and obtained some optimality results under r-preinvexity. Lee and Ho [15] established necessary and sufficient conditions for efficiency of multiobjective fractional programming problems involving r-invex functions, they also discussed Wolfe and Mond–Weir duality in this setting, Antczak [16] introduced p-invex sets and (p,r)-invex functions as a generalization of invex and preinvex functions. Ahmad et al. [10] worked out the duality in nondifferentiable minimax fractional programming with $B-(p,r)$-invexity. Recently, Jayswal et al. [17] investigated the duality for semi infinite programming problems involving (H_p,r)-invexity. Motivated by Jyaswal et al. [17] and Ahmad et al. [10], in this paper we investigate the duality for minimax fractional programming involving (H_p,r)-invexity.

We consider the following nondifferentiable minimax fractional programming problem

$$\text{Minimize} \quad \psi(x) = \sup_{y\in Y} \frac{f(x,y) + (x^TCx)^{\frac{1}{2}}}{g(x,y) - (x^TDx)^{\frac{1}{2}}} \quad \text{(NFP)}$$

$$\text{Subject to} \quad h(x) \leqslant 0, \quad x \in R^n,$$

where Y is a compact subset of R^l,$f(.,.)$, $g(.,.) : R^n \times R^l \to R$, $h(.,.) : R^n \to R^m$ are C^1 functions. C and D are $n\times n$ positive semidefinite symmetric matrices. Throughout this paper, we assume that $g(x,y) - (x^TDx)^{\frac{1}{2}} > 0$ and $f(x,y) + (x^TCx)^{\frac{1}{2}} \geqslant 0$, for all $(x,y) \in R^n \times R^l$.

2. Preliminaries

We start this section with the following some definitions

Definition 2.1. [18]. The weighted r-mean of a_1 and a_2 $(a_1,a_2 > 0)$ is given by

$$M_r(a_1,a_2;\lambda) = \begin{cases} (\lambda a_1^r + (1-\lambda)a_2^r)^{\frac{1}{r}} & \text{for } r\neq 0, \\ a_1^{\lambda} a_2^{1-\lambda} & \text{for } r = 0, \end{cases}$$

where $\lambda \in (0,1)$ and $r \in R$.

Definition 2.2. A subset $X \subseteq R^n$ is said to be $H_p - invex$ set, if for any $x,\ u \in X$, there exists a vector function $H_p : X \times X \times [0,1] \to R^n$, such that

$$H_p(x,u;0) = e^u, H_p(x,u;\lambda) \in R^n_+,$$

$$lnH_p(x,u;\lambda) \in X, \quad \forall \lambda \in [0,1], \quad p \in R.$$

Note 2.1. It is understood that the logarithm and the exponentials appearing in the above definition are taken to be component wise.

Throughout the paper, we take X to be a H_p-invex set unless otherwise specified, H_p-right differentiable at 0 with respect to the variable λ for each given pair $x,\ u \in X$ and $f : X \to R$ is differentiable function on X. The symbol $H'_p(x,u;0+) = \left(H'_{p^1}(x,u;0+),\ldots,H'_{p^n}(x,u;0+)\right)^T$ denotes the right derivative of H_p at 0 with respect to the variable λ for each given pair $x,\ u \in X$, $\nabla f(x) = (\nabla_1 f(x),\ldots,\nabla_n f(x))^T$ denotes the differential of f at x, and so $\frac{\nabla f(u)}{e^u} = \left(\frac{\nabla_1 f(u)}{e^u_1},\ldots\frac{\nabla_n f(u)}{e^u_n}\right)^T$.

Note 2.2. All the theorems in the subsequent parts of this paper will be proved only in the case when $r\neq 0$ and $r > 0$ (in the case when $r < 0$, the direction of some of the inequalities in the proof of the theorems should be changed to the opposite one).

Definition 2.3. A differentiable function $f : X \to R$ is said to be (strictly) (H_p,r)-invex at $u \in X$, if for all $x \in X$, one of the relations

$$\frac{1}{r}[e^{r(f(x)-f(u))} - 1] \geqslant \frac{\nabla f(u)^T}{e^u} H'_p(x,u;0+)(>) \quad \text{for } r\neq 0,$$

$$f(x) - f(u) \geqslant \frac{\nabla f(u)^T}{e^u} H'_p(x,u;0+)\ (>) \quad \text{for} \quad r = 0,$$

hold.

If the above inequalities are satisfied at any point $u \in X$ then f is said to be (H_p,r)-invex (strictly (H_p,r)-invex) on X.

Now we define the generalized (H_p,r)-invex functions as follows.

Definition 2.4. A differentiable function $f : X \to R$ is said to be (H_p,r)-pseudo invex at $u \in X$, if for all $x \in X$, the relations

$$\frac{\nabla f(u)^T}{e^u} H'_p(x,u;0+) \geqslant 0 \quad \Rightarrow \quad \frac{1}{r}[e^{r(f(x)-f(u))} - 1] \geqslant 0, \quad \text{for} \quad r\neq 0,$$

$$\frac{\nabla f(u)^T}{e^u} H'_p(x,u;0+) \geqslant 0 \quad \Rightarrow \quad f(x) - f(u) \geqslant 0, \quad \text{for} \quad r = 0,$$

hold.

Definition 2.4. A differentiable function $f : X \to R$ is said to be (Strictly) (H_p,r)-quasi invex at $u \in X$, if for all $x \in X$, the relations

$$\frac{1}{r}[e^{r(f(x)-f(u))} - 1] \leqslant 0, \quad \Rightarrow \quad \frac{\nabla f(u)^T}{e^u} H'_p(x,u;0+)\ (<) \leqslant 0 \quad \text{for} \quad r\neq 0,$$

$$f(x) - f(u) \leqslant 0, \quad \Rightarrow \quad \frac{\nabla f(u)^T}{e^u} H'_p(x,u;0+)\ (<) \leqslant 0 \quad \text{for} \quad r = 0,$$

hold.

3. Notations and preliminaries

Let $S = \{x \in R^n : h(x) \leqslant 0\}$ denotes the set of all feasible solutions (NFP). An point $x \in S$ is called the feasible point of (NFP). For each $(x,y) \in R^n \times R^l$, we define

$J(x) = \{j \in M = \{1, 2, \dots, m\},\ h_j(x) = 0\}$,

where $J = \{1, 2, \dots, m\}$,

$$Y(x) = \left\{ y \in Y : \frac{f(x,y) + (x^T Cx)^{\frac{1}{2}}}{g(x,y) - (x^T Dx)^{\frac{1}{2}}} = \sup_{z \in Y} \frac{f(x,z) + (x^T Cx)^{\frac{1}{2}}}{g(x,z) - (x^T Dx)^{\frac{1}{2}}} \right\},$$

$K(x) = \{(s, t, \bar{y}) \in N \times R^s_+ \times R^{ls} : 1 \leqslant s \leqslant n+1,\ t = (t_1, t_2, \dots t_s) \in R^s_+$ with $\sum_{i=1}^s t_i = 1,\ \bar{y} = (\bar{y}_1, \dots, \bar{y}_s)$ where $\bar{y}_i \in Y(x),\ i = 1, 2 \dots, s\}$.

Since f and g are C^1 functions and Y is compact in R^l, it follows that for each $x^* \in S$, $Y(x^*) \neq \phi$, and for any $\bar{y}_i \in Y(x^*)$, we have positive constant

$$K_0 = \psi(x^*, \bar{y}_i) = \frac{f(x^*, \bar{y}_i) + (x^{*T} Cx^*)^{\frac{1}{2}}}{g(x^*, \bar{y}_i) - (x^{*T} Dx^*)^{\frac{1}{2}}}.$$

Generalized Schwartz Inequality

Let A be a positive semidefinite matrix of order n. Then, for all, $x,\ w \in R^n$,

$$x^T Aw \leqslant (x^T Ax)^{1/2} (w^T Aw)^{1/2}. \tag{1}$$

Equality holds if for some $\lambda \geqslant 0$,

$Ax = \lambda Aw$.

Evidently, if $(w^T Aw)^{1/2} \leqslant 1$, we have

$x^T Aw \leqslant (x^T Ax)^{1/2}$.

If the functions f, g and h in problem (NFP) are continuously differentiable with respect to $x \in R^n$. Lai et al. [8] proved the following first order necessary condition for optimality of (NFP), which will be required to prove the strong duality theorem.

Theorem 1 (*Necessary Condition*). *Let x^* be a solution (local or global) of (NFP) satisfying $x^{*T} Cx^* > 0$, $x^{*T} Dx^* > 0$, and let $\nabla h_j(x^*)$, $j \in J(x^*)$ be linearly independent . Then there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$, $k_0 \in R_+$, w, $v \in R^n$ and $\mu^* \in R^m_+$ such that*

$$\sum_{i=1}^{s^*} t_i^* \{ (\nabla f(x^*, \bar{y}_i^*) + Cw - k_0 (\nabla g(x^*, \bar{y}_i^*) - Dv) \} + \nabla \sum_{j=1}^m \mu_j^* h_j(x^*) = 0, \tag{2}$$

$$f(x^*, \bar{y}_i^*) + (x^{*T} Cx^*)^{1/2} - k_0 \left(g(x^*, \bar{y}_i^*) - (x^{*T} Dx^*)^{1/2} \right) = 0, \quad i = 1, 2, \dots s^*, \tag{3}$$

$$\sum_{j=1}^m \mu_j^* h_j(x^*) = 0, \tag{4}$$

$$t_i^* \geqslant 0, \sum_{i=1}^{s^*} t_i^* = 1, \tag{5}$$

$$w^T Cw \leqslant 1, v^T Dv \leqslant 1, \quad (x^{*T} Cx^*)^{1/2} = x^{*T} Cw, \times (x^{*T} Dx^*)^{1/2} = x^{*T} Dv. \tag{6}$$

Now we discuss the sufficiency of the problem in the following Theorem

Theorem 2 (*Sufficient condition*). *Le x^* be a feasible solution of (NFP) and there exist a positive integer s, $1 \leqslant s \leqslant n+1$, $t^* \in R^s_+$, $\bar{y}_i \in y(x^*)$, $(i = 1, 2, \dots s)$, $k_0 \in R_+$, w, $v \in R^n$ and $\mu^* \in R^m_+$, satisfying the relation (2)–(6). Assume that*

(i) $\left[\sum_{i=1}^s t_i^* (f(., \bar{y}_i) + (.)^T Cw - k_0 (g(., \bar{y}_i) - (.)^T Dv)\right]$ is (H_p, r)- invex at x^*, and
(ii) $\sum_{j=1}^m \mu_j h_j(.)$ is (H_p, r)- quasiinvex at x^*.

Then x^* is an optimal solution of (NFP).

Proof. Suppose x^* is not an optimal solution of (NFP). Then there exists $\bar{x} \in s$, such that

$$\sup_{y \in Y} \frac{f(\bar{x}, y) + (\bar{x}^T C\bar{x})^{\frac{1}{2}}}{g(\bar{x}, y) - (\bar{x}^T D\bar{x})^{\frac{1}{2}}} < \sup_{y \in Y} \frac{f(x^*, y) + (x^{*T} Cx^*)^{\frac{1}{2}}}{g(x^*, y) - (x^{*T} Dx^*)^{\frac{1}{2}}}.$$

We note that

$$\sup_{y \in Y} \frac{f(x^*, y) + (x^{*T} Cx^*)^{\frac{1}{2}}}{g(x^*, y) - (x^{*T} Dx^*)^{\frac{1}{2}}} = \frac{f(x^*, \bar{y}_i) + (x^{*T} Cx^*)^{\frac{1}{2}}}{g(x^*, \bar{y}_i) - (x^{*T} Dx^*)^{\frac{1}{2}}} = k_0,$$

for $\bar{y}_i \in Y(x^*)$, $i = 1, 2, \dots s$ and

$$\sup_{y \in Y} \frac{f(\bar{x}, \bar{y}_i) + (\bar{x}^T C\bar{x})^{\frac{1}{2}}}{g(\bar{x}, \bar{y}_i) - (\bar{x}^T D\bar{x})^{\frac{1}{2}}} \leqslant \sup_{y \in Y} \frac{f(\bar{x}, y) + (\bar{x}^T C\bar{x})^{\frac{1}{2}}}{g(\bar{x}, y) - (\bar{x}^T D\bar{x})^{\frac{1}{2}}}.$$

Thus we have

$$\frac{f(\bar{x}, \bar{y}_i) + (\bar{x}^T C\bar{x})^{\frac{1}{2}}}{g(\bar{x}, \bar{y}_i) - (\bar{x}^T D\bar{x})^{\frac{1}{2}}} < k_0,$$

for $i = 1, 2, \dots, s$, or equivalently,

$$f(\bar{x}, \bar{y}_i) + (\bar{x}^T C\bar{x})^{\frac{1}{2}} - k_0 \left(g(\bar{x}, \bar{y}_i) - (\bar{x}^T D\bar{x})^{\frac{1}{2}} \right) < 0. \tag{7}$$

From (1), (3), (5)–(7), we obtain

$$\begin{aligned} &\sum_{i=1}^s t_i^* \{ f(\bar{x}, \bar{y}_i) + \bar{x}^T Cw - k_0 (g(\bar{x}, \bar{y}_i) - \bar{x}^T Dv) \} \\ &\leqslant \sum_{i=1}^s t_i^* \left\{ f(\bar{x}, \bar{y}_i) + (\bar{x}^T C\bar{x})^{\frac{1}{2}} - k_0 \left(g(\bar{x}, \bar{y}_i) - (\bar{x}^T D\bar{x})^{\frac{1}{2}} \right) \right\} \\ &< 0 = \sum_{i=1}^s t_i^* \left\{ f(x^*, \bar{y}_i) + (x^{*T} Cx^*)^{\frac{1}{2}} - k_0 \left(g(x^*, \bar{y}_i) - (x^{*T} Dx^*)^{\frac{1}{2}} \right) \right\} \\ &= \sum_{i=1}^s t_i^* \{ f(x^*, \bar{y}_i) + x^{*T} Cw - k_0 (g(x^*, \bar{y}_i) - x^{*T} Dv) \}. \end{aligned} \tag{8}$$

As $\sum_{i=1}^s t_i^* (f(., \bar{y}_i) + (.)^T Cw - k_0 (g(., \bar{y}_i) - (.)^T Dv)$ is (H_p, r)-invex function at x^*, we have

$$\frac{1}{r} \left[e^{r \left[\sum_{i=1}^s t_i^* (f(x, \bar{y}_i) + x^T Cw - k_0 (g(x, \bar{y}_i) - x^T Dv)) - \sum_{i=1}^s t_i^* (f(x^*, \bar{y}_i) + x^{*T} Cw - k_0 (g(x^*, \bar{y}_i) - x^{*T} Dv)) \right]} - 1 \right] \geqslant \frac{1}{e^{x^*}} \left\{ \sum_{i=1}^s t_i^* (\nabla f(x^*, \bar{y}_i) + Cw - k_0 (\nabla g(x^*, \bar{y}_i) - Dv)) \right\}^T H_p'(x, x^*, 0+)$$

holds for all $x \in S$ and so $x = \bar{x}$, the above inequality together with the inequality (8) gives

$$\frac{1}{e^{x^*}}\left\{\sum_{i=1}^{s} t_i^*(\nabla f(x^*,\bar{y}_i)+Cw-k_0(\nabla g(x^*,\bar{y}_i)-Dv))\right\}^T H_p'(\bar{x},x^*,0+) < 0. \tag{9}$$

By the feasibility of $\bar{x}$ for (NFP), $\mu^* \geqslant 0$ and (4), we get

$$\sum_{j=1}^{m}\mu_j^* h_j(\bar{x}) - \sum_{j=1}^{m}\mu_j^* g_j(x^*) \leqslant 0.$$

Since $r > 0$, using the fundamental properties of exponential function, the above inequality yields

$$\frac{1}{r}\left[e^{r\sum_{j=1}^{m}\left(\mu_j^* h_j(\bar{x})-\mu_j^* h_j(x^*)\right)} - 1\right] \leqslant 0. \tag{10}$$

The inequality (10) together with the assumption (ii) implies

$$\frac{1}{e^{x^*}}\sum_{j=1}^{m}\nabla\mu_j^* h_j(x^*)^T H_p'(\bar{x},x^*,0+) \leqslant 0. \tag{11}$$

By adding (9) and (11), we have

$$\frac{1}{e^{x^*}}\left\{\sum_{i=1}^{s} t_i^*\{(\nabla f(x^*,\bar{y}_i)+Cw-k_0(\nabla g(x^*,\bar{y}_i)-Dv)\} + \nabla\sum_{j=1}^{m}\mu_j^* h_j(x^*)\right\}^T H_p'(\bar{x},x*,0+) < 0,$$

which contradict (2). Hence the result. □

4. Duality results

In this section, we consider the following dual to (NFP)

$$\max_{(s,t,\bar{y})\in K(z)} \quad \sup_{(z,\mu,k,v,w)\in H_1(s,t,\bar{y})} k, \tag{FD}$$

where $H_1(s,t,\bar{y})$ denotes the set of all $(z,\mu,k,v,w) \in R^n \times R_+^m \times R_+ \times R^n \times R^n$ satisfying,

$$\sum_{i=1}^{s} t_i(\nabla f(z,\bar{y}_i)+Cw-k(\nabla g(z,\bar{y}_i)-Dv)\} + \nabla\sum_{j=1}^{m}\mu_j h_j(z) = 0, \tag{12}$$

$$\sum_{i=1}^{s} t_i\{(f(z,\bar{y}_i)z^T Cw - k(g(z,\bar{y}_i) - z^T Dv)\} \geqslant 0, \tag{13}$$

$$\sum_{j=1}^{m}\mu_j h_j(z), \geqslant 0. \tag{14}$$

$$(s,t,\bar{y}) \in k(z), \tag{15}$$

$$w^T Cw \leqslant 1, v^T Dv \leqslant 1. \tag{16}$$

If, for a triplet, $(s,t,\bar{y}) \in k(z)$, the set $H_1(s,t,\bar{y}) = \phi$, then we define the supremum over it to be $-\infty$.

Theorem 3 (*Weak Duality*). *Let x and $(z,\mu,k,v,w,s,t,\bar{y})$ be feasible solutions of (NFP) and (FD), respectively. Assume that*

(i) $\left[\sum_{i=1}^{s} t_i(f(.,\bar{y}_i)+(.)^T Cw - k(g(.,\bar{y}_i)-(.)^T Dv)\right]$ is (H_p,r)- invex at z, and
(ii) $\sum_{j=1}^{m}\mu_j h_j(.)$ is (H_p,r)- quasiinvex at z.

Then,

$$\sup_{y\in Y}\frac{f(x,y)+(x^T Cx)^{\frac{1}{2}}}{g(x,y)-(x^T Dx)^{\frac{1}{2}}} \geqslant k. \tag{17}$$

Proof. Suppose to the contrary that

$$\sup_{y\in Y}\frac{f(x,y)+(x^T Cx)^{\frac{1}{2}}}{g(x,y)-(x^T Dx)^{\frac{1}{2}}} < k.$$

Then we have

$$f(x,\bar{y}_i)+(x^T Cx)^{\frac{1}{2}} - k(g(x,\bar{y}_i)-(x^T Dx)^{\frac{1}{2}})) < 0,$$

for all $\bar{y}_i \in Y$.

It follows from (5) that

$$\sum_{i=1}^{s} t_i\left\{f(x,\bar{y}_i)+(x^T Cx)^{\frac{1}{2}} - k(g(x,\bar{y}_i)-(x^T Dx)^{\frac{1}{2}})\right\} < 0. \tag{18}$$

As same line of proof of inequality (8), from (1), (13), (16) and (18) , we have

$$\sum_{i=1}^{s} t_i\{f(x,\bar{y}_i)+x^T Cw-k(g(x,\bar{y}_i)-x^T Dv)\} - \sum_{i=1}^{s} t_i\{f(z,\bar{y}_i)+z^T Cw-k(g(z,\bar{y}_i)-z^T Dv)\} < 0. \tag{19}$$

Since $\sum_{i=1}^{s} t_i^*(f(.,\bar{y}_i)+(.)^T Cw - k_0(g(.,\bar{y}_i)-(.)^T Dv)$ is (H_p,r)-invex at z, then we have

$$\frac{1}{r}\left[e^{r\left[\sum_{i=1}^{s} t_i(f(x,\bar{y}_i)+x^T Cw-k(g(x,\bar{y}_i)-x^T Dv)) - \sum_{i=1}^{s} t_i(f(z,\bar{y}_i)+z^T Cw-k(g(z,\bar{y}_i)-z^T Dv))\right]} - 1\right] \geqslant \frac{1}{e^z}\left\{\sum_{i=1}^{s} t_i(\nabla f(z,\bar{y}_i)+Cw-k(\nabla g(z,\bar{y}_i)-Dv))\right\}^T H_p'(x,z,0+).$$

From (19) together with the above inequality, we get

$$\frac{1}{e^z}\left\{\sum_{i=1}^{s} t_i(\nabla f(z,\bar{y}_i)+Cw-k(\nabla g(z,\bar{y}_i)-Dv))\right\}^T H_p'(x,z,0+) < 0. \tag{20}$$

By the feasibility of x for (NFP), $\mu \geqslant 0$ and (14), we get

$$\sum_{j=1}^{m}\mu_j h_j(x) - \sum_{j=1}^{m}\mu_j g_j(z) \leqslant 0.$$

Since $r > 0$, using the fundamental properties of exponential functions, the above inequality yields

$$\frac{1}{r}\left[e^{r\sum_{j=1}^{m}(\mu_j h_j(x)-\mu_j h_j(z))} - 1\right] \leqslant 0. \tag{21}$$

The above inequality together with the assumption (ii) implies

$$\frac{1}{e^z}\sum_{j=1}^{m}\nabla\mu_j h_j(z)^T H_p'(x,z,0+) \leqslant 0. \qquad (22)$$

Thus by (20) and (22), we obtain the following inequality

$$\frac{1}{e^z}\left\{\sum_{i=1}^{s} t_i\{(\nabla f(z,\bar{y}_i)+Cw-k(\nabla g(z,\bar{y}_i)-Dv)\}+\nabla\sum_{j=1}^{m}\mu_j h_j(z)\right\}^T$$
$$H_p'(x,z,0+)<0,$$

which contradict (12). Hence (17) holds. □

We can prove the following theorem similar as Theorem 3.

Theorem 4 (*Weak Duality*). *Let x and $(z,\mu,k,v,w,s,t,\bar{y})$ be feasible solutions of (NFP) and (FD), respectively. Assume that*

(i) $\left[\sum_{i=1}^{s} t_i(f(.,\bar{y}_i)+(.)^T Cw-k(g(.,\bar{y}_i)-(.)^T Dv)\right]$ is (H_p,r)- psedoinvex at z, and

(ii) $\sum_{j=1}^{m}\mu_j h_j(.)$ is (H_p,r)- quasiinvex at z.

Then,

$$\sup_{y\in Y}\frac{f(x,y)+(x^T Cx)^{\frac{1}{2}}}{g(x,y)-(x^T Dx)^{\frac{1}{2}}} \geqslant k.$$

Theorem 5 (*Strong Duality*). *Let x^* be an optimal solution of (NFP) and $\nabla h_j(x^*)$, $j\in J(x^*)$ is linearly independent. Then there exist $(\bar{s},\bar{t},\bar{y})\in K(x^*)$ and $(x^*,\bar{\mu},\bar{k},\bar{v},\bar{w})\in H_1(\bar{s},\bar{t},\bar{y}^*)$ such that $(x^*,\bar{\mu},\bar{k},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^*)$ is a feasible solution of (FD). In addition, if the hypothesis of weak duality theorem are satisfied for all feasible solutions $(z,\mu,k,v,w,s,t,\bar{y})$ of (FD), then $(x^*,\bar{\mu},\bar{k},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^*)$ is an optimal solution of (FD), and the two objectives have the same optimal values.*

Proof. If x^* be an optimal solution and $\nabla h_j(x^*)$, $j\in J(x^*)$ is linearly independent, then by Theorem 1, there exist $(\bar{s},\bar{t},\bar{y}^*)\in K(x^*)$ and such that $(x^*,\bar{\mu},\bar{k},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^*)$ is feasible for (FD) and problem (NFP) and (FD) have same objective values and

$$\frac{f(x^*,\bar{y}_i^*)+(x^{*T}Cx^*)^{\frac{1}{2}}}{g(x^*,\bar{y}_i^*)-(x^{*T}Dx^*)^{\frac{1}{2}}}=\bar{k}.$$

The optimality of this feasible solution for (FD) thus follows from Theorem 3. □

Theorem 6 (*Strict Converse duality*). *Let x^* and $(z^*,\mu^*,k^*,v^*,w^*,s^*,t^*,\bar{y}^*)$ be optimal solution of (NFP) and (FD), respectively, suppose that*

(i) $\nabla h_j(x^*)$, $j\in J(x^*)$ are linearly independent,

(ii) $\left[\sum_{i=1}^{s^*} t_i^*\left(f(.,\bar{y}_i^*)+(.)^T Cw^*-k^*(g(.,\bar{y}_i^*)-(.)^T Dv^*\right)\right]$ is (H_p,r)- invex at z^*, and

(iii) $\sum_{j=1}^{m}\mu_j^* h_j(.)$ is (H_p,r)- quasiinvex at z^*.

Then $z^*=x^*$,

Proof. We shall assume that $x^*\neq z^*$ and reach a contradiction. From the strong duality Theorem (Theorem 5), it follows that

$$\frac{f(x^*,\bar{y}_i^*)+(x^{*T}Cx^*)^{\frac{1}{2}}}{g(x^*,\bar{y}_i^*)-(x^{*T}Dx^*)^{\frac{1}{2}}}=\bar{k}. \qquad (23)$$

Thus, we have

$$\left[f(x^*,\bar{y}_i^*)+(x^{*T}Cx^*)^{\frac{1}{2}}-k^*\left(g(x^*,\bar{y}_i^*)-(x^{*T}Dx^*)^{\frac{1}{2}}\right)\leqslant 0, \qquad (24)$$

for all $\bar{y}_i^*\in Y(x^*)$, $i=1,2,\ldots,s^*$.

Now, proceeding as in Theorem 3, we get

$$\frac{1}{e^{z^*}}\left\{\sum_{i=1}^{s^*} t_i^*\left(\nabla f(z^*,\bar{y}_i^*)+Cw^*-k^*\left(\nabla g(z^*,\bar{y}_i^*)-Dv^*\right)\right)\right\}^T$$
$$H_p'(x^*,z^*,0+)<0, \qquad (25)$$

and

$$\frac{1}{e^{z^*}}\sum_{j=1}^{m}\nabla\mu_j^* h_j(z^*)^T H_p'(x^*,z^*,0+)\leqslant 0, \qquad (26)$$

adding (25) and (26), we get the contradiction of (12), hence $x^*=z^*$. □

5. Conclusion and further development

In this paper, we have established, optimality condition for a class of nondifferentiable minimax fractional programming problems. Further, weak, strong and strict converse duality theorems are discussed for nondifferentiable minimax fractional programming problems in the framework of $(H_p,r)-$invexity. This paper generalized the results of Jayswal et al. [17].

The question arises as to whether the results developed in this paper hold for the following complex nondifferentiable minimax fractional problem.

$$\text{Minimize}\quad \psi(\xi)=\sup_{v\in W}\frac{Re\left[f(\xi,v)+(z^T Cz)^{\frac{1}{2}}\right]}{Re\left[g(\xi,v)-(z^T Dz)^{\frac{1}{2}}\right]}$$
$$\text{Subject to}\quad -h(z)\in S,\quad \xi\in C^{2n},$$

where $\xi=(z,\bar{z}), v=(w,w)$ for $z\in C^n, w\in C^l, f(.,.)$, $g(.,.): C^{2n}\times C^{2l}\to C$ are analytic with respect to W, W is a specified compact subset in C^{2l}, S is a polyhedral cone in C^m, and $g: C^{2n}\to C^m$ is analytic. Also $C,D\in C^{n\times n}$ are positive semidefinite Hermitian matrices.

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