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Non-commutativity in a time-dependent background

Louise Dolan [a], Chiara R. Nappi [b]

[a] *Department of Physics, University of North Carolina, Chapel Hill, NC 27599-3255, USA*
[b] *Department of Physics, Jadwin Hall Princeton University, Princeton, NJ 08544-0708, USA*

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Abstract

We compute a time-dependent non-commutativity parameter in a model with a time-dependent background, a spacetime metric of the plane wave type supported by a Neveu–Schwarz two-form potential. This model is the open string version of the WZW model based on a non-semi-simple group previously studied by Nappi and Witten. Like its closed string counterpart, it is exactly conformally invariant to all orders in α'. We quantize the sigma-model in light-cone gauge, compute the worldsheet propagator, and use it to derive the non-commutativity parameter.

1. Introduction

Non-commutativity in string theory is a very interesting topic, as it may have important implications for the structure of spacetime. Non-commutativity has emerged in the context of open strings, starting from the treatment of open string field theory in [1]. More recently, it has reappeared in the context of Matrix theory compactified on a torus [2,3], and in the low energy description of strings in an electromagnetic background [4,5].

It is interesting to find other models in which non-commutativity emerges. In most of the examples currently known, the non-commutativity parameter is constant. An obvious task is to look for time-dependent non-commutativity parameters, especially given the recent interest in strings on time-dependent backgrounds [6–19].

In this Letter we study an exactly conformally invariant open string model, whose target space has a plane wave metric supported by a time-dependent Neveu–Schwarz two-form potential. This background was studied by Nappi and Witten [20] for closed strings. Here we are looking at the open string version, and by computing the worldsheet propagator we can derive a time-dependent non-commutativity parameter. It is important that the background is of the Neveu–Schwarz type: plane waves with Ramond fields remain commutative as the Ramond background amounts to the addition of a mass term to the action in light-cone gauge. In our case, for large values of the time parameter, our model reduces to a neutral string in a constant background B field [4,21], hence, it is a good candidate for spacetime non-commutativity.

E-mail addresses: dolan@physics.unc.edu (L. Dolan), cnappi@princeton.edu (C.R. Nappi).

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In Section 2, we show the open string model is conformally invariant to all orders in α', and quantize the model in light-cone gauge. The mode expansion of a closed string version of this model has been explicitly exhibited in [22,23]. We compute the open mode expansion as a power series in a suitable parameter μ. This expansion is adequate to show non-commutativity. In Section 3 the worldsheet propagator is derived on the disk. In Section 4 we evaluate the propagator on the boundaries and compute a time-dependent non-commutativity parameter. The techniques used in this calculation are similar to those of [21] which analyzes strings in a $U(1)\times U(1)$ background.

2. An exactly conformally invariant time-dependent background

The Polyakov action coupling a string to a general metric and background Neveu–Schwarz field is

$$S=\int_\Sigma d\tau\, d\sigma\big[\sqrt{-\gamma}\,\gamma^{\alpha\beta}G_{MN}\partial_\alpha X^M\partial_\beta X^M+B_{MN}\epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\big] \tag{2.1}$$

where we choose the string worldsheet Σ with Lorentz signature, and have rescaled the scalar worldsheet fields by $(2\sqrt{\pi\alpha'})^{-1}$ so that the X^M are dimensionless. We consider the time-dependent background provided by the Nappi–Witten WZW model based on a non-semi-simple group, and adopt the same notation as in [20], with $X^M=(a_1,a_2,u,v)$, and u being identified with the time in the target space

$$G_{MN}=\begin{pmatrix}1&0&\frac{a_2}{2}&0\\0&1&-\frac{a_2}{2}&0\\\frac{a_2}{2}&-\frac{a_1}{2}&b&1\\0&0&1&0\end{pmatrix},\qquad B_{MN}=\begin{pmatrix}0&u&0&0\\-u&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{2.2}$$

The Lorentz signature target space metric G_{MN} can be recognized as a plane wave metric [20]. The time-dependence is the u-dependence of B_{12}. Nappi and Witten checked that this model is exactly conformally invariant (i.e., to all orders in α') by showing the one-loop β function equations for the closed string backgrounds were satisfied, and then proving there were no higher order graphs.

In this Letter, since we are interested in non-commutativity, we consider open string boundary conditions. We can show exact conformal invariance also in this case. Indeed, the background (2.2) satisfies the Born–Infeld field equations

$$(D_MF_{NL})\big(1-F^2\big)^{-1LM}=0, \tag{2.3}$$

where $(1-F^2)^{-1LM}=(1+F)^{-1LP}G_{PN}(1-F)^{-1NM}$ and $(1-F)_{MN}\equiv G_{MN}-2\pi\alpha' F_{MN}$. In our case $F_{MN}=B_{MN}$. For (2.2) the non-vanishing components of the Ricci tensor and affine connections are $R_{uu}=-\frac{1}{2}$, $\Gamma^i_{uj}=\frac12\epsilon^i_{\ j}$, $\Gamma^v_{ui}=-\frac{a^i}{4}$. It follows that $(D_MF_{NL})(1-F^2)^{-1LM}=\epsilon_{ij}(1-F^2)^{-1ju}=0$. Moreover the higher order in α' contributions vanish as in the closed string case [20,24].

As in [20], the sigma model action is (2.1):

$$S=\int_\Sigma d\tau\, d\sigma\big[\sqrt{-\gamma}\,\gamma^{\alpha\beta}\big(\partial_\alpha a^i\partial_\beta a^i+2\partial_\alpha u\partial_\beta v+b\partial_\alpha u\partial_\beta u+\epsilon_{ij}\partial_\alpha u\partial_\beta a^i a^j\big)+\epsilon^{\alpha\beta}\epsilon_{ij}u\partial_\alpha a^i\partial_\beta a^j\big]. \tag{2.4}$$

Although this action has a cubic interaction, if one treats it as a closed string theory, it is possible to find an exact mode expansion in the light-cone gauge [22,23]. However, in considering it as an open string theory, one has different boundary conditions which make the solution more complicated. Consequently, we will solve the theory in light-cone gauge only via a power series expansion. For simplicity, we work to lowest order in μ, where μ is a dimensionless constant, as this is sufficient to prove non-commutativity. It is quite possible that another version of this model, differing from (2.4) via boundary terms, would lead to an exact mode expansion.

To implement light-cone gauge, we find the Virasoro constraints from varying (2.4) with respect to $\gamma_{\alpha\beta}$. In orthonormal gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$, they are given by

$$\partial_\alpha X^M \partial_\beta X^M G_{MN} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^M \partial_\delta X^N G_{MN} = 0 \tag{2.5}$$

for the background (2.2). Here $\eta^{\alpha\beta}$ is the Minkowski worldsheet metric $\eta^{\tau\tau} = -1, \eta^{\sigma\sigma} = 1$. We will use $\Box \equiv -\partial_\tau^2 + \partial_\sigma^2$. In orthonormal gauge, (2.1) becomes

$$S = \int_\Sigma d\tau\, d\sigma \big[\eta^{\alpha\beta}\big(\partial_\alpha a^i \partial_\beta a^i + 2\partial_\alpha u \partial_\beta v + b\partial_\alpha u \partial_\beta u + \epsilon_{ij}\partial_\alpha u \partial_\beta a^i a^j\big) + \epsilon^{\alpha\beta}\epsilon_{ij} u \partial_\alpha a^i \partial_\beta a^j\big] \tag{2.6}$$

where $\epsilon^{\tau\sigma} = 1$, and for the open string $-\infty \leqslant \tau \leqslant \infty, 0 \leqslant \sigma \leqslant \pi$. The equations of motion and Neumann boundary conditions obtained by extremizing (2.6) with respect to $X^M(\sigma, \tau)$ are

$$\begin{aligned}
&\Box a^i + \frac{1}{2}\epsilon_{ij} a^j \Box u + \epsilon_{ij}\big(\eta^{\alpha\beta} + \epsilon^{\alpha\beta}\big)\partial_\alpha u \partial_\beta a^j = 0,\\
&\partial_\sigma a_i + \frac{1}{2}\partial_\sigma u \epsilon_{ij} a^j - \epsilon_{ij} u \partial_\tau a^j\big|_{\sigma=0,\pi} = 0,\\
&\Box v + b\Box u + \frac{1}{2}\epsilon_{ij} a^j \Box a^i - \frac{1}{2}\epsilon_{ij}\epsilon^{\alpha\beta}\partial_\alpha a^i \partial_\beta a^j = 0,\\
&\partial_\sigma v + b\partial_\sigma u + \frac{1}{2}\epsilon_{ij} a^j \partial_\sigma a^i\big|_{\sigma=0,\pi} = 0,\\
&\Box u = 0, \qquad \partial_\sigma u\big|_{\sigma=0,\pi} = 0.
\end{aligned} \tag{2.7}$$

As in flat target space, here we can use the residual worldsheet gauge invariance to choose the light-cone gauge condition: $u = \mu\tau$, for μ is a dimensionless constant. In this gauge we can solve the constraints (2.5) for the dependent variable v:

$$\begin{aligned}
&\mu\partial_\tau v = -\frac{1}{2}\partial_\tau a^i \partial_\tau a^i - \frac{1}{2}\partial_\sigma a^i \partial_\sigma a^i - \frac{b}{2}\mu^2 - \frac{1}{2}\mu\epsilon_{ij}\partial_\tau a^i a^j,\\
&\mu\partial_\sigma v = -\partial_\tau a^i \partial_\sigma a^i - \frac{\mu}{2}\epsilon_{ij}\partial_\sigma a^i a^j.
\end{aligned} \tag{2.8}$$

The equations of motion and boundary conditions for the transverse fields a^i written in terms of $X \equiv a^1 + ia^2$ and $\widetilde{X} \equiv a^1 - ia^2$ become:

$$\begin{aligned}
&\Box X - i\mu(\partial_\sigma X - \partial_\tau X) = 0, \qquad \Box\widetilde{X} + i\mu\big(\partial_\sigma \widetilde{X} - \partial_\tau \widetilde{X}\big) = 0,\\
&[\partial_\sigma X + i\mu\tau\partial_\tau X]|_{\sigma=0,\pi} = 0, \qquad \big[\partial_\sigma \widetilde{X} - i\mu\tau\partial_\tau \widetilde{X}\big]\big|_{\sigma=0,\pi} = 0,
\end{aligned} \tag{2.9}$$

where $\Box \equiv -\partial_\tau^2 + \partial_\sigma^2 = 4z\bar{z}\partial_z\partial_{\bar{z}}$.

For large τ (so that τ can be considered constant), notice the similarity of the boundary condition in (2.9) with the boundary condition for an open string in a background B field. Since in the latter case the non-commutativity parameter is proportional to the background, this suggests we should expect here a non-commutativity parameter which depends on time.

The solution of (2.9) is given by the normal mode expansion for the transverse coordinates X and $\widetilde{X}$, to first order in μ:

$$\begin{aligned}
X(\sigma,\tau) = {} & x_0 + a_0\left[\tau + \mu\left(-i\tau\sigma + \frac{i}{2}\tau^2\right)\right]\\
&+ \sum_{n\neq 0} a_n e^{-in\tau}\left[\frac{i}{n}\cos n\sigma + \mu\left(\left(-\frac{1}{2n^2} - i\frac{\tau}{n}\right)\sin n\sigma + \left(\frac{i}{2n^2} + \frac{(\sigma-\tau)}{2n}\right)\cos n\sigma\right)\right] + O\big(\mu^2\big),
\end{aligned}$$

$$\widetilde{X}(\sigma,\tau) = \tilde{x}_0 + \tilde{a}_0\left[\tau - \mu\left(-i\tau\sigma + \frac{i}{2}\tau^2\right)\right] + \sum_{n\neq 0}\tilde{a}_n e^{-in\tau}\left[\frac{i}{n}\cos n\sigma - \mu\left(\left(-\frac{1}{2n^2} - i\frac{\tau}{n}\right)\sin n\sigma + \left(\frac{i}{2n^2} + \frac{(\sigma-\tau)}{2n}\right)\cos n\sigma\right)\right] + O\left(\mu^2\right). \tag{2.10}$$

We have derived (2.10) as follows. In (2.9) substitute $X(\sigma,\tau) = e^{i\frac{\mu}{2}(\tau+\sigma)}\phi(\sigma,\tau)$, and find

$$\Box\phi = 0,$$
$$\left[(\partial_\sigma + i\mu\tau\partial_\tau)\phi + i\frac{\mu}{2}(1+i\mu\tau)\phi\right]\Big|_{\sigma=0,\pi} = 0. \tag{2.11}$$

One such solution is $\phi(\sigma,\tau) = x_0 e^{-i\frac{\mu}{2}(\tau+\sigma)}$, corresponding to the constant mode $X(\sigma,\tau) = x_0$. A general solution to the wave equation $\Box\phi = 0$ is

$$\phi(\sigma,\tau) = f(\tau+\sigma) + g(\tau-\sigma). \tag{2.12}$$

So the constant solution above corresponds to $\phi(\sigma,\tau) = f(\tau+\sigma) = x_0 e^{-i\frac{\mu}{2}(\tau+\sigma)}$, and $g(\tau-\sigma) = 0$. To generate the solutions which provide the coefficients of a_0 and a_n in the normal mode expansion of $X(\sigma,\tau)$, we will try to find solutions $\phi(\sigma,\tau) = f(\tau+\sigma) + g(\tau-\sigma)$ satisfying the boundary conditions (2.11) via the power series expansions

$$f(\tau+\sigma) = \sum_{p=0}^{\infty} C_p(\tau+\sigma)^p$$
$$g(\tau-\sigma) = \sum_{p=0}^{\infty} D_p(\tau-\sigma)^p \tag{2.13}$$

and

$$f_n(\tau+\sigma) = e^{-in(\tau+\sigma)}\sum_{p=0}^{\infty} C_p(n)(\tau+\sigma)^p,$$
$$g_n(\tau-\sigma) = e^{-in(\tau-\sigma)}\sum_{p=0}^{\infty} D_p(n)(\tau-\sigma)^p, \tag{2.14}$$

respectively. A solution of (2.11), in the form of (2.13) is

$$\mu\phi(\sigma,\tau) = \mu\tau + \mu^2\left[-i\frac{3}{2}\tau\sigma\right] + \mu^3\left[\frac{1}{2}\tau^2\sigma + \frac{1}{6}\sigma^3 - \frac{9}{8}\tau\sigma^2 - \frac{3}{8}\tau^3 - \frac{\pi}{4}\left(\tau^2+\sigma^2\right)\right] + i\mu^4\left[-\frac{1}{6}\tau^4 + \frac{21}{16}\tau^3\sigma - \tau^2\sigma^2 + \frac{21}{16}\tau\sigma^3 - \frac{1}{6}\sigma^4 + \pi\left(-\frac{3}{8}\tau^3 + \frac{5}{8}\tau^2\sigma - \frac{9}{8}\tau\sigma^2 + \frac{5}{24}\sigma^3\right) + \frac{\pi^2}{24}\left(\tau^2+\sigma^2\right)\right] + O\left(\mu^5\right), \tag{2.15}$$

where the functions f and g are given by

$$\mu f(\tau) = \frac{\mu}{2}\tau - i\frac{3}{8}\mu^2\tau^2 - \mu^3\frac{\pi}{8}\tau^2 - \frac{5}{48}\mu^3\tau^3 + i\frac{31}{3\cdot 128}\mu^4\tau^4 + i\mu^4\left(-\frac{\pi}{12}\tau^3 + \frac{\pi^2}{48}\tau^2\right) + O\left(\mu^5\right),$$
$$\mu g(\tau) = \frac{\mu}{2}\tau + i\frac{3}{8}\mu^2\tau^2 - \mu^3\frac{\pi}{8}\tau^2 - \frac{13}{48}\mu^3\tau^3 - i\frac{95}{3\cdot 128}\mu^4\tau^4 + i\mu^4\left(-\frac{7\pi}{24}\tau^3 + \frac{\pi^2}{48}\tau^2\right) + O\left(\mu^5\right). \tag{2.16}$$

These expressions are derived iteratively, by considering the solution of (2.11) to some order μ^p, and then integrating the boundary condition to find the solution to order μ^{p+1}. Since finding a general form inarbitrary p, and summing these series to a closed form is difficult, we work to first order in μ. Note that although τ, σ could be rescaled to essentially eliminate μ, we keep it here to track the order in the power series solution of (2.11). The series in (2.16) are reminiscent of hypergeometric functions. To derive the coefficient of a_n, we use the ansatz (2.14) to find

$$\phi_n(\sigma,\tau) = ie^{-in\tau}\left[\cos n\sigma + \mu\left(\left(-\tau + \frac{i}{2n}\right)\sin n\sigma + \left(-i\sigma + \frac{1}{2n}\right)\cos n\sigma\right) + O\big(\mu^2\big)\right], \tag{2.17}$$

where $\phi_n(\sigma,\tau) = f_n(\tau+\sigma) + g_n(\tau-\sigma)$ with

$$f_n(\tau) = ie^{-in\tau}\left[\frac{1}{2} + \mu\left(-\frac{i}{2}\tau\right) + O\big(\mu^2\big)\right],$$
$$g_n(\tau) = ie^{-in\tau}\left[\frac{1}{2} + \mu\left(\frac{i}{2}\tau + \frac{1}{2n}\right) + O\big(\mu^2\big)\right]. \tag{2.18}$$

We then construct the normal mode expansion that satisfies (2.9) from

$$X(\sigma,\tau) = x_0 + e^{i\frac{\mu}{2}(\tau+\sigma)}a_0\phi(\sigma,\tau) + e^{i\frac{\mu}{2}(\tau+\sigma)}\sum_{n\neq 0} a_n\phi_n(\sigma,\tau). \tag{2.19}$$

From (2.15) and (2.17), we see that $X(\sigma,\tau)$ is given by an expansion where the coefficients of a_0, a_n are themselves a double power series in σ and τ. Although our open string model satisfies an equation of motion that can be simply related to the one-dimensional wave equation (2.9), the particular boundary condition that is required substantially complicates the form of the solution. (2.10) is reproduced by expanding (2.19) to first order in μ, using (2.15) and (2.17). Let $\mu \to -\mu$ to find $\widetilde{X}(\sigma,\tau)$.

To quantize the theory in standard form, we reinsert the scale $2\sqrt{\pi\alpha'}$ so that X, $\widetilde{X}$ become fields with length dimension, and find the canonical momenta:

$$P(\sigma,\tau) = -\frac{\delta S}{\delta\partial_\tau X} = \frac{1}{4\pi\alpha'}\left(\partial_\tau\widetilde{X} + i\frac{\mu}{2}\widetilde{X} - i\mu\tau\partial_\sigma\widetilde{X}\right),$$
$$\widetilde{P}(\sigma,\tau) = -\frac{\delta S}{\delta\partial_\tau \widetilde{X}} = \frac{1}{4\pi\alpha'}\left(\partial_\tau X - i\frac{\mu}{2}X + i\mu\tau\partial_\sigma X\right). \tag{2.20}$$

To first order in μ, we can invert the normal mode expansions in (2.10) as:

$$\left(1+\frac{\mu}{2n}\right)a_n = \frac{1}{2\pi\sqrt{2\alpha'}}\int_0^\pi d\sigma\, \cos n\sigma\Big[-in\big[X(\sigma,0) + X(-\sigma,0)\big] + \big[4\pi\alpha'\big[\widetilde{P}(\sigma,0) + \widetilde{P}(-\sigma,0)\big]\big]\Big],$$
$$\left(1-\frac{\mu}{2n}\right)\tilde{a}_n = \frac{1}{2\pi\sqrt{2\alpha'}}\int_0^\pi d\sigma\, \cos n\sigma\Big[-in\big[\widetilde{X}(\sigma,0) + \widetilde{X}(-\sigma,0)\big] + \big[4\pi\alpha'\big[P(\sigma,0) + P(-\sigma,0)\big]\big]\Big] \tag{2.21}$$

for $n \neq 0$ and

$$x_0 = \frac{1}{2\pi}\int_0^\pi d\sigma\big[X(\sigma,0) + X(-\sigma,0)\big],$$
$$\tilde{x}_0 = \frac{1}{2\pi}\int_0^\pi d\sigma\big[\widetilde{X}(\sigma,0) + \widetilde{X}(-\sigma,0)\big],$$

$$\sqrt{2\alpha'}\,a_0 - i\frac{\mu}{2}x_0 = 2\alpha'\int_0^\pi d\sigma\big[\widetilde{P}(\sigma,0)+\widetilde{P}(-\sigma,0)\big],$$

$$\sqrt{2\alpha'}\,\tilde{a}_0 + i\frac{\mu}{2}\tilde{x}_0 = 2\alpha'\int_0^\pi d\sigma\big[P(\sigma,0)+P(-\sigma,0)\big]. \tag{2.22}$$

The commutation relations which follow from canonical quantization $[X(\sigma,\tau),P(\sigma',\tau)] = i\delta(\sigma-\sigma')$, $[\widetilde{X}(\sigma,\tau),\widetilde{P}(\sigma',\tau)] = i\delta(\sigma-\sigma')$ are:

$$\begin{aligned}
&[a_m,\tilde{a}_n] = 2(m-\mu)\delta_{m,-n}, \qquad [a_m,a_n] = [\tilde{a}_m,\tilde{a}_n] = 0,\\
&[x_0,\tilde{x}_0] = 0, \qquad [a_n,x_0] = [a_n,\tilde{x}_0] = [\tilde{a}_n,x_0] = [\tilde{a}_n,\tilde{x}_0] = 0 \quad \text{for } n\neq 0,\\
&[x_0,\tilde{a}_0] = i2\sqrt{2\alpha'} = [\tilde{x}_0,a_0], \qquad [x_0,a_0] = [\tilde{x}_0,\tilde{a}_0] = 0.
\end{aligned} \tag{2.23}$$

3. The propagator on the disk

Having found a mode expansion, we compute the propagator, along the lines of [21]. In $z,\bar{z}$ coordinates (where z is in the upper half plane, since $0\leqslant\sigma\leqslant\pi$), the equation of motion and boundary conditions for the propagator are:

$$\begin{aligned}
&4z\bar{z}\partial_z\partial_{\bar{z}}X - 2\mu\bar{z}\partial_{\bar{z}}X = 0, \qquad 4z\bar{z}\partial_z\partial_{\bar{z}}\widetilde{X} + 2\mu\bar{z}\partial_{\bar{z}}\widetilde{X} = 0,\\
&(\partial_z-\partial_{\bar{z}})X + \frac{\mu}{2}\ln z\bar{z}(\partial_z+\partial_{\bar{z}})X|_{z=\bar{z}} = 0, \qquad (\partial_z-\partial_{\bar{z}})\widetilde{X} - \frac{\mu}{2}\ln z\bar{z}(\partial_z+\partial_{\bar{z}})\widetilde{X}|_{z=\bar{z}} = 0,\\
&4\partial_z\partial_{\bar{z}}\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle - 2\mu z^{-1}\partial_{\bar{z}}\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle = -2\pi\alpha'\delta^2(z-\zeta),\\
&\left[(\partial_z-\partial_{\bar{z}})\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle + \frac{\mu}{2}\ln z\bar{z}(\partial_z+\partial_{\bar{z}})\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle\right]\bigg|_{z=\bar{z}} = 0.
\end{aligned} \tag{3.1}$$

We will compute the propagator on the disk, and will use $z = e^{i(\tau+\sigma)}$, $\bar{z} = e^{i(\tau-\sigma)}$, $\zeta = e^{i(\tau'+\sigma')}$ and $\bar{\zeta} = e^{i(\tau'-\sigma')}$. In the above boundary conditions, the notation $|_{z=\bar{z}}$ denotes $z=|z|, \bar{z}=|z|$ at the $\sigma=0$ endpoint and $z = |z|e^{i\pi}, \bar{z} = |z|e^{-i\pi}$ at $\sigma=\pi$. Assuming the commutation relations in (2.23), then for $|z|>|\zeta|$, the propagator to order μ is

$$\begin{aligned}
&\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle\\
&\quad= \sqrt{2\alpha'}\,[a_0,\tilde{x}_0]\left(\tau+\mu\left(-i\tau\sigma+\frac{i}{2}\tau^2\right)\right)\\
&\qquad+ 2\alpha'\sum_{n=1}^{\infty}[a_n,\tilde{a}_m]e^{-in\tau}e^{-im\tau'}\\
&\qquad\quad\times\left[-\frac{1}{nm}\cos n\sigma\cos m\sigma' + i\frac{\mu}{m}\cos m\sigma'\left(\left(-\frac{1}{2n^2}-\frac{i\tau}{n}\right)\sin n\sigma + \left(\frac{i}{2n^2}+\frac{(\sigma-\tau)}{2n}\right)\cos n\sigma\right)\right.\\
&\qquad\qquad\left. - i\frac{\mu}{n}\cos n\sigma\left(\left(-\frac{1}{2m^2}-\frac{i\tau'}{m}\right)\sin m\sigma' + \left(\frac{i}{2m^2}+\frac{(\sigma'-\tau')}{2m}\right)\cos m\sigma'\right)\right] + \mu(c_1\tau+c_0)
\end{aligned}$$

$$
\begin{aligned}
&= -i4\alpha'\left(\tau + \mu\left(-i\tau\sigma + \frac{i}{2}\tau^2\right)\right) \\
&\quad + 4\alpha' \sum_{n=1}^{\infty} e^{-in(\tau-\tau')} \\
&\qquad \times \left[\frac{1}{n}\cos n\sigma \cos n\sigma' + i\mu \cos n\sigma'\left(\left(\frac{1}{2n^2} + \frac{i\tau}{n}\right)\sin n\sigma - \left(\frac{i}{2n^2} + \frac{(\sigma-\tau)}{2n}\right)\cos n\sigma\right)\right. \\
&\qquad\quad \left. - i\mu \cos n\sigma\left(\left(\frac{1}{2n^2} - \frac{i\tau'}{n}\right)\sin n\sigma' + \left(\frac{i}{2n^2} - \frac{(\sigma'-\tau')}{2n}\right)\cos n\sigma'\right) - \frac{\mu}{n^2}\cos n\sigma \cos n\sigma'\right] \\
&\quad + \mu(c_1\tau + c_0). \qquad (3.2)
\end{aligned}
$$

We are free to add the function $\mu(c_1\tau + c_0)$ to the expression since it does not affect the equation of motion or the boundary condition for the propagator to first order in μ. For $|z| > |\zeta|$, the expression for $\langle \widetilde{X}(z,\bar{z}) X(\zeta,\bar{\zeta})\rangle$ is given by letting $\mu \to -\mu$ in the above propagator. In the $\mu \to 0$ limit, these propagators reduce to the open bosonic string propagator $\lim_{\mu\to 0}\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\rangle = -2\alpha'(\ln|z-\zeta| + \ln|z-\bar{\zeta}|)$.

4. Time-dependent non-commutativity

To evaluate the non-commutativity parameter as defined from time ordering [4,25], we consider the propagator on the worldsheet boundary at $\sigma = 0$, then $z = |z| = e^{i\tau} \equiv \mathcal{T}$, and $\zeta = e^{i(\tau'+\sigma')} = |\zeta| = e^{i\tau'} = \mathcal{T}'$, so $\mathcal{T}, \mathcal{T}' > 0$. We will also consider the propagator at $\sigma = \pi$, then $z = |z|e^{i\pi} = \mathcal{T}$ and $\zeta = |\zeta|e^{i\pi} = \mathcal{T}'$ so here $\mathcal{T}, \mathcal{T}' < 0$. Note that $\mathcal{T}$ is different from the worldsheet time τ

$$
\begin{aligned}
&\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle\big|_{\sigma=0} \\
&\quad = -i4\alpha'\left(\tau + \mu\frac{i}{2}\tau^2\right) + \mu(c_1\tau + c_0) - 4\alpha'\ln\big(1 - e^{-i(\tau-\tau')}\big) - 2\alpha'\mu i(\tau-\tau')\ln\big(1 - e^{-i(\tau-\tau')}\big) \\
&\quad = -4\alpha'\ln(\mathcal{T} - \mathcal{T}') + \mu\left(-2\alpha'\ln^2\mathcal{T} - 2\alpha'\ln\left(\frac{\mathcal{T}}{\mathcal{T}'}\right)\ln\left(1 - \frac{\mathcal{T}'}{\mathcal{T}}\right) + (-c_1 i\ln\mathcal{T} + c_0)\right) \\
&\big\langle \widetilde{X}(z,\bar{z})X(\zeta,\bar{\zeta})\big\rangle\big|_{\sigma=0} \\
&\quad = -i4\alpha'\left(\tau - \mu\frac{i}{2}\tau^2\right) - \mu(c_1\tau + c_0) - 4\alpha'\ln\big(1 - e^{-i(\tau-\tau')}\big) + 2\alpha'\mu i(\tau-\tau')\ln\big(1 - e^{-i(\tau-\tau')}\big). \qquad (4.1)
\end{aligned}
$$

Then at $\sigma = 0$:

$$
\begin{aligned}
\big[X(\mathcal{T}), \widetilde{X}(\mathcal{T})\big] &= T\big(X(\mathcal{T})\widetilde{X}\big(\mathcal{T}^-\big) - X(\mathcal{T})\widetilde{X}\big(\mathcal{T}^+\big)\big) \\
&\equiv \lim_{\epsilon\to 0}\big(\big\langle X(\mathcal{T})\widetilde{X}(\mathcal{T}-\epsilon)\big\rangle - \big\langle \widetilde{X}(\mathcal{T}+\epsilon)X(\mathcal{T})\big\rangle\big) \quad (\text{for } \epsilon > 0) \\
&= \mu(-4i\alpha')\big(\pi\ln\mathcal{T} - i\ln^2\mathcal{T}\big) \\
&= \mu 4\alpha'\big(\pi\tau + \tau^2\big) \equiv \Theta, \qquad (4.2)
\end{aligned}
$$

where we chose $c_1 = 2\pi\alpha'$, $c_0 = 0$, and use $\lim_{\epsilon\to 0}(\ln(1+\epsilon)\ln\epsilon) = 0$. The non-commutativity parameter Θ is time-dependent.

At $\sigma = \pi$:

$$
\begin{aligned}
\big\langle X(z,\bar{z})\widetilde{X}(\zeta,\bar{\zeta})\big\rangle\big|_{\sigma=\pi} = &-i4\alpha'\left(\tau + \mu\left(-i\tau\pi + \frac{i}{2}\tau^2\right)\right) + \mu(c_1\tau + c_0) \\
&- 4\alpha'\ln\big(1 - e^{i(\tau'-\tau)}\big) - 2\alpha'\mu i(\tau-\tau')\ln\big(1 - e^{-i(\tau-\tau')}\big), \qquad (4.3)
\end{aligned}
$$

$$\begin{aligned}
\langle \widetilde{X}(z,\bar z)X(\zeta,\bar\zeta)\rangle\big|_{\sigma=\pi} &= -i4\alpha'\left(\tau-\mu\left(-i\tau\pi+\frac{i}{2}\tau^2\right)\right)+\mu(c_1\tau+c_0)\\
&\quad -4\alpha'\ln\bigl(1-e^{i(\tau'-\tau)}\bigr)+2\alpha'\mu i(\tau-\tau')\ln\bigl(1-e^{-i(\tau-\tau')}\bigr),\\
\bigl[X(\mathcal{T}),\widetilde{X}(\mathcal{T})\bigr] &= T\bigl(X(\mathcal{T})\widetilde{X}(\mathcal{T}^-)-X(\mathcal{T})\widetilde{X}(\mathcal{T}^+)\bigr)\\
&\equiv \lim_{\epsilon\to 0}\bigl(\langle X(\mathcal{T})\widetilde{X}(\mathcal{T}-\epsilon)\rangle-\langle \widetilde{X}(\mathcal{T}+\epsilon)X(\mathcal{T})\rangle\bigr)\quad (\text{for } \epsilon>0)\\
&= (-i4\alpha')\mu\bigl[-\pi\ln\mathcal{T}-i\ln^2\mathcal{T}\bigr]\\
&= \mu 4\alpha'\bigl(-\pi\tau+\tau^2\bigr).
\end{aligned}\tag{4.4}$$

Thus for small μ, we have:

$$\begin{aligned}
\Theta &= \mu 4\alpha'\bigl(\pi\tau+\tau^2\bigr) \quad \text{at } \sigma=0,\\
\Theta &= \mu 4\alpha'\bigl(-\pi\tau+\tau^2\bigr) \quad \text{at } \sigma=\pi.
\end{aligned}\tag{4.5}$$

For small τ, the theta parameter at the $\sigma=0$ end of the string is minus that at the $\sigma=\pi$ end. This is the case for the neutral string in a constant background B field as well. In fact, although we have worked only to lowest order in μ, we can see directly from the equations of motion and boundary conditions (in $z,\bar z$) variables in (3.1), that in the limit of large z, i.e., large $i\tau$, a limit for which $z^{-1}\to 0$, that the system reduces to the neutral string with the identification $-\mu\tau=B$, a constant. (In the large τ limit, we note that $\ln|z|$ is approximately constant, in the sense that it is changing slowly, i.e., its derivative $|z|^{-1}$ is small. Therefore, for large τ the non-commutativity parameter becomes constant, and our model is similar to the neutral string.) For large τ, using the neutral string expressions, we find the non-commutativity parameter be time-dependent:

$$\begin{aligned}
\Theta &= -4\alpha'\pi B = 4\alpha'\mu\pi\tau \quad \text{at } \sigma=0,\\
\Theta &= 4\alpha'\pi B = -4\alpha'\mu\pi\tau \quad \text{at } \sigma=\pi.
\end{aligned}\tag{4.6}$$

We have shown that our model exhibits non-commutativity for both small and large τ. The expectation is that the model will remain non-commutative with a time-dependent non-commutativity parameter for all times.

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