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A time-domain homogenization technique for lamination stacks in dual finite element formulations☆

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Abstract

A time-domain homogenization technique is developed to take the eddy currents in lamination stacks into account with dual 3-D magnetodynamic **b**- and **h**-conform finite element formulations. The lamination stack is considered as a source region carrying predefined current density and magnetic flux density distributions describing the eddy currents and skin effect in each lamination. These distributions are related and are approximated with sub-basis functions. The stacked laminations are then converted into continuums with which terms are associated for considering the eddy current loops produced by parallel fluxes, through the homogenization of the sub-basis function contributions.

Keywords: Eddy currents; Finite element method; Homogenization; Lamination stack

1. Introduction

Iron cores in electrical devices are usually made of lamination stacks in order to reduce the eddy current losses due to time-varying flux excitations. When simulating such devices using the finite element (FE) method, it is usually impossible to model the eddy currents in each separate lamination. Commonly these currents are first completely ignored, whereupon the Joule losses may be estimated from the results of the eddy current free model [2]. This nevertheless leads to neglecting the flux reduction due to the eddy currents, while this reduction can be significant for high frequency excitations.

A technique is proposed here to directly take these losses into account with both dual 3-D magnetodynamic **b**- and **h**-conform FE formulations. The technique is a time-domain extension, particularly to allow nonlinear analyses, of the frequency-domain method proposed in [5,7] and an alternative to the method proposed in [6] for homogenizing the laminations in a formulation. It also extends the methods proposed in [1,8] to high frequency excitations and in [3] to 3-D applications. It consists in considering the lamination stack as an homogenized source region carrying predefined current density distributions, as well as the associated magnetic flux density distributions, describing the eddy currents

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and skin effect in each lamination. The expressions of such related current and magnetic flux densities and their use in dual formulations constitute the key objective of this paper. Results obtained with both formulations are validated for test problems.

2. Magnetodynamic problem in lamination stacks

A bounded domain Ω of the 3-D Euclidean space is considered, in which the magnetodynamic problem is defined. Massive conductors, subject to eddy currents, belong to $\Omega_c \subset \Omega$ and source conductors, carrying a given current density $\mathbf{j}_s$, are regions of $\Omega_s \subset \Omega$.

The direct way to consider the eddy currents in a lamination stack is to model this stack as a set of massive conductors (i.e., a subset of Ω_c) separated by insulating layers. This is nevertheless generally unfeasible in view of the large number of laminations encountered in iron cores. An alternative method consists in considering the lamination stack as an homogenized region (i.e., rather a subset of Ω_s) through predefined current density and magnetic flux density distributions in each lamination. These distributions can be expressed in terms of adequate basis functions and unknown coefficients.

3. Predefined eddy current density distribution

3.1. Eddy current density versus the magnetic flux density

A lamination stack region Ω_{ls} is considered (Fig. 1), with transversal gaps between the laminations; longitudinal gaps are not considered here and would ask for an extension of the proposed method. Each lamination has a thickness d, an electric conductivity σ and a magnetic permeability μ. It is described by a local coordinate system $(\mathbf{i}_\alpha, \mathbf{i}_\beta, \mathbf{i}_\gamma)$, for which the directions $\mathbf{i}_\alpha$ and $\mathbf{i}_\beta$ are parallel to the lamination plane, while $\mathbf{i}_\gamma$ is perpendicular to it. The direction $\mathbf{i}_\alpha$ is considered as the *a priori* unknown direction of the magnetic flux density $\mathbf{b}_\alpha = \mu\mathbf{h}_\alpha$ parallel to the lamination, and consequently $\mathbf{i}_\beta$ is the main direction of the eddy current loops generated by variations of $\mathbf{b}_\alpha$, with the associated current density $\mathbf{j}_\beta$.

The effect of a varying magnetic flux density perpendicular to the laminations can be taken into account in the magnetodynamic problem *via* an anisotropic conductivity having a zero component along $\mathbf{i}_\gamma$. Its consideration in the developed formulation would lead to non-zero net currents for $\mathbf{j}_\beta$.

The current density $\mathbf{j}_\beta$ has to undergo a pretreatment for avoiding, at the discrete level, the discretization of each lamination separately. This pretreatment concerns the strongly expressed equations in the considered formulation and consequently differs for **b**- and **h**-conform formulations. On one hand, for a **b**-conform formulation, the Faraday and the flux conservation equations have to be expressed strongly, while the Ampere equation is rather expressed in a weak form. On the other hand, for an **h**-conform formulation, the Ampere equation is the one to be strongly expressed, while the others have to be written in a weak form.

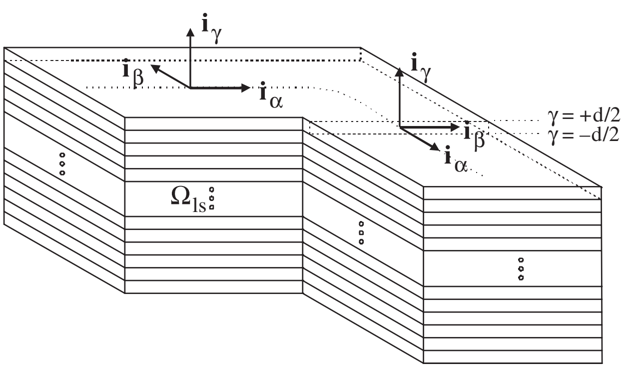

Fig. 1. Lamination stack Ω_{ls} with its local coordinate system $(\mathbf{i}_\alpha, \mathbf{i}_\beta, \mathbf{i}_\gamma)$ associated with each lamination.

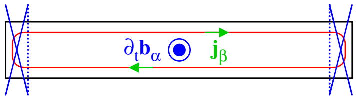

Fig. 2. Magnetic flux density $\mathbf{b}_\alpha$ and associated current density $\mathbf{j}_\beta$ in the cross section $(\mathbf{i}_\beta, \mathbf{i}_\gamma)$ of a lamination stack Ω_{ls} for either (2) or (3); fringing effects are neglected.

On one hand, by the 1-D Faraday equation neglecting fringing effects of $\mathbf{j}_\beta$ (Fig. 2), one has for one lamination, with thickness coordinate γ,

$$\partial_\gamma \mathbf{e}_\beta = \mathbf{i}_\gamma \times \partial_t \mathbf{b}_\alpha, \tag{1}$$

where $\mathbf{e}_\beta$ is the β-directed electric field. Taking the primitive of (1) and the Ohm law, the current density $\mathbf{j}_\beta$ can be expressed in terms of a primitive in γ of $\mathbf{b}_\alpha$, i.e.,

$$\mathbf{j}_\beta = \sigma \mathbf{e}_\beta = \sigma \mathbf{i}_\gamma \times \partial_t \int \mathbf{b}_\alpha \, \mathrm{d}\gamma. \tag{2}$$

On the other hand, by the 1-D Ampere equation neglecting fringing effects of $\mathbf{j}_\beta$ as well (Fig. 2), one has for one lamination,

$$\mathbf{j}_\beta = -\mathbf{i}_\gamma \times \partial_\gamma \mathbf{h}_\alpha = -\mathbf{i}_\gamma \times \partial_\gamma (\mu^{-1} \mathbf{b}_\alpha). \tag{3}$$

3.2. Magnetic flux density approximate expansion

Field $\mathbf{b}_\alpha$ in either (2) or (3) is unknown and an approximate expansion limited to a certain order is given along the thickness of each lamination, i.e., for its component b_α, with $\mathbf{b}_\alpha = b_\alpha \mathbf{i}_\alpha$,

$$b_\alpha(\gamma, t) = \sum_{n \in S} b_n(t) \eta_n(\gamma), \tag{4}$$

where functions $\eta_n(\gamma)$ define sub-basis functions for $\mathbf{b}_\alpha$, which must differ from the constant function because of the skin effect, and S defines a set of expansion orders. Field $\mathbf{b}_\alpha$ has been considered as a constant in [1,5,8] for a low frequency model and is now intended to be given a higher approximation order. For a given problem, an optimum order can be selected via a p-adaptation process based on the relative error between solutions of successive orders p and $p+1$.

The so-called skin effect sub-basis functions can be freely chosen as polynomials [6] or as more sophisticated expressions closer to the expected actual distribution of $\mathbf{b}_\alpha$ (e.g., hyperbolic functions). In case no net current exists in the laminations, even polynomial basis functions can be considered [6].

The purpose of expansion (4) is to avoid the definition of a FE mesh of each lamination approximating accurately enough the skin effect. Here, this distribution will rather be considered by a pretreatment of sub-basis functions in the FE formulation, as will be shown, instead of using explicit classical basis functions of a sufficient number of thin FEs along each lamination thickness.

Coefficients b_n are peculiar to each lamination. For the lamination stack, they have to be converted to a continuum, or homogenized, continuously varying from one lamination to the next. This means that other basis functions are needed for the approximation of this continuum along direction $\mathbf{i}_\gamma$; a variation along the longitudinal direction $\mathbf{i}_\alpha$ should be allowed as well. These basis functions will be associated with the so-defined homogenized FE mesh.

Both expressions (2) and (3) cannot be simultaneously satisfied with the discrete expansion (4) of $\mathbf{b}_\alpha$. One of them can be exactly satisfied while the other can be only satisfied weakly, depending on the wished conformity in the FE formulation. For a $\mathbf{b}$-conform formulation, the strong expression of the flux conservation, i.e., $\operatorname{div} \mathbf{b} = 0$, has to be applied to (4) for giving particular properties to coefficients b_n. An adequate choice for the sub-basis functions will be shown to simplify, among other things, the expression of such properties. For an $\mathbf{h}$-conform formulation, the strong expression of the Ampere equation will have to be satisfied as well.

4. Homogenization process in FE formulations

4.1. **b**-Conform magnetic vector potential formulation

The general expression of the electric field **e** via a magnetic vector potential **a** involves the gradient of an electric scalar potential v in Ω_c, i.e.,

$$\mathbf{e} = -\partial_t \mathbf{a} - \operatorname{grad} v \quad \text{in } \Omega_c, \tag{5}$$

with

$$\mathbf{b} = \operatorname{curl} \mathbf{a} \quad \text{in } \Omega, \tag{6}$$

so that the Faraday equation is satisfied; **b** is the magnetic flux density and v can usually be fixed to zero in passive conductors. The magnetodynamic (**b**-conform) **a** formulation is obtained from the weak form of the Ampere equation, i.e. [5],

$$(\mu^{-1} \operatorname{curl} \mathbf{a}, \operatorname{curl} \mathbf{a}')_{\Omega} + (\sigma \partial_t \mathbf{a}, \mathbf{a}')_{\Omega_c} - (\mathbf{j}_s, \mathbf{a}')_{\Omega_s} = 0, \quad \forall \mathbf{a}' \in F_a(\Omega), \tag{7}$$

where $F_a(\Omega)$ is a function space defined on the whole studied domain Ω and containing the basis functions for **a** and test function $\mathbf{a}'$.

The lamination stack is considered as a source region (i.e., in Ω_s, rather than a set of massive conductors, in Ω_c) carrying the predefined current density (2) in place of source current $\mathbf{j}_s$ in (7), as well as the associated predefined magnetic flux density $\mathbf{b}_\alpha$.

Both the first and third terms in (7) have to give contributions similar to these given in case of a massive conductor region [5]. This is achieved by using (2), (4), as well as $\mathbf{e}_\beta = -\partial_t \mathbf{a}_\beta$ leading to $\mathbf{a}_\beta = -\mathbf{i}_\gamma \times \int \mathbf{b}_\alpha \, d\gamma$.

One has, for the first term of (7),

$$(\mu^{-1} \operatorname{curl} \mathbf{a}, \operatorname{curl} \mathbf{a}')_{\Omega_{ls}} = \sum_i (\mu^{-1} \mathbf{b}_\alpha, \mathbf{b}'_\alpha)_{\Omega_{ls,i}} = \sum_i \left(\mu^{-1} \sum_n b_n \eta_n, \sum_m b'_m \eta_m \right)_{\Omega_{ls,i}}, \tag{8}$$

where the contribution of Ω_{ls} is split up into a sum of contributions of each of its laminations $\Omega_{ls,i}$ (i.e., the ith lamination in Ω_{ls}). Then, by summing on i the integrals in all the laminations $\Omega_{ls,i}$, and averaging the contributions of the sub-basis functions products, (8) becomes

$$(\mu^{-1} \operatorname{curl} \mathbf{a}, \operatorname{curl} \mathbf{a}')_{\Omega_{ls}} = \sum_m \sum_n (\bar{N}_{mn} \bar{\mathbf{b}}_n, \bar{\mathbf{b}}'_m)_{\bar{\Omega}_{ls}} = \sum_m \sum_n (\bar{N}_{mn} \operatorname{curl} \bar{\mathbf{a}}_n, \operatorname{curl} \bar{\mathbf{a}}'_m)_{\bar{\Omega}_{ls}}, \tag{9}$$

with coefficients $\bar{N}_{mn}$ of a matrix $\bar{\mathbf{N}}$ defined by

$$\bar{N}_{mn} = d^{-1} \int_{-d/2}^{d/2} \mu^{-1} \eta_m(\gamma) \eta_n(\gamma) \, d\gamma, \quad \forall m, n \in S, \tag{10}$$

where an upper bar symbol denotes an homogenized entity along the lamination or stack thickness. The coefficients b_n of expansion (4) for each lamination are indeed transformed to a continuum $\bar{\mathbf{b}}_n$ when all the laminations are considered together, i.e., through the integration in $\bar{\Omega}_{ls}$, defined as the homogenized laminated region.

The homogenized magnetic flux density $\bar{\mathbf{b}}_n$ can be expressed as the curl of a magnetic vector potential $\bar{\mathbf{a}}_n$, considered as homogenized as well. In this way, the conservation of the magnetic flux density, i.e., $\operatorname{div} \mathbf{b} = 0$, is satisfied for the whole expansion (4), as will be shown. At the discrete level, the vector potential is approximated with edge FEs. Particular definition of sub-basis functions $\eta_n(\gamma)$ will be shown to conveniently allow the transposition of the flux conservation.

The third term of (7) is developed analogously, this time with primitives of $\eta_n(\gamma)$, i.e.,

$$-(\mathbf{j}_s, \mathbf{a}')_{\Omega_{ls}} = -\sum_i (\mathbf{j}_\beta, \mathbf{a}')_{\Omega_{ls,i}} = -\sum_i \left(\sigma \mathbf{i}_\gamma \times \partial_t \int \mathbf{b}_\alpha \, d\gamma, -\mathbf{i}_\gamma \times \int \mathbf{b}'_\alpha \, d\gamma\right)_{\Omega_{ls,i}}$$

$$= \sum_i \left(\sigma \partial_t \int \mathbf{b}_\alpha \, d\gamma, \int \mathbf{b}'_\alpha \, d\gamma\right)_{\Omega_{ls,i}} = \sum_i \left(\sigma \partial_t \int \sum_n b_n \eta_n \, d\gamma, \int \sum_m b'_m \eta_m \, d\gamma\right)_{\Omega_{ls,i}}$$

$$= \sum_i \left(\sigma \partial_t \sum_n b_n \int \eta_n \, d\gamma, \sum_m b'_m \int \eta_m \, d\gamma\right)_{\Omega_{ls,i}}. \tag{11}$$

Then, by summing on i the integrals in all the laminations $\Omega_{ls,i}$, and averaging the contributions of the primitive sub-basis functions products, (11) becomes

$$-(\mathbf{j}_s, \mathbf{a}')_{\Omega_{ls}} = \sum_m \sum_n (\bar{\mathrm{P}}_{mn} \partial_t \bar{\mathbf{b}}_n, \bar{\mathbf{b}}'_m)_{\bar{\Omega}_{ls}} = \sum_m \sum_n (\bar{\mathrm{P}}_{mn} \partial_t \, \mathrm{curl}\, \bar{\mathbf{a}}_n, \mathrm{curl}\, \bar{\mathbf{a}}'_m)_{\bar{\Omega}_{ls}} \tag{12}$$

with coefficients $\bar{\mathrm{P}}_{mn}$ of a matrix $\bar{\mathbf{P}}$ defined by

$$\bar{\mathrm{P}}_{mn} = \mathrm{d}^{-1} \int_{-d/2}^{d/2} \sigma \left(\int \eta_m(\gamma) \, d\gamma\right) \left(\int \eta_n(\gamma) \, d\gamma\right) d\gamma, \quad \forall m, n \in S. \tag{13}$$

The magnetic permeability μ has generally to be kept in integral (10) in order to consider its possible nonlinear nature, i.e., $\mu^{-1} = \mu^{-1}(b_\alpha)$. For a non-constant permeability, each coefficient $\bar{N}_{mn}$ depends on each particular evaluation point in $\bar{\Omega}_{ls}$. In this way, the nonlinear behavior is averaged along the lamination thickness through (10).

4.2. **h**-*Conform magnetic field formulation*

The magnetodynamic (**h**-conform) **h**-ϕ formulation is obtained from the weak form of the Faraday equation, i.e. [4],

$$\partial_t (\mu \mathbf{h}, \mathbf{h}')_\Omega + (\sigma^{-1} \mathrm{curl}\, \mathbf{h}, \mathrm{curl}\, \mathbf{h}')_{\Omega_c} + (\sigma^{-1} \mathbf{j}_s, \mathrm{curl}\, \mathbf{h}')_{\Omega_s} = 0, \quad \forall \mathbf{h}' \in F_{h\phi}(\Omega), \tag{14}$$

where $F_{h\phi}(\Omega)$ is a function space defined on Ω and containing the basis functions for the magnetic field $\mathbf{h}$, coupled to a magnetic scalar potential ϕ, as well as for the test function $\mathbf{h}'$.

Both first and third terms in (14) have to give contributions similar to those given in case of a massive conductor region. This is achieved by using approximation (4) for $\mathbf{h}_\alpha = \mu^{-1} \mathbf{b}_\alpha$ and the associated curl for approximating $\mathbf{j}_\beta$ through (3), i.e., for the components of these quantities,

$$h_\alpha(\gamma, t) = \sum_{n \in S} h_n(t) \eta_n(\gamma), \tag{15}$$

$$j_\beta(\gamma, t) = -\sum_{n \in S} h_n(t) \partial_\gamma \eta_n(\gamma), \tag{16}$$

with $h_n = \mu^{-1} b_n$. In this way, the Ampere equation is exactly satisfied in each lamination. The Faraday equation will be rather weakly satisfied through (14). The terms for $n > 0$ in (15) can be interpreted as defining a source magnetic field of which the curl is the source current density $\mathbf{j}_\beta$, while the term for $n = 0$ defines a curl-free component of $\mathbf{h}_\alpha$.

One has, for the first term of (14) expressed by the contribution of all separate laminations $\Omega_{ls,i}$,

$$\partial_t (\mu \mathbf{h}, \mathbf{h}')_{\Omega_{ls}} = \sum_i (\mu \mathbf{h}_\alpha, \mathbf{h}'_\alpha)_{\Omega_{ls,i}} = \sum_i \left(\mu \sum_n h_n \eta_n, \sum_m h_m \eta_m\right)_{\Omega_{ls,i}} \tag{17}$$

which becomes, by summing the integrals in all the laminations $\Omega_{ls,i}$,

$$\partial_t (\mu \mathbf{h}, \mathbf{h}')_{\Omega_{ls}} = \sum_m \sum_n (\bar{N}_{h,mn} \bar{\mathbf{h}}_n, \bar{\mathbf{h}}'_m)_{\bar{\Omega}_{ls}}, \tag{18}$$

with coefficients $\bar{N}_{h,mn}$ of a matrix $\bar{\mathbf{N}}_h$ defined by

$$\bar{N}_{h,mn} = d^{-1} \int_{-d/2}^{d/2} \mu \eta_m(\gamma)\eta_n(\gamma)\,\mathrm{d}\gamma, \quad \forall m, n \in S. \tag{19}$$

The homogenized magnetic field $\bar{\mathbf{h}}_0$ can be expressed as the gradient of a magnetic scalar potential, considered as homogenized as well. For a nonlinear material, $\mu = \mu(h_\alpha)$ in (19).

The third term of (14) is developed analogously, this time with derivatives of $\eta_n(\gamma)$, i.e.,

$$(\sigma^{-1}\mathbf{j}_\mathrm{s}, \operatorname{curl}\mathbf{h}')_{\Omega_\mathrm{ls}} = \sum_i (\sigma^{-1}\mathbf{j}_\beta, \mathbf{j}'_\beta)_{\Omega_{\mathrm{ls},i}} = \sum_i \left(\sigma^{-1}\sum_n h_n \partial_\gamma \eta_n, \sum_m h'_m \partial_\gamma \eta_m\right)_{\Omega_{\mathrm{ls},i}} \tag{20}$$

which becomes, by summing the integrals in all the laminations $\Omega_{\mathrm{ls},i}$,

$$(\sigma^{-1}\mathbf{j}_\mathrm{s}, \operatorname{curl}\mathbf{h}')_{\Omega_\mathrm{ls}} = \sum_m \sum_n (\bar{P}_{h,mn}\bar{\mathbf{h}}_n, \bar{\mathbf{h}}'_m)_{\bar{\Omega}_\mathrm{ls}}, \tag{21}$$

with coefficients $\bar{P}_{h,mn}$ of a matrix $\bar{\boldsymbol{P}}_h$ defined by

$$\bar{P}_{h,mn} = d^{-1} \int_{-d/2}^{d/2} \sigma^{-1} \partial_\gamma \eta_m(\gamma) \partial_\gamma \eta_n(\gamma)\,\mathrm{d}\gamma, \quad \forall m, n \in S. \tag{22}$$

5. Skin effect sub-basis functions

Examples of polynomial skin effect sub-basis functions $\eta_n(\gamma_\mathrm{r})$ are developed here. These can be limited to even orders in case no net current flows in the laminations, which gives, for the expansion order set in (4), $S = \{0, 2, 4, \ldots\}$.

For convenience, coordinate γ is defined as equal to zero at the mid-thickness of each lamination. It can be also associated with a reference coordinate γ_r, varying from -1 to 1 along the thickness of each lamination, i.e., defined by $\gamma_\mathrm{r} = \gamma 2/d$.

The sub-basis functions can be worthily given the property of orthogonality, i.e.,

$$\int_{-1}^{+1} \eta_i(\gamma_\mathrm{r})\eta_j(\gamma_\mathrm{r})\,\mathrm{d}\gamma_\mathrm{r} = 0 \quad \text{for } i \neq j, i, j \in S, \tag{23}$$

which, in particular, defines a zero average value for each function η_n of order n greater or equal to 1; function η_0 is constant and equal to 1. Consequently, the coefficient b_0 in (4) is directly the mean magnetic flux density. For linear magnetic materials, this property also gives matrix $\bar{\mathbf{N}}$ a diagonal form.

Even functions $\eta_n(\gamma_\mathrm{r})$, $n \in S = \{0, 2, 4, \ldots\}$, can then be determined as

$$\eta_0(\gamma_\mathrm{r}) = 1, \quad \eta_2(\gamma_\mathrm{r}) = \tfrac{1}{2}(3\gamma_\mathrm{r}^2 - 1), \quad \eta_4(\gamma_\mathrm{r}) = \tfrac{1}{8}(35\gamma_\mathrm{r}^4 - 30\gamma_\mathrm{r}^2 + 3), \ldots\ . \tag{24}$$

The associated primitive functions $\tau_n(\gamma_\mathrm{r}) = \int \eta_n(\gamma_\mathrm{r})\,\mathrm{d}\gamma_\mathrm{r}$, $n \in S$, are

$$\tau_0(\gamma_\mathrm{r}) = \gamma_\mathrm{r}, \quad \tau_2(\gamma_\mathrm{r}) = \tfrac{1}{2}(\gamma_\mathrm{r}^3 - \gamma_\mathrm{r}), \quad \tau_4(\gamma_\mathrm{r}) = \tfrac{1}{8}(7\gamma_\mathrm{r}^5 - 10\gamma_\mathrm{r}^3 + 3\gamma_\mathrm{r}), \ldots\ . \tag{25}$$

The associated derivative functions $\kappa_n(\gamma_\mathrm{r}) = \partial_{\gamma_\mathrm{r}} \eta_n(\gamma_\mathrm{r})$, $n \in S$, are

$$\kappa_0(\gamma_\mathrm{r}) = 0, \quad \kappa_2(\gamma_\mathrm{r}) = 3\gamma_\mathrm{r}, \quad \kappa_4(\gamma_\mathrm{r}) = \tfrac{5}{2}(7\gamma_\mathrm{r}^3 - 3\gamma_\mathrm{r}), \ldots\ . \tag{26}$$

The corresponding functions written in terms of γ are given by

$$\eta_n(\gamma) = \eta_n(\gamma_\mathrm{r}), \quad \tau_n(\gamma) = \frac{d}{2}\tau_n(\gamma_\mathrm{r}), \quad \kappa_n(\gamma) = \frac{2}{d}\kappa_n(\gamma_\mathrm{r}). \tag{27}$$

Thanks to their property of orthogonality, sub-basis functions of additional orders satisfying (23) can be determined without difficulties. They will act in a hierarchical way in the expansion of $\mathbf{b}_\alpha$, only adding coefficients in matrices $\bar{\mathbf{N}}$, $\bar{\mathbf{P}}$, $\bar{\mathbf{N}}_h$ and $\bar{\mathbf{P}}_h$, without modifying the already calculated contributions.

For $S=\{0,2,4\}$ and constant μ and σ, matrices $\bar{\mathbf{N}}$ and $\bar{\mathbf{P}}$ are (indexes for rows and columns follow those of set S)

$$\bar{\mathbf{N}}=\mu^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{5} & 0\\ 0 & 0 & \frac{1}{9}\end{pmatrix},\quad \bar{\mathbf{P}}=\sigma\frac{d^2}{4}\begin{pmatrix}\frac{1}{3} & -\frac{1}{15} & 0\\ -\frac{1}{15} & \frac{2}{105} & -\frac{1}{315}\\ 0 & -\frac{1}{315} & \frac{2}{693}\end{pmatrix},\tag{28}$$

of which the coefficients can be directly used in terms (9) and (12) of the magnetic vector potential formulation (7). Matrices $\bar{\mathbf{N}}_h$ and $\bar{\mathbf{P}}_h$, giving the coefficients to be used in terms (18) and (21) of the magnetic field formulation (14), are

$$\bar{\mathbf{N}}_h=\mu\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{5} & 0\\ 0 & 0 & \frac{1}{9}\end{pmatrix},\quad \bar{\mathbf{P}}_h=\sigma^{-1}\frac{4}{d^2}\begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 3\\ 0 & 3 & 10\end{pmatrix}.\tag{29}$$

6. Constraints on the homogenized magnetic flux density

As stated before, for the **b**-conform formulation, the conservation of the magnetic flux density **b** has to be expressed in a strong way. In case there is no component of **b** perpendicular to the laminations, i.e., $\mathbf{b}_\gamma=0$ and $\mathbf{b}=\mathbf{b}_\alpha$ given by (6), one has, in each lamination $\Omega_{\mathrm{ls},i}$,

$$\operatorname{div}\mathbf{b}_\alpha=\operatorname{div}\left(\sum_{n\in S}\mathbf{b}_n\eta_n(\gamma)\right)=\sum_{n\in S}\operatorname{div}\mathbf{b}_n\eta_n(\gamma)=0.\tag{30}$$

Then, thanks to the orthogonality property (23) of the sub-basis functions η_n, (30) implies that

$$\operatorname{div}\mathbf{b}_n=0,\quad \forall n\in S,\tag{31}$$

which holds in any lamination but also for the homogenized coefficients $\bar{\mathbf{b}}_n$, i.e.,

$$\operatorname{div}\bar{\mathbf{b}}_n=0,\quad \forall n\in S.\tag{32}$$

A non-zero perpendicular component $\mathbf{b}_\gamma$, with $\mathbf{b}=\mathbf{b}_\alpha+\mathbf{b}_\gamma$ (or $\mathbf{b}=\bar{\mathbf{b}}_\alpha+\mathbf{b}_\gamma$), only gives a complementary contribution to coefficient $\mathbf{b}_0$ (or $\bar{\mathbf{b}}_0$), all together satisfying

$$\operatorname{div}(\mathbf{b}_0+\mathbf{b}_\gamma)=0\quad\text{or}\quad\operatorname{div}(\bar{\mathbf{b}}_0+\mathbf{b}_\gamma)=0,\tag{33}$$

with all the other relations in (31) or (32), for $n\neq 0$, unchanged.

It should be noted that, in this case, odd order sub-basis functions have to be considered in the expansion of $\mathbf{b}_\alpha$. Indeed, the eddy currents induced by time variations of $\mathbf{b}_\gamma$ have to be given even order distributions. Relations (32) and (33b) justify that $\bar{\mathbf{b}}_0+\mathbf{b}_\gamma$ and all the other coefficients $\bar{\mathbf{b}}_n$ can be expressed as the curl of magnetic vector potentials. The restriction of coefficients $\bar{\mathbf{b}}_n$, $\forall n\in S$, in a plane perpendicular to the laminations (with *a priori* unknown direction $\mathbf{i}_\alpha$) can be obtained through the use of a tensorial conductivity in (13), with non-zero values only in the directions parallel to each lamination. Both contributions $\bar{\mathbf{b}}_0$ and $\mathbf{b}_\gamma$ are naturally mixed through the weak formulation.

7. Application

The homogenized formulations have been applied to a test problem consisting of a 3-D stack of rectangular laminations. The laminations are characterized by a constant relative permeability $\mu_{\mathrm{r}}=2000$ and a conductivity $\sigma=10^7\,\mathrm{S\,m^{-1}}$. Two values are considered for their thickness: $d=0.3$ and 0.6 mm. The thickness of the stack is 1.8 mm and its width is 10 mm. It is excited by an enforced magnetomotive force (MMF) either sinusoidal or triangular, giving a flux parallel to the laminations. The frequency of the sinusoidal MMF varies from 50 to 50 kHz, with associated skin depth from 0.5 to 0.016 mm. The one of the triangular MMF is either 50 Hz, 1 or 5 kHz.

The homogenized solutions, with polynomials orders p equal to 0 and 2 in (4), are compared with the solution of the direct method considering the stack as a set of insulated massive conductors (Fig. 3 for a sinusoidal MMF and Fig. 4

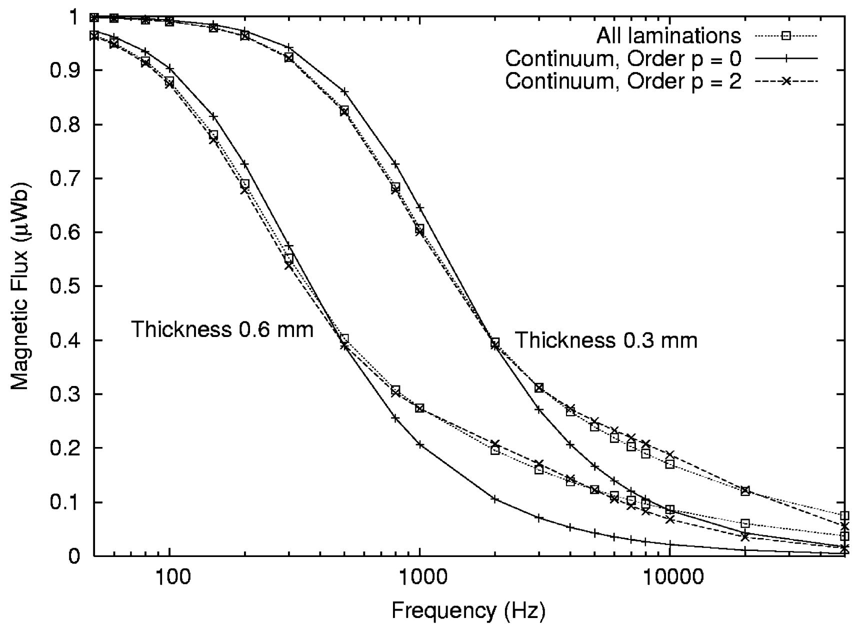

Fig. 3. Amplitude of the magnetic flux in the lamination stack versus the frequency with the direct method and the homogenized or continuum model.

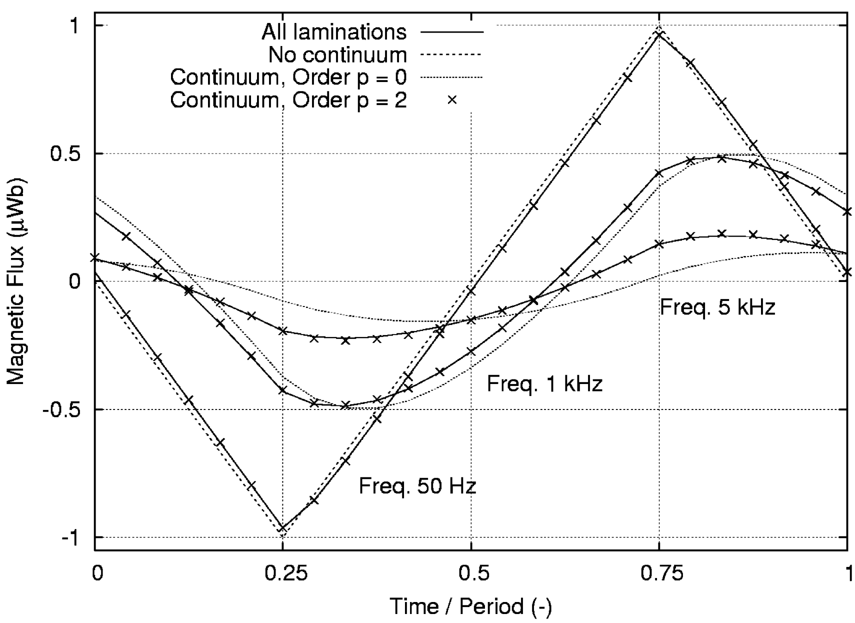

Fig. 4. Magnetic flux versus normalized time with the direct method, no consideration of eddy current (no continuum; a direct image of the MMF is obtained), and the homogenized or continuum model ($d = 0.3$ mm); triangular MMF of different frequencies.

for a triangular MMF). A very good agreement is obtained for $p = 2$ up to quite a high frequency, while the accuracy is lower for $p = 0$. The mesh of the massive conductors is sufficiently fine to take the small skin depth into account for high frequencies (in addition, insulating layers have to be considered between the laminations), while the mesh of the continuum does not need to satisfy this constraint and is therefore coarse for any frequency.

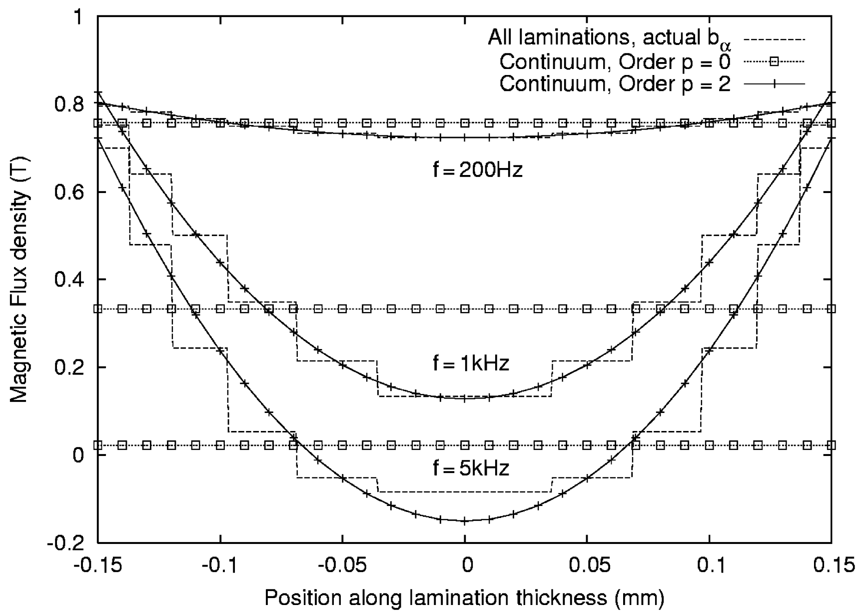

Fig. 5. Magnetic flux density along the thickness of a lamination, at a certain time instant, for different frequencies with the direct method and the homogenized or continuum model ($d = 0.3$ mm).

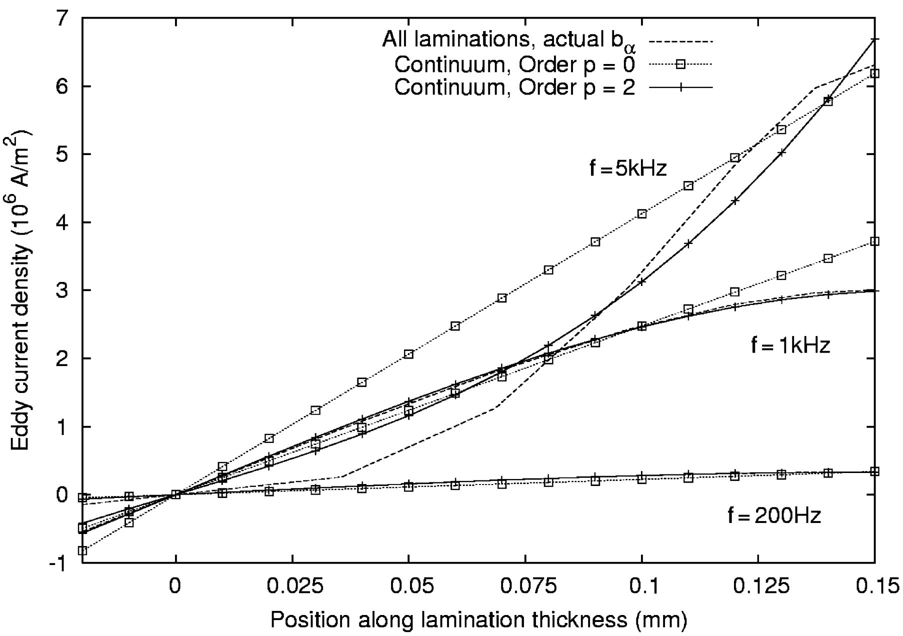

Fig. 6. Eddy current density along the thickness of a lamination, at a certain time instant, for different frequencies (sinusoidal MMF) with the direct method and the homogenized or continuum model ($d = 0.3$ mm).

Other comparisons can also be made. The distribution of $\mathbf{b}_\alpha$ can be post-computed by (4), using the homogenized coefficients $\bar{\mathbf{b}}_n$ and can be compared to the magnetic flux density obtained with the direct method in massive conductor regions (Fig. 5). A similar comparison can be done for the current density $\mathbf{j}_\beta$, post-computed through (2) or (3) and (4) (Fig. 6). Such comparisons point out the accuracy obtained in function of the approximation order p of $\mathbf{b}_\alpha$ in (4).

8. Conclusions

A time-domain technique has been developed to take the eddy currents in lamination stacks into account with dual 3-D magnetodynamic **b**- and **h**-conform FE formulations. In order to avoid the explicit definition of all laminations, each laminated region is converted into a continuum in which the eddy current effects are taken into account thanks to adapted terms in the weak formulations. For that, each lamination stack is considered as a source region carrying predefined current density and magnetic flux density distributions describing the eddy currents and skin effect in each lamination. The current density is expressed in terms of the magnetic flux density, this latter being approximated with sub-basis functions of which the contributions are averaged in the weak formulations.

The developed method appears attractive for directly taking into account the eddy current effects which are particularly significant for high frequency components. The time-domain analysis makes possible the consideration of these effects for both nonlinear and transient phenomena.

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