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The Gegenbauer polynomials and typically real functions

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Abstract

Linear and nonlinear coefficient problems for some class of typically real functions are studied. Different inequalities for the Gegenbauer polynomials appear to be very useful.

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1. Introduction

The so-called class T_{R} of typically real functions

$$f(z) = z + a_2 z^2 + \cdots, \quad z \in \mathbb{D} = \{z\colon |z| < 1\}, \tag{1}$$

which are holomorphic in $\mathbb{D}$, real for $z \in (-1,1)$ and satisfy the condition

$$\operatorname{Im} f(z) \operatorname{Im} z > 0 \quad \text{for } z \in \mathbb{D} \setminus (-1,1) \tag{2}$$

plays an important role in the geometric theory of holomorphic functions in the unit disk $\mathbb{D}$.

The class T_{R} was introduced by Rogosinski [11] and intensively studied because of the Bieberbach conjecture and the relation $T_{\mathrm{R}} = \overline{\mathrm{co}}\, S_{\mathrm{R}}$, where S_{R} denotes the class of univalent functions in $\mathbb{D}$ with real coefficients, and $\overline{\mathrm{co}}\, S_{\mathrm{R}}$ denotes the closed convex hull of S_{R}.

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Robertson [10] proved an integral representation for T_{R}, namely $f \in T_{\mathrm{R}}$ if and only if it has the representation

$$f(z) = \int_{-1}^{1} \frac{z}{1 - 2xz + z^2}\, \mathrm{d}\mu(x), \quad z \in \mathbb{D}, \tag{3}$$

where μ is a probability measure on $[-1, 1]$.

One can observe from (3) and (1) that

$$a_n = \int_{-1}^{1} U_{n-1}(x)\, \mathrm{d}\mu(x),$$

where U_n is the Tchebycheff polynomial of the second kind given by the formula

$$U_n(x) = U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \theta \in [0, \pi].$$

The notion of the class T_{R} has been extended in [15] to the class $T_{\mathrm{R}}(\lambda)$, $\lambda > 0$, which is defined by the integral formula

$$f(z) = \int_{-1}^{1} \frac{z}{(1 - 2xz + z^2)^{\lambda}}\, \mathrm{d}\mu(x), \quad z \in \mathbb{D}, \tag{4}$$

where μ is a probability measure on $[-1, 1]$. Of course, we have $T_{\mathrm{R}}(1) \equiv T_{\mathrm{R}}$ and if f given by (4) has the form (1) then we have

$$a_n = \int_{-1}^{1} C_{n-1}^{(\lambda)}(x)\, \mathrm{d}\mu(x), \tag{5}$$

where $C_n^{(\lambda)}(x)$ is the Gegenbauer polynomial of degree n, and in particular:

$$\begin{aligned} &C_0^{(\lambda)}(x) = 1, \quad C_1^{\lambda}(x) = 2\lambda x, \quad C_2^{\lambda}(x) = 2\lambda(\lambda+1)x^2 - \lambda, \\ &3C_3^{\lambda}(x) = 4\lambda(\lambda+1)(\lambda+2)x^3 - 6\lambda(\lambda+1)x, \\ &6C_4^{\lambda}(x) = 4\lambda(\lambda+1)(\lambda+2)(\lambda+3)x^4 - 12\lambda(\lambda+1)(\lambda+2)x^2 + 3\lambda(\lambda+1). \end{aligned} \tag{6}$$

One can easily see that the class $T_{\mathrm{R}}(\lambda)$ is a compact and convex set in the linear space $H_0(\mathbb{D})$ of holomorphic functions f in $\mathbb{D}$ (which have the form (1)) endowed with the topology of local uniform convergence on compact subsets of $\mathbb{D}$. The importance of the class $T_{\mathrm{R}}(\lambda)$ follows as well from the paper of Hallenbeck [5] who studied the extreme points of some families of univalent functions and proved that ($\overline{\mathrm{co}}\, A =$ closed convex hull of A, $\mathrm{ext}\, A =$ the set of the extremal points of A):

$$\overline{\mathrm{co}}\, S_{\mathrm{R}}^*(1-\lambda) = T_{\mathrm{R}}(\lambda) \text{ and } \mathrm{ext}\, \overline{\mathrm{co}}\, S_{\mathrm{R}}^*(1-\lambda) = \left\{ \frac{z}{(1 - 2xz + z^2)^{\lambda}};\ x \in [-1, 1] \right\}. \tag{7}$$

$S_{\mathrm{R}}^*(\alpha)$, $0 \leqslant \alpha < 1$, denotes the class of holomorphic functions (1) which are univalent and starlike of order α, $\alpha \in [0, 1)$ in $\mathbb{D}$ and have real coefficients. It is well known that f is starlike of order α,

$\alpha \in [0,1)$ in $\mathbb{D}$ if and only if, it satisfies the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad \text{for } z \in \mathbb{D}. \tag{8}$$

One can easily observe that the kernel function in (4):

$$s_\lambda(z;x) := \frac{z}{(1-2xz+z^2)^\lambda}, \quad x \in [-1,1],\ z \in \mathbb{D} \tag{9}$$

is starlike of order $(1-\lambda)$ and has real coefficients. Moreover, we have $T_{\mathrm{R}}(\lambda_1) \subset T_{\mathrm{R}}(\lambda_2)$ for $0 < \lambda_1 < \lambda_2 \leqslant 1$. One can consider the class $T_{\mathrm{R}}(\lambda)$ as well for $\lambda > 1$. Then the kernel function $s_\lambda(z;x)$ is still starlike but of negative order. However, such functions were studied as well and share some properties with those of positive order, e.g., [5,13].

In this note we study some coefficients functionals within the class $T_{\mathrm{R}}(\lambda)$. The properties of the Gegenbauer polynomials and the representation (4) will play the key role. In the case of some linear functionals the problem has an easy solution thanks to the Krein–Millman theorem and the simple form of the extreme points of $T_{\mathrm{R}}(\lambda)$ given in (7). When dealing with nonlinear problems one can apply some other arguments (e.g., [12]). The results obtained not only extend and generalize known results for T_{R} (a direct generalization of some results from [1] – Theorems 1 and 2), but also give some new information about the class $T_{\mathrm{R}}(\lambda)$ (Theorems 3 and 4).

2. Lemmas

Lemma 1 (Rainville [9]). *For any $\lambda > 0$, $x \in [-1,1]$ and integer n the following sharp estimate holds*:

$$|C_n^{(\lambda)}(x)| \leqslant \frac{(2\lambda)_n}{n!} = C_n^{(\lambda)}(1). \tag{10}$$

More precisely we have the interesting:

Lemma 2 (Lohöfer [8]). *For any $\lambda > 0$, $x \in [-1,1]$ and integer n the following inequality holds*:

$$|C_n^{(\lambda)}(x)| \leqslant x^2 d_{2n,2\lambda} + (1-x^2) d_{n,\lambda}, \tag{11}$$

where

$$d_{n,\lambda} = \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}. \tag{12}$$

We observe that $d_{2n,\lambda} = |C_{2n}^{(\lambda)}(0)|$.

Inequality (11) will be sufficient in what follows only for even n. For odd n we have the following:

Lemma 3. *For any $\lambda > 0$, $x \in [-1,1]$ and $n = 2m-1$, $m = 1,2,\ldots$ we have*:

$$|C_{2m-1}^{(\lambda)}(x)| \leqslant \frac{(2\lambda)_{2m-1}}{(2m-1)!}|x| \quad \textit{if } \lambda \geqslant 1 \tag{13}$$

and

$$|C_{2m-1}^{(\lambda)}(x)| \leqslant 2\frac{(\lambda)_m}{(m-1)!}|x| \quad \textit{if } 0<\lambda \leqslant 1. \tag{14}$$

Proof. We will use the formula for $C_{2m+1}^{(\lambda)}(x)$ connected by quadratic transformation with Jacobi polynomials $P_n^{(\alpha,\beta)}$ and the corresponding bound for Jacobi polynomials [14,6]:

$$C_{2m+1}^{(\lambda)}(x) = \frac{(\lambda)_{m+1}}{(\frac{1}{2})_{m+1}} x P_m^{(\lambda-1/2,1/2)}(2x^2-1), \quad \lambda>0,\ x \in [-1,1], \tag{15}$$

$$|P_m^{(\alpha,\beta)}(x)| \leqslant \begin{pmatrix} m+q \\ m \end{pmatrix}, \quad q=\max(\alpha,\beta) \geqslant -\frac{1}{2}. \tag{16}$$

If $\lambda \geqslant 1$, then $q=\max(\lambda-\frac{1}{2},\frac{1}{2})=\lambda-\frac{1}{2}$ and

$$\begin{pmatrix} m+q \\ m \end{pmatrix} = \frac{(\lambda+\frac{1}{2})_m}{m!}.$$

From (15) and (16) we get

$$|C_{2m+1}^{(\lambda)}(x)| \leqslant |x|\frac{(\lambda)_{m+1}}{(\frac{1}{2})_{m+1}}\begin{pmatrix} m+q \\ m \end{pmatrix} = |x|\frac{(\lambda)_{m+1}}{(\frac{1}{2})_{m+1}} \cdot \frac{(\lambda+\frac{1}{2})_m}{m!}$$

$$= |x| \cdot \frac{2^{m+1}(\lambda)_{m+1}(2\lambda+1)(2\lambda+3)\cdot \ldots \cdot(2\lambda+2m-1)}{(2m+1)!!m!2^m} = \frac{(2\lambda)_{2m+1}}{(2m+1)!}|x|.$$

The case $0<\lambda\leqslant 1$ follows in similar way, by taking $q=\frac{1}{2}$.

Another type of inequality for Gegenbauer polynomials ("the Bernstein-type inequality") recently obtained by Förster [3], will be needed as well.

Lemma 4 (Förster [3, p. 65]). *For any $\lambda \geqslant 1$, $x \in [-1,1]$ and integer n the following inequality holds*:

$$(1-x^2)|C_n^{(\lambda)}(x)| \leqslant (2\lambda-1)\left\{1-\frac{2\lambda(\lambda-1)}{(n+\lambda)^2+\lambda(\lambda-1)}\right\}^{1/2} \cdot L(\lambda,n,\lambda) =: H(n,\lambda), \tag{17}$$

where

$$L(\lambda,n,\lambda) = \begin{cases} \dfrac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}, & n \textit{ even} \\[2ex] \dfrac{n+1}{(n^2+2\lambda n+\lambda)^{1/2}} \cdot \dfrac{\Gamma(\frac{1}{2}(n+1)+\lambda)}{\Gamma(\lambda)\Gamma(\frac{1}{2}(n+1)+1)} & n \textit{ odd}. \end{cases}$$

Lemma 5 (Föster [3, p. 62]). *If $\lambda=2$, then for any $x\in[-1,1]$ and $n\in\mathbb{N}$ the following inequality holds*:

$$(1-x^2)|C_n^{(2)}(x)|\leqslant \tfrac{3}{2}(n+2).$$

3. Main results

We start with the following remark (see [1]) concerning the regular curve Γ: $w(x)=C_{k-1}^{(\lambda)}(x)+\mathrm{i}C_{n-1}^{(\lambda)}(x)$, $x\in[-1,1]$ on the complex w – plane. The convex hull of the curve Γ is the following set of points:

$$W=\left\{w:\ \ w=\sum_{j=1}^{2}\mu_j w(x_j):\ \mu_1+\mu_2=1,\ \ \mu_1\geqslant 0,\mu_2\geqslant 0,-1\leqslant x_1\leqslant x_2\leqslant 1\right\}.$$

Therefore, by (4) and (5) and the Carathéodory theorem, the region of variability of the functional

$$\Omega(f)=\left\{w=a_k+\mathrm{i}a_n=\int_{-1}^{1}[C_{k-1}^{(\lambda)}(x)+\mathrm{i}C_{n-1}^{(\lambda)}(x)]\,\mathrm{d}\mu(x);\ f\in T_{\mathrm{R}}(\lambda)\right\} \tag{18}$$

is the convex hull of the curve Γ, i.e., the set W.

Therefore, when considering the extremal values of the continuous functional $J(f)$ depending on a_k and a_n, $2\leqslant k<n$, k,n fixed integers one can restrict consideration to the functions

$$f(z)=\mu_1\frac{z}{(1-2x_1z+z^2)^{\lambda}}+\mu_2\frac{z}{(1-2x_2z+z^2)^{\lambda}}, \tag{19}$$

where $\mu_1\geqslant 0$, $\mu_2\geqslant 0$, $\mu_1+\mu_2=1$ and $-1\leqslant x_1\leqslant x_2\leqslant 1$.

(a) Putting $k=3$ and $n>3$, we have

Theorem 1. *For any $f\in T_{\mathrm{R}}(\lambda)$, $\lambda>0$ the following sharp bounds hold*:

$$|a_{2m}|\leqslant\frac{(2\lambda)_{2m-1}}{(2m-1)!}\sqrt{\frac{a_3+\lambda}{2\lambda(\lambda+1)}}\quad \textit{if}\ \ \lambda\geqslant 1, \tag{20}$$

$$|a_{2m}|\leqslant 2\frac{(\lambda)_m}{(m-1)!}\sqrt{\frac{a_3+\lambda}{2\lambda(\lambda+1)}}\quad \textit{if}\ \ 0<\lambda\leqslant 1, \tag{21}$$

$$|a_{2m-1}|\leqslant d_{m-1,\lambda}+(d_{2m-2,2\lambda}-d_{m-1,\lambda})\frac{a_3+\lambda}{2\lambda(\lambda+1)},\qquad \lambda>0, \tag{22}$$

where $d_{m,\lambda}$ is given by (12). The extremal function has the form

$$s_{\lambda}(z;1)=\frac{z}{(1-z)^{2\lambda}}=z+\sum_{n=2}^{\infty}A_n(\lambda)z^n, \tag{23}$$

where

$$A_n(\lambda) = \frac{(2\lambda)_{n-1}}{(n-1)!} = C_{n-1}^{(\lambda)}(1)$$

and $s_\lambda(z;x)$ *is given by* (9).

Proof. First of all, we observe that if $f(z)$ is given by (19) then

$$a_3 = \sum_{j=1}^{2} \mu_j C_2^{(\lambda)}(x_j) = \sum_{j=1}^{2} \mu_j [2\lambda(\lambda+1)x_j^2 - \lambda] = 2\lambda(\lambda+1) \sum_{j=1}^{2} \mu_j x_j^2 - \lambda \tag{24}$$

and therefore

$$0 \leqslant a_3 + \lambda \leqslant 2\lambda(\lambda+1).$$

From (5) and (19), for even coefficients, we obtain, using Lemma 3:

$$|a_{2m}| = \left| \sum_{j=1}^{2} \mu_j C_{2m-1}^{(\lambda)}(x_j) \right| \leqslant \sum_{j=1}^{2} \mu_j \frac{(2\lambda)_{2m-1}}{(2m-1)!} |x_j|$$

$$= \frac{(2\lambda)_{2m-1}}{(2m-1)!} \sum_{j=1}^{2} \mu_j |x_j| \quad \text{in the case } \lambda \geqslant 1,$$

and

$$|a_{2m}| \leqslant 2 \frac{(\lambda)_m}{(m-1)!} \sum_{j=1}^{2} \mu_j |x_j| \quad \text{in the case } 0 < \lambda \leqslant 1.$$

Using the Cauchy–Schwarz inequality we have from (19) and (24)

$$\sum_{j=1}^{2} \mu_j |x_j|^2 \leqslant \left(\sum_{j=1}^{2} \mu_j \right)^{1/2} \left(\sum_{j=1}^{2} \mu_j x_j^2 \right)^{1/2} = \sqrt{\frac{a_3 + \lambda}{2\lambda(\lambda+1)}}$$

and then (20) and (21) follow.

In the case of odd coefficients we apply Lemma 2, and obtain

$$|a_{2m-1}| = \left| \sum_{j=1}^{2} \mu_j C_{2m-2}^{(\lambda)}(x_j) \right| \leqslant \sum_{j=1}^{2} \mu_j \{ d_{m-1,\lambda} + (d_{2m-2,2\lambda} - d_{m-1,\lambda}) x_j^2 \}$$

$$= d_{m-1,\lambda} + (d_{2m-2,2\lambda} - d_{m-1,\lambda}) \sum_{j=1}^{2} \mu_j x_j^2$$

which implies (22), by (24). □

(b) Putting $k = 5$ and $n > 5$, we have

Theorem 2. *For any $f \in T_R(\lambda)$ the following sharp bounds hold*:

$$|a_{2m}| \leqslant \frac{(2\lambda)_{2m-1}}{(2m-1)!} \frac{1}{\sqrt{2(\lambda+3)}} \left\{ 3 + \left[\frac{6(\lambda+3)^2 a_5 + 3\lambda(\lambda+1)(\lambda+3)(2\lambda+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)} \right]^{1/2} \right\}^{1/2}, \quad (25)$$

$$|a_{2m}| \leqslant 2\frac{(\lambda)_m}{(m-1)!} \frac{1}{\sqrt{2(\lambda+3)}} \left\{ 3 + \frac{6(\lambda+3)^2 a_5 + 3\lambda(\lambda+1)(\lambda+3)(2\lambda+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)} \right\}^{1/2}. \quad (26)$$

Inequality (25) is valid for $\lambda \geqslant 1$ and (26) for $\lambda \in (0, 1]$.

$$|a_{2m-1}| \leqslant d_{m-1,\lambda} + (d_{2m-2,2\lambda} - d_{m-1,\lambda}) \times \left\{ 3 + \left[\frac{6(\lambda+3)^2 a_5 + 3\lambda(\lambda+1)(\lambda+3)(2\lambda+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)} \right]^{1/2} \right\}, \quad \lambda > 0. \quad (27)$$

The extremal function has the form (23).

Proof. For the extremal function (19) we have

$$a_5 = \sum_{j=1}^{2} \mu_j C_4^{(\lambda)}(x_j),$$

where $\mu_1, \ \mu_2 \geqslant 0, \ \mu_1 + \mu_2 = 1, \ -1 \leqslant x_1 \leqslant x_2 \leqslant 1$ and $C_4^{(\lambda)}(x)$ is given by (6).

After simple calculations we get

$$6(\lambda+3)^2 a_5 + 3\lambda(\lambda+1)(\lambda+3)(2\lambda+3) = \lambda(\lambda+1)(\lambda+2)(\lambda+3) \sum_{j=1}^{2} \mu_j [2(\lambda+3)x_j^2 - 3]^2. \quad (28)$$

Suppose $\lambda \geqslant 1$. Then, by Lemma 3, we have

$$|a_{2m}| \leqslant \frac{(2\lambda)_{2m-1}}{(2m-1)!} \sum_{j=1}^{2} \mu_j |x_j| \leqslant \frac{(2\lambda)_{2m-1}}{(2m-1)!} \left(\sum_{j=1}^{2} \mu_j x_j^2 \right)^{1/2}$$

$$\leqslant \frac{(2\lambda)_{2m-1}}{(2m-1)!} \frac{1}{\sqrt{2(\lambda+3)}} \left\{ \left[\sum_{j=1}^{2} \mu_j (2(\lambda+3)x_j^2 - 3)^2 \right]^{1/2} + 3 \right\}^{1/2},$$

where we applied the Cauchy–Schwarz inequality twice. Substituting in the last line, the expression in the square brackets from (28) we get (25). Inequality (26) is obtained in an analogous way by using (14).

In the case of odd coefficients we have by (19), (6) and (11):

$$|a_{2m-1}| \leqslant \sum_{j=1}^{2} \mu_j [d_{m-1,\lambda} + (d_{2m-2,2\lambda} - d_{m-1,\lambda}) x_j^2]$$

$$= d_{m-1,\lambda} + \frac{d_{2m-2,2\lambda} - d_{m-1,\lambda}}{2(\lambda+3)} \sum_{j=1}^{2} \mu_j (2\lambda+3) x_j^2.$$

Application of the Cauchy–Schwarz inequality and (28) complete the proof. □

Remark. One can get a sharp bound for a_n depending on a_2 or a_4, etc., which is exact only for the "Koebe-type" function $s_\lambda(z;1)$ given by (23). For example, we can find the relation:

$$3[a_4 + (\lambda+1)a_2] = 4\lambda(\lambda+1)(\lambda+2)\sum_{j=1}^{2} \mu_j x_j^3$$

which easily implies the sharp bound

$$|a_4 + (\lambda+1)a_2| \leqslant \tfrac{4}{3}\lambda(\lambda+1)(\lambda+2).$$

The precise bound for a_n depending on every prescribed value of a_2 is much more complicated and will be considered elsewhere.

One more result where the "Koebe-type" function (23) is extremal is the following

Theorem 3. *For any $f \in T_{\mathrm{R}}(\lambda)$, $\lambda \geqslant 1$, $n = 1, 2, \ldots$ we have the sharp bound*:

$$|a_{n+2} - a_n| \leqslant 2\frac{(2\lambda-1)_n}{(n+1)!}(n+\lambda). \tag{29}$$

Proof. The following recurrence formula can be found in [9]:

$$(n+\lambda)C_{n+1}^{(\lambda-1)}(x) = [C_{n+1}^{(\lambda)}(x) - C_{n-1}^{(\lambda)}(x)](\lambda-1), \quad \lambda > 1.$$

Applying (5), we obtain the identity

$$(\lambda-1)(a_{n+2} - a_n) = (n+\lambda)\int_{-1}^{1} C_{n+1}^{(\lambda-1)}(x)\,\mathrm{d}\mu(x).$$

(29) follows from this and Lemma 1. □

In the limiting case $\lambda = 1$ we have the result of Golusin [4].

In contrast to the bound contained in Theorems 1–3, the next result shows the situation in which the function $s_\lambda(z;1)$ is not extremal. The problem leads us to the application of "the Bernstein-type inequality" for the Gegenbauer polynomials even in the case of typically-real functions (the case $\lambda = 1$). We have

Theorem 4. *For any* $f \in T_{\mathrm{R}}(\lambda)$, $\lambda > 0$ *and any integer* $n \geqslant 2$ *we have the following bound*:

$$|(n+2\lambda)(n+2\lambda-1)a_n - n(n+1)a_{n+2}| \leqslant 4\lambda(n+\lambda)H(n,\lambda), \tag{30}$$

where $H(n,\lambda)$ *is given by* (17).

Remark. The result is "almost" sharp for even n.

Proof. The following relation [9, p. 283] for the Gegenbauer polynomials

$$2\lambda(1-x^2)C_{n-1}^{(\lambda+1)}(x) = (n+2\lambda)xC_n^{(\lambda)}(x) - (n+1)C_{n+1}^{(\lambda)}(x)$$

and the three-term recurrence formula for the Gegenbauer polynomials [9, p. 279]

$$xC_n^{(\lambda)}(x) = \frac{n+1}{2(\lambda+n)}C_{n+1}^{(\lambda)}(x) + \frac{2\lambda+n-1}{2(\lambda+n)}C_{n-1}^{(\lambda)}(x)$$

give

$$4\lambda(\lambda+n)(1-x^2)C_{n-1}^{(\lambda+1)}(x) = (n+2\lambda)(2\lambda+n-1)C_{n-1}^{(\lambda)}(x) - n(n+1)C_{n+1}^{(\lambda)}(x).$$

After integration we get:

$$(n+2\lambda)(n+2\lambda-1)a_n - n(n+1)a_{n+2} = 4\lambda(n+\lambda)\int_{-1}^{1}(1-x^2)C_{n-1}^{(\lambda+1)}(x)\,\mathrm{d}\mu(x). \tag{31}$$

Finally, we are led to apply the best-known bound for

$$(1-x^2)|C_{n-1}^{(\lambda+1)}(x)|, \quad x \in [-1,1],\ \lambda > 0,\ n \in \mathbb{N}.$$

There are many of these and related inequalities (e.g., [3,7,8]). The "nearly" best bound (17) found recently [3] is useful in our situation. From Lemma 4 (bound (17)) we find (30).

Application of the Förster inequality (17) implies the result. □

Corollary. *For any* $f \in T_{\mathrm{R}}$ *we have the following bound*:

$$|(n+2)a_n - na_{n+2}| \leqslant 6(n+1), \quad n \geqslant 2.$$

For the initial coefficients and $f \in T_{\mathrm{R}} = T_{\mathrm{R}}(1)$ one can find from the formula (31) after some calculations of the exact extremal values for $(1-x^2)C_{n-1}^{(2)}(x)$.

Theorem 5. *For* $f \in T_{\mathrm{R}}$ *we have the sharp bounds*:

$$|3a_1 - a_3| \leqslant 4 \left(\textit{the extremal function } f(z) = \frac{z}{1+z^2}\right),$$

$$|4a_2 - 2a_4| \leqslant \frac{32\sqrt{3}}{9}\left(\textit{the extremal function } f(z) = \frac{z}{1 \pm 2\frac{1}{\sqrt{3}}z + z^2}\right),$$

$$|5a_3 - 3a_5| \leqslant \frac{25}{3}\left(\textit{the extremal function } f(z) = \frac{z}{1 \pm \sqrt{(7/3)}z + z^2}\right). \tag{32}$$

4. Uncited reference

[2]

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