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The split-step backward Euler method for linear stochastic delay differential equations☆

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ABSTRACT

In this paper, the numerical approximation of solutions of linear stochastic delay differential equations (SDDEs) in the Itô sense is considered. We construct split-step backward Euler (SSBE) method for solving linear SDDEs and develop the fundamental numerical analysis concerning its strong convergence and mean-square stability. It is proved that the SSBE method is convergent with strong order $\gamma = \frac{1}{2}$ in the mean-square sense. The conditions under which the SSBE method is mean-square stable (MS-stable) and general mean-square stable (GMS-stable) are obtained. Some illustrative numerical examples are presented to demonstrate the order of strong convergence and the mean-square stability of the SSBE method.

1. Introduction

Consider a scalar linear system of the Itô stochastic delay differential equations

$$\begin{cases} \mathrm{d}y(t) = (ay(t) + by(t-\tau))\mathrm{d}t + (cy(t) + dy(t-\tau))\mathrm{d}W(t), & t \geq 0, \\ y(t) = \psi(t), & t \in [-\tau, 0]. \end{cases} \tag{1}$$

Here $a, b, c, d \in \mathbb{R}$, τ is a positive fixed delay, $W(t)$ is a one-dimensional standard Wiener process and $\psi(t)$ is a $C([-\tau, 0]; \mathbb{R})$-valued initial segment.

Stochastic delay differential equations (SDDEs) serve as models of physical processes whose time evolution depends on their past history with noise disturbance. In many fields of science there has been an increasing interest in the investigation of SDDEs, in particular, in the combined effects of noise and delay in dynamical systems. One can see, for example, [5,6]. The fundamental theory of existence and uniqueness of solution of SDDEs has been studied in [14,16].

SDDEs arising in many applications cannot be solved explicitly. Hence, one needs to develop effective numerical methods for such systems. The development of a numerical approximation scheme for SDDEs is relatively new, compared with that for deterministic delay differential equations (DDEs) or for stochastic ordinary differential equations (SODEs)—see, e.g., [3, 11], respectively. Baker and Buckwar gave some results of convergence for explicit single-step methods in [1], and Mao [15] proved that the numerical solutions produced by the Euler–Maruyama method converge to the true solutions under the local Lipschitz condition. Hu et al. [10] developed a strong Milstein approximation scheme for solving SDDEs with convergence

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order 1. Liu et al. [13] studied the convergence and stability of the semi-implicit Euler method and Wang and Zhang [18] analyzed the stability of the Milstein method for linear SDDEs. Wang [17] studied the convergence and stability of some numerical methods for nonlinear SDDEs. Küchler and Platen [12] introduced an approach for derivation of discrete time approximations for solutions of SDDEs. Baker and Buckwar [2] obtained conditions for the pth mean stability of a solution of SDDEs. Buckwar and Winkler [6] constructed multi-step Maruyama methods for SDDEs.

Higham et al. [9] introduced SSBE method for nonlinear SODEs firstly. The authors obtained that the strong convergence order of the SSBE method is $\gamma = \frac{1}{2}$ under one-sided Lipschitz condition, which is weaker than the global Lipschitz condition. In [8], the authors generalized the SSBE method to nonlinear stochastic differential equations with Poisson jump and obtained analogous results with that in [9].

In this paper, we construct SSBE method for linear SDDEs. The convergence and mean-square stability of the SSBE method are studied. The structure of the paper is organized as follows. In Section 2, we introduce some notations and hypotheses of Eq. (1) and discuss the properties of its analytical solution. Furthermore, the SSBE scheme for linear SDDE (1) is constructed in this section. In Section 3, convergence result of the SSBE method and its proof are presented. We prove that it is convergent with strong order $\frac{1}{2}$ in the mean-square sense. In Section 4, we investigate the MS-stability and GMS-stability of the SSBE method. Numerical results are reported in Section 5.

2. Definitions and preliminary results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions (that is, it is increasing and right continuous, and each $\{\mathcal{F}_t\}$, $t \in [0, T]$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$). Let $W(t)$, $t \geq 0$ be $\{\mathcal{F}_t\}$-adapted and independent of $\mathcal{F}_0$. Moreover, we assume $\psi(t)$, $t \in [-\tau, 0]$ to be $\mathcal{F}_0$-measurable and right continuous with norm $\|\psi\| = \sup_{-\tau\leq t\leq 0} |\psi(t)|$ and $\mathbb{E}\|\psi\|^2 < \infty$, where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$.

Under the above assumptions, linear equation (1) has a unique strong solution $y(t)$: $[-\tau, +\infty) \to \mathbb{R}$, which satisfies Eq. (1) and $y(t)$ is a measurable, sample-continuous and $\mathcal{F}_t$-adapted process. This result can be found in [14,16].

Now, we present some lemmas which will be used in the following sections.

Lemma 2.1 (*[7]*). *If the constants a, b, c, d satisfy*

$$a < -|b| - \frac{1}{2}\left(|c| + |d|\right)^2, \tag{2}$$

then the solution of Eq. (1) is asymptotically stable in the mean-square sense, that is,

$$\lim_{t\to\infty} \mathbb{E}|y(t)|^2 = 0. \tag{3}$$

Lemma 2.2 (*[14]*). *For any given $0 < T < \infty$, there exist positive numbers C_1, C_2 and M such that the solution of Eq. (1) satisfies*

$$\mathbb{E}(\sup_{-\tau\leq t\leq T} |y(t)|^2) \leq C_1(1 + \mathbb{E}\|\psi\|^2) \tag{4}$$

for all $t \in [-\tau, T]$,

$$\mathbb{E}|y(t) - y(s)|^2 \leq C_2(t - s) \tag{5}$$

for any $0 \leq s < t \leq T$, $t - s < 1$, and

$$\mathbb{E}|ay(t) + by(t - \tau)| \leq \sqrt{2M(1 + \mathbb{E}\|\psi\|^2)} \tag{6}$$

for all $t \in [0, T]$.

Now, we construct the SSBE scheme for solving linear SDDEs. This stochastic numerical method can be regarded as a generalization of the SSBE scheme for nonlinear SODEs [9] and of the SSBE scheme for nonlinear stochastic differential equations with Poisson jumps [8].

We define a mesh with a uniform stepsize h on the interval $[0, T]$ and $h = T/N$, $t_n = n \cdot h$, $n = 0, \ldots, N$. We assume that there is an integer number m such that the delay can be expressed in terms of the stepsize h as $\tau = m \cdot h$. We construct the SSBE method for solving Eq. (1) by $Y_k = \psi(kh)$ when $k = -m, -m + 1, \ldots, 0$ and when $k \geq 0$,

$$\begin{cases} Y_k^* = Y_k + h[aY_k^* + bY_{k-m+1}], & \text{(a)} \\ Y_{k+1} = Y_k^* + (cY_k^* + dY_{k-m+1})\Delta W_k, & \text{(b)} \end{cases} \tag{7}$$

where Y_k is the numerical approximation of $y(t_k)$ with $t_k = kh$. Moreover, the increments $\Delta W_k := W(t_{k+1}) - W(t_k)$, are independent $N(0, h)$-distributed Gaussian random variables. If $1 - ah \neq 0$, we can obtain the sequences $\{Y_k^*, k \geq 0\}$ and $\{Y_k, k \geq 1\}$ via (7), when given

$$Y_k = \psi(t_{-k}) \quad \text{for } k \in \mathcal{J} \text{ where } \mathcal{J} := \{0, 1, \ldots, m\}.$$

3. Convergence of the SSBE method

In the following of this paper, we always assume that $1 - ah \neq 0$ holds. By (7) we have

$$Y_{k+1} = Y_k + \left(\frac{a}{1-ah}Y_k + \frac{b}{1-ah}Y_{k+1-m}\right)h + \left[\frac{c}{1-ah}Y_k + \left(\frac{bch}{1-ah} + d\right)Y_{k+1-m}\right]\Delta W_k. \tag{8}$$

Definition 3.1 ([4]). The *local error* of SSBE method (7) for the approximation of the solution $y(t)$ of Eq. (1), for $k = 0, 1, \ldots, N-1$, is defined as

$$\begin{aligned}\delta_{k+1} := y(t_{k+1}) - \Bigg\{ & y(t_k) + \left(\frac{a}{1-ah}y(t_k) + \frac{b}{1-ah}y(t_{k+1-m})\right)h \\ & + \left(\frac{c}{1-ah}y(t_k) + \left(\frac{bch}{1-ah} + d\right)y(t_{k+1-m})\right)\Delta W_k\Bigg\},\end{aligned} \tag{9}$$

where $y(t_k)$ denotes the value of the exact solution of Eq. (1) at the mesh-point t_k.

Definition 3.2 ([4]). The *global error* of SSBE method (7) for the approximation of the solution $y(t)$ of Eq. (1), for $k = 1, 2, \ldots, N$, is defined as

$$\varepsilon_k = y(t_k) - Y_k. \tag{10}$$

Note that ε_k is $\mathcal{F}_{t_k}$-measurable since both $y(t_k)$ and Y_k are $\mathcal{F}_{t_k}$-measurable random variables.

Definition 3.3 ([4]). SSBE method (7) is *consistent with order p_1 in the mean and with order p_2 in the mean-square sense* if the following estimates hold with $p_2 \geq \frac{1}{2}$ and $p_1 \geq p_2 + \frac{1}{2}$:

$$\max_{1\leq k\leq N} |\mathbb{E}(\delta_k)| \leq Ch^{p_1}, \quad \text{as } h \to 0, \tag{11}$$

and

$$\max_{1\leq k\leq N} \left(\mathbb{E}(\delta_k)^2\right)^{\frac{1}{2}} \leq C'h^{p_2}, \quad \text{as } h \to 0, \tag{12}$$

where the positive constants C and C' do not depend on h, but may depend on T and the initial segment ψ of Eq. (1).

Theorem 3.1. *SSBE method (7) is consistent with order $p_1 = \frac{3}{2}$ in the mean and $p_2 = 1$ in the mean-square sense.*

Proof. Without loss of generality, we assume that $0 < h < 1$ and $1 - ah \geq \frac{1}{2}$ hold. When $0 \leq s \leq t \leq T$, we have

$$y(t) - y(s) = \int_s^t [ay(r) + by(r-\tau)]\mathrm{d}r + \int_s^t [cy(r) + dy(r-\tau)]\mathrm{d}W(r). \tag{13}$$

First, we prove the consistent order $p_1 = \frac{3}{2}$ in the mean sense. By (1), (9) and (13), we can derive that

$$\begin{aligned}\delta_{k+1} &= y(t_k) + \int_{t_k}^{t_{k+1}} (ay(t) + by(t-\tau))\mathrm{d}t + \int_{t_k}^{t_{k+1}} (cy(t) + dy(t-\tau))\mathrm{d}W(t) \\ &\quad - \left\{y(t_k) + \left(\frac{a}{1-ah}y(t_k) + \frac{b}{1-ah}y(t_{k+1-m})\right)h + \left(\frac{c}{1-ah}y(t_k) + \left(\frac{bch}{1-ah} + d\right)y(t_{k+1-m})\right)\Delta W_k\right\} \\ &= \int_{t_k}^{t_{k+1}} \left(ay(t) - \frac{a}{1-ah}y(t_k)\right) + \left(by(t-\tau) - \frac{b}{1-ah}y(t_{k+1-m})\right)\mathrm{d}t \\ &\quad + \int_{t_k}^{t_{k+1}} \left(cy(t) - \frac{c}{1-ah}y(t_k)\right) + \left(dy(t-\tau) - \left(\frac{bch}{1-ah} + d\right)y(t_{k+1-m})\right)\mathrm{d}W(t) \\ &= \int_{t_k}^{t_{k+1}} a(y(t) - y(t_k)) + b(y(t-\tau) - y(t_{k+1-m}))\mathrm{d}t - \frac{a^2h^2}{1-ah}y(t_k) \\ &\quad - \frac{abh^2}{1-ah}y(t_{k+1-m}) + c\int_{t_k}^{t_{k+1}} \left[(y(t) - y(t_k)) - \frac{ah}{1-ah}y(t_k)\right]\mathrm{d}W(t) \\ &\quad + \int_{t_k}^{t_{k+1}} \left[d(y(t-\tau) - y(t_{k+1-m})) - \frac{bch}{1-ah}y(t_{k+1-m})\right]\mathrm{d}W(t)\end{aligned}$$

$$
\begin{aligned}
&= a\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}(ay(r)+by(r-\tau))\mathrm{d}r\mathrm{d}t + a\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}(cy(r)+dy(r-\tau))\mathrm{d}W(r)\mathrm{d}t\\
&\quad + b\int_{t_k}^{t_{k+1}}\int_{t_k-\tau}^{t-\tau}(ay(r)+by(r-\tau))\mathrm{d}r\mathrm{d}t + b\int_{t_k}^{t_{k+1}}\int_{t_k-\tau}^{t-\tau}(cy(r)+dy(r-\tau))\mathrm{d}W(r)\mathrm{d}t\\
&\quad + b\int_{t_k}^{t_{k+1}}(y(t_{k-m})-y(t_{k+1-m}))\mathrm{d}t - \frac{a^2h^2}{1-ah}y(t_k) - \frac{abh^2}{1-ah}y(t_{k+1-m})\\
&\quad + c\int_{t_k}^{t_{k+1}}[(y(t)-y(t_k)) - \frac{ah}{1-ah}y(t_k)]\mathrm{d}W(t)\\
&\quad + \int_{t_k}^{t_{k+1}}[d(y(t-\tau)-y(t_{k+1-m})) - \frac{bch}{1-ah}y(t_{k+1-m})]\mathrm{d}W(t).
\end{aligned} \tag{14}
$$

Taking mathematical expectation and using the properties of the Itô integral, we can obtain that

$$
\begin{aligned}
|\mathbb{E}(\delta_{k+1})| &\le |a|\mathbb{E}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}|ay(r)+by(r-\tau)|\mathrm{d}r\mathrm{d}t + \frac{a^2}{1-ah}\mathbb{E}|y(t_k)|h^2\\
&\quad + |b|\mathbb{E}\int_{t_k}^{t_{k+1}}\int_{t_k-\tau}^{t-\tau}|ay(r)+by(r-\tau)|\mathrm{d}r\mathrm{d}t + \left|\frac{ab}{1-ah}|\mathbb{E}|y(t_{k+1-m})\right|h^2\\
&\quad + |b|\mathbb{E}\int_{t_k}^{t_{k+1}}|y(t_{k-m})-y(t_{k+1-m})|\mathrm{d}t\\
&\le |a|\mathbb{E}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}|ay(r)+by(r-\tau)|\mathrm{d}r\mathrm{d}t + 2a^2\mathbb{E}|y(t_k)|h^2\\
&\quad + |b|\mathbb{E}\int_{t_k}^{t_{k+1}}\int_{t_k-\tau}^{t-\tau}|ay(r)+by(r-\tau)|\mathrm{d}r\mathrm{d}t + 2|ab|\mathbb{E}|y(t_{k+1-m})|h^2\\
&\quad + |b|\mathbb{E}\int_{t_k}^{t_{k+1}}|y(t_{k-m})-y(t_{k+1-m})|\mathrm{d}t.
\end{aligned} \tag{15}
$$

It follows from inequalities (4) and (5) that

$$
\mathbb{E}\int_{t_k}^{t_{k+1}}|y(t_{k+1-m})-y(t_{k-m})|\mathrm{d}t \le \sqrt{C_2}h^{\frac{3}{2}}, \tag{16}
$$

$$
\mathbb{E}|y(t_k)| \le \sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}, \tag{17}
$$

and

$$
\mathbb{E}|y(t_{k+1-m})| \le \sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}. \tag{18}
$$

Substituting (6) and (16)–(18) into (15), we can obtain that

$$
\begin{aligned}
|\mathbb{E}(\delta_{k+1})| &\le |a|\sqrt{2M(1+\mathbb{E}\|\psi\|^2)}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\mathrm{d}r\mathrm{d}t + 2a^2\sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}h^2\\
&\quad + |b|\sqrt{2M(1+\mathbb{E}\|\psi\|^2)}\int_{t_k}^{t_{k+1}}\int_{t_k-\tau}^{t-\tau}\mathrm{d}r\mathrm{d}t + |b|\sqrt{C_2}h^{\frac{3}{2}} + 2|ab|\sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}h^2\\
&= \frac{1}{2}|a|\sqrt{2M(1+\mathbb{E}\|\psi\|^2)}h^2 + 2a^2\sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}h^2 + \frac{1}{2}|b|\sqrt{2M(1+\mathbb{E}\|\psi\|^2)}h^2 + |b|\sqrt{C_2}h^{\frac{3}{2}}\\
&\quad + 2|ab|\sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}h^2.
\end{aligned}
$$

Let

$$
\bar{K} = \max\left\{\frac{1}{2}(|a|\vee|b|)\sqrt{2M(1+\mathbb{E}\|\psi\|^2)},\ |b|\sqrt{C_2},\ 2(a^2\vee|ab|)\sqrt{C_1(1+\mathbb{E}\|\psi\|^2)}\right\},
$$

where $a\vee b$ denotes the maximum of a and b. Then we have

$$
|\mathbb{E}(\delta_{k+1})| \le 5\bar{K}h^{\frac{3}{2}}.
$$

That is to say, the SSBE method is consistent with order $p_1 = \frac{3}{2}$ in the mean sense.

Then, we prove the consistent order $p_2 = 1$ in the mean-square sense. When $t_k \le t < t_{k+1}$, let

$$\xi(t) = a(y(t) - y(t_k)) + b(y(t-\tau) - y(t_{k+1-m})) - \frac{a^2h}{1-ah}y(t_k) - \frac{abh}{1-ah}y(t_{k+1-m})$$

and

$$\eta(t) = c(y(t) - y(t_k)) + d(y(t-\tau) - y(t_{k+1-m})) - \frac{ach}{1-ah}y(t_k) - \frac{bch}{1-ah}y(t_{k+1-m}).$$

It follows from (14) that

$$\delta_{k+1} = \int_{t_k}^{t_{k+1}} \xi(t)\mathrm{d}t + \int_{t_k}^{t_{k+1}} \eta(t)\mathrm{d}W(t). \tag{19}$$

Taking the square of both sides of (19) and then taking mathematical expectation, by the Bunyakovsky–Schwarz inequality, we have that

$$\begin{aligned}\mathbb{E}|\delta_{n+1}|^2 &= \mathbb{E}\left(\int_{t_k}^{t_{k+1}} \xi(t)\mathrm{d}t\right)^2 + \mathbb{E}\left(\int_{t_k}^{t_{k+1}} \eta(t)\mathrm{d}W(t)\right)^2 + 2\mathbb{E}\left(\int_{t_k}^{t_{k+1}} \xi(t)\mathrm{d}t\int_{t_k}^{t_{k+1}} \eta(t)\mathrm{d}W(t)\right)\\ &\le \mathbb{E}\left(\int_{t_k}^{t_{k+1}} \xi(t)\mathrm{d}t\right)^2 + \int_{t_k}^{t_{k+1}} \mathbb{E}|\eta(t)|^2\mathrm{d}t + 2\left[\mathbb{E}\left(\int_{t_k}^{t_{k+1}} \xi(t)\mathrm{d}t\right)^2\right]^{\frac{1}{2}}\left[\mathbb{E}\left(\int_{t_k}^{t_{k+1}} \eta(t)\mathrm{d}W(t)\right)^2\right]^{\frac{1}{2}}\\ &\le 2h\int_{t_k}^{t_{k+1}} \mathbb{E}|\xi(t)|^2\mathrm{d}t + 2\int_{t_k}^{t_{k+1}} \mathbb{E}|\eta(t)|^2\mathrm{d}t.\end{aligned}$$

Making use of the elementary inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, we obtain that

$$\begin{aligned}|\xi(t)|^2 \le\ & 4a^2|y(t)-y(t_k)|^2 + 8b^2|y(t-\tau)-y(t_{k-m})|^2 + 8b^2|y(t_{k+1-m})-y(t_{k-m})|^2\\ &+ 16a^4h^2|y(t_k)|^2 + 16a^2b^2h^2|y(t_{k+1-m})|^2\end{aligned}$$

and

$$\begin{aligned}|\eta(t)|^2 \le\ & 4c^2|y(t)-y(t_k)|^2 + 8d^2|y(t-\tau)-y(t_{k-m})|^2 + 8d^2|y(t_{k+1-m})-y(t_{k-m})|^2\\ &+ 16a^2c^2h^2|y(t_k)|^2 + 16b^2c^2h^2|y(t_{k+1-m})|^2.\end{aligned}$$

It then follows from the inequalities (4)–(6) that

$$\mathbb{E}|\xi(t)|^2 \le 4a^2C_2(t-t_k) + 8b^2C_2(t-t_k) + 8b^2C_2h + 16a^4h^2C_1(1+\mathbb{E}\|\psi\|^2) + 16a^2b^2h^2C_1(1+\mathbb{E}\|\psi\|^2)$$

and

$$\mathbb{E}|\eta(t)|^2 \le 4c^2C_2(t-t_k) + 8d^2C_2(t-t_k) + 8d^2C_2h + 16a^2c^2h^2C_1(1+\mathbb{E}\|\psi\|^2) + 16b^2c^2h^2C_1(1+\mathbb{E}\|\psi\|^2).$$

Setting $K = \max\{a^2, b^2, c^2, d^2\}$, we can estimate

$$\mathbb{E}|\xi(t)|^2 \le [20KC_2 + 32K^2C_1(1+\mathbb{E}\|\psi\|^2)]h$$

and

$$\mathbb{E}|\eta(t)|^2 \le [20KC_2 + 32K^2C_1(1+\mathbb{E}\|\psi\|^2)]h.$$

So, we have

$$\mathbb{E}|\delta_{n+1}|^2 \le 4[20KC_2 + 32K^2C_1(1+\mathbb{E}\|\psi\|^2)]h^2.$$

That is to say, the SSBE method is consistent with order $p_2 = 1$ in the mean-square sense. The proof is completed. □

The following theorem shows the strong order of convergence of SSBE method (7).

Theorem 3.2. *The numerical solution produced by the SSBE method* (7) *is convergent to the exact solution of Eq.* (1) *on the mesh-points in the mean-square sense with strong order* $\gamma = \frac{1}{2}$, *i.e., there exists a positive constant* C_0 *such that*

$$\max_{1\le k\le N} (\mathbb{E}(\varepsilon_k)^2)^{\frac{1}{2}} \le C_0h^{\frac{1}{2}}, \quad \text{as } h \to 0. \tag{20}$$

Proof. According to Eq. (1), SSBE method (7) and Theorem 3.1, we can easily see that all conditions of Theorem 5 in [4] are satisfied. Thus, this conclusion can be considered as a corollary of Theorem 5 in [4]. □

4. Mean-square stability of the SSBE method

We investigate the mean-square stability of the SSBE scheme in this section.

Definition 4.1 (*[13]*). Under condition (2), a numerical method is said to be *mean-square stable*, if there exists a $h_0(a, b, c, d) > 0$, such that any application of the method to Eq. (1) generates numerical approximations Y_n, which satisfy

$$\lim_{k\to\infty} \mathbb{E}|Y_k|^2 = 0$$

for all $h \in (0, h_0(a, b, c, d))$ with $h = \tau/m$.

Definition 4.2 (*[13]*). Under condition (2), a numerical method is said to be *general mean-square stable*, if any application of the method to Eq. (1) generates numerical approximations Y_n, which satisfy

$$\lim_{k\to\infty} \mathbb{E}|Y_k|^2 = 0$$

for any stepsize $h = \tau/m > 0$.

We now state the main theorem of this section.

Theorem 4.1. *Assume condition* (2) *is satisfied.*
(i) *If* $ad - bc = 0$ *and* $4|b|c^2 + b^2 - a^2 \le 0$, *then the SSBE method is GMS-stable.*
(ii) *If* $ad - bc = 0$ *and* $4|b|c^2 + b^2 - a^2 > 0$, *then the SSBE method is MS-stable and the stepsize satisfies*

$$h \in (0, h_1(a, b, c, d)),$$

where

$$h_1(a, b, c, d) = \frac{-\left[2a + 2|b| + (|c| + |d|)^2\right]}{4|b|c^2 + b^2 - a^2}.$$

(iii) *If* $ad - bc \neq 0$, *then the SSBE method is MS-stable and the stepsize satisfies*

$$h \in (0, h_2(a, b, c, d)),$$

where

$$h_2(a, b, c, d) = \frac{-[2|b|c^2 - 2a|cd| + b^2 - 2ad^2 + 2bcd - a^2] + \sqrt{\Delta}}{2(ad - bc)^2}.$$

Here,

$$\Delta = [2|b|c^2 - 2a|cd| + b^2 - 2ad^2 + 2bcd - a^2]^2 - 4(ad - bc)^2[2a + 2|b| + (|c| + |d|)^2].$$

Proof. In view of $a < 0$, we can see from ((7)-a) that

$$Y_k^* = \frac{1}{1 - ah}(Y_k + bhY_{k+1-m}).$$

Substituting this into ((7)-b) gives that

$$Y_{k+1} = \frac{1 + c\Delta W_k}{1 - ha}(Y_k + bhY_{k-m+1}) + dY_{k-m+1}\Delta W_k. \tag{21}$$

Noting $\mathbb{E}(\Delta W_k) = 0$, $\mathbb{E}[(\Delta W_k)^2] = h$ and Y_k, Y_{k+1-m} are $\mathcal{F}_{t_k}$-measurable, we have that

$$\mathbb{E}Y_{k+1}^2 = \frac{1 + c^2h}{(1 - ha)^2}(\mathbb{E}Y_k^2 + 2bh\mathbb{E}Y_kY_{k-m+1} + b^2h^2\mathbb{E}Y_{k-m+1}^2) + d^2h\mathbb{E}Y_{k-m+1}^2 + 2\frac{cdh}{1 - ah}(\mathbb{E}Y_kY_{k-m+1} + bh\mathbb{E}Y_{k-m+1}^2).$$

It follows from the inequality $2\beta\gamma xy \le |\beta\gamma|(x^2 + y^2)$ that

$$\mathbb{E}Y_{k+1}^2 \le P(a, b, c, d; h)\mathbb{E}Y_k^2 + Q(a, b, c, d; h)\mathbb{E}Y_{k-m+1}^2,$$

where

$$P(a, b, c, d; h) = \frac{1 + c^2h}{(1 - ha)^2}(1 + |b|h) + \frac{|cd|}{1 - ah}h;$$

$$Q(a, b, c, d; h) = \frac{1 + c^2h}{(1 - ha)^2}(|b|h + b^2h^2) + d^2h + \frac{|cd|}{1 - ah}h + \frac{2bcd}{1 - ah}h^2.$$

Hence

$$\mathbb{E}Y_{k+1}^2 \le [P(a,b,c,d;h) + Q(a,b,c,d;h)] \max\{\mathbb{E}Y_k^2, \mathbb{E}Y_{k-m+1}^2\}.$$

By recursive calculation we conclude that $\mathbb{E}Y_{k+1}^2 \to 0 (k \to \infty)$ if

$$P(a,b,c,d;h) + Q(a,b,c,d;h) < 1, \tag{22}$$

which is equivalent to

$$\frac{1+c^2h}{(1-ha)^2}(1+2|b|h+b^2h^2) + 2\frac{|cd|}{1-ah}h + d^2h + \frac{2bcd}{1-ah}h^2 < 1,$$

i.e.,

$$(ad-bc)^2h^2 + [2|b|c^2 - 2a|cd| + b^2 - 2ad^2 + 2bcd - a^2]h + [2a + 2|b| + (|c|+|d|)^2] < 0. \tag{23}$$

Now consider the following function on h

$$f(h) = (ad-bc)^2h^2 + [2|b|c^2 - 2a|cd| + b^2 - 2ad^2 + 2bcd - a^2]h + [2a + 2|b| + (|c|+|d|)^2].$$

In the case of $ad - bc = 0$, the function $f(h)$ reduces to

$$\begin{aligned} f(h) &= [2|b|c^2 - 2a|cd| + b^2 - 2ad^2 + 2bcd - a^2]h + [2a + 2|b| + (|c|+|d|)^2] \\ &= [4|b|c^2 + b^2 - a^2]h + [2a + 2|b| + (|c|+|d|)^2]. \end{aligned}$$

(i) When $4|b|c^2 + b^2 - a^2 \le 0$, considering (2), we can see that $f(h) < 0$ for any $h > 0$. Thus, (23) holds for any $h > 0$ with $h = \tau/m$, i.e., the SSBE method is GMS-stable.

(ii) When $4|b|c^2 + b^2 - a^2 > 0$, noting that (2), we can see that if $h < \frac{-[2a+2|b|+(|c|+|d|)^2]}{4|b|c^2+b^2-a^2}$, $f(h) < 0$ holds. Therefore, (23) holds for $h \in (0, h_1(a,b,c,d))$, i.e., the SSBE method is MS-stable.

In the case of $ad - bc \neq 0$, in view of (2), we know that

$$\Delta = [2|b|c^2 - 2a|cd| + b^2 - 2ad^2 + 2bcd - a^2]^2 - 4(ad-bc)^2[2a + 2|b| + (|c|+|d|)^2] > 0$$

always holds. We can easily obtain that $f(h) < 0$ holds when $h \in (0, h_2(a,b,c,d))$. Consequently, the SSBE method is MS-stable in this case. The proof is completed. □

Remark 4.1. When the stochastic differential delay system (1) reduces to the stochastic differential system

$$\begin{cases} dy(t) = ay(t)dt + cy(t)dW(t), & t \ge 0, \\ y(0) = y_0. \end{cases} \tag{24}$$

Theorem 4.1(i) illustrates that the SSBE method is GMS-stable if system (24) is asymptotically stable in the mean-square sense.

5. Numerical experiments

In this section we give several illustrative numerical examples of applying the SSBE method to linear SDDEs. Our objective is to illustrate intuitively the rate of strong convergence obtained in previous Section 3 and the mean-square stability obtained in previous Section 4. Furthermore, we compare the restrictions on stepsize of the MS-stable SSBE method with that of the Euler–Maruyama method (*cf.* [7]) or with that of the semi-implicit Euler method (*cf.* [13]).

We apply the SSBE method to the following linear stochastic delay differential system

$$\begin{cases} dy(t) = (ay(t) + by(t-1))dt + (cy(t) + dy(t-1))dW(t), & t \ge 0, \\ y(t) = t+1, & t \in [-1, 0]. \end{cases} \tag{25}$$

According to [11], when $t \in [0,1]$, the solution of (25) is given by

$$y(t) = \Phi_{t,0}\left(1 + \int_0^t \Phi_{s,0}^{-1}(b-cd)s\mathrm{d}s + \int_0^t ds\Phi_{s,0}^{-1}\mathrm{d}W(s)\right), \tag{26}$$

where

$$\Phi_{t,0} = \exp\left(\int_0^t \left(a - \frac{1}{2}c^2\right)\mathrm{d}s + \int_0^t c\mathrm{d}W(s)\right).$$

When $t \in [1,2]$, we obtain the explicit solution by using (26) as a new initial function. In the same way, step by step, we can obtain the explicit solution on the subsequent intervals. In our experiments, we use the SSBE scheme to compute an 'explicit solution' with stepsize $h = \frac{1}{1024}$.

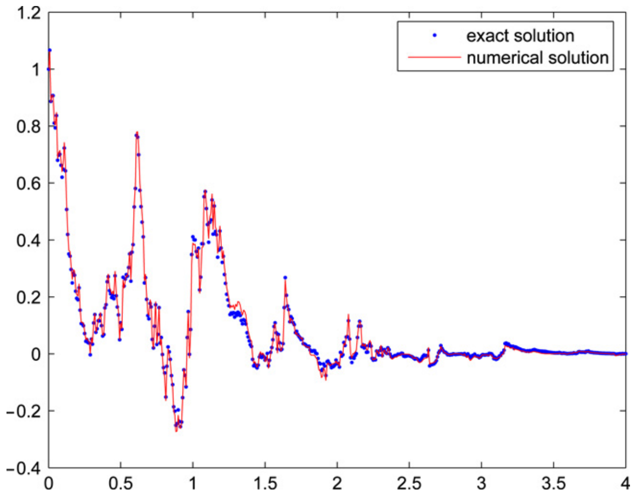

Fig. 1. The stable analytical solution and numerical solution of the SSBE method.

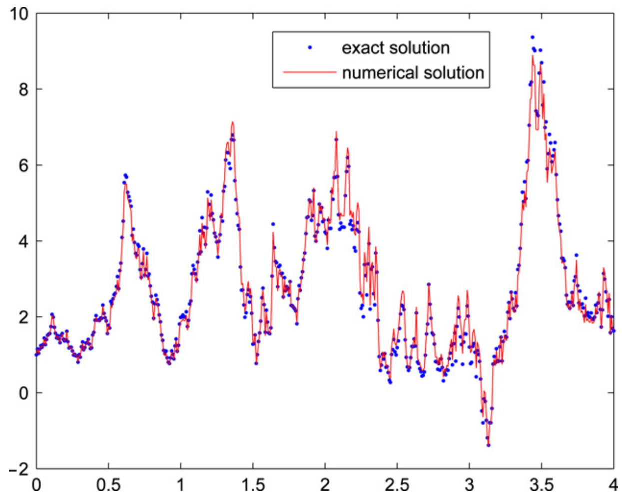

Fig. 2. The unstable analytical solution and numerical solution of the SSBE method.

Table 1
The convergence of the SSBE method applied to Eq. (25)

Stepsize	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$
ε	0.000074	0.000184	0.000409	0.001915	0.006980

First, we consider the theoretical order of strong convergence. In this case we choose the coefficients of system (25) as $a = -5$, $b = 1$, $c = d = 0.5$. The mean-square error $\mathbb{E}|y(T) - Y_N|^2$ at the final time $T = 2$ is estimated in the following way. A set of 20 blocks each containing 100 outcomes (ω_{ij}: $1 \le i \le 20$, $1 \le j \le 100$) are simulated and for each block the estimator

$$\varepsilon_i = \frac{1}{100}\sum_{j=1}^{100} |y(T, \omega_{ij}) - Y_N(\omega_{ij})|^2$$

is formed. In Table 1, ε denotes the mean of this estimator, which is itself estimated in the usual way:

$$\frac{1}{20}\sum_{i=1}^{20} \varepsilon_i .$$

We draw the numerical solution obtained from the SSBE method with stepsize $h = \frac{1}{128}$ together with the explicit solution of the test equation $a = -10$, $b = 2$, $c = 1$, $d = 1$ in Fig. 1 and $a = 0.1$, $b = 1$, $c = 1$, $d = 1$ in Fig. 2. We note the difference

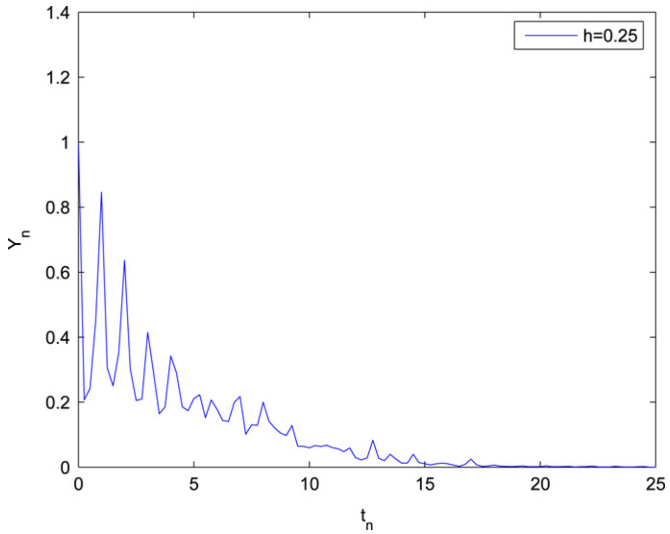

Fig. 3. Simulations with fixed stepsize $h = 0.25$ for **Case 1** of (25).

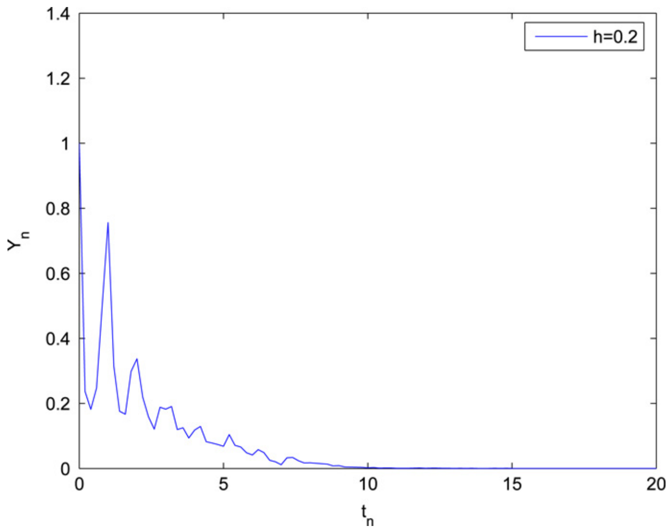

Fig. 4. Simulations with fixed stepsize $h = 0.2$ for **Case 2** of (25).

of stability between Figs. 1 and 2, which is caused by different values of the coefficients of the test equation. Owing to the convergence of SSBE method, the figures illustrate that the numerical solution has the same stability property as its analytical solution.

Next, we show the influence of stepsize h on mean-square stability of the SSBE method and compare our results with that obtained in [7,13]. The data used in all figures are obtained by the mean-square of data by 100 trajectories, that is, ω_i: $1 \le i \le 100$, $Y_n = \frac{1}{100}\sum_{i=1}^{100} |Y_n(\omega_i)|^2$. In all figures t_n denotes the mesh-point.

Case 1. We choose the coefficients of the test equation (25) as $a = -9$, $b = 7$, $c = 0$ and $d = 1$. This test equation was investigated in [7]. By Theorem 4.1 we know that the SSBE method is MS-stable when the stepsize $h \in (0, 0.2974)$. Fig. 3 illustrates the MS-stability of numerical solution obtained by the SSBE scheme when $h = \frac{1}{4}$. However, applied to the same test equation, the Euler–Maruyama method is unstable when the stepsize $h = \frac{1}{4}$ (*cf.* [7]).

Case 2. We choose the coefficients of the test Eq. (25) as $a = -10$, $b = 7$, $c = 1$ and $d = 0.5$. By Theorem 4.1, the SSBE method is MS-stable when the stepsize $h \in (0, 0.2217)$. Fig. 4 illustrates the MS-stability of numerical solution obtained by the SSBE scheme when $h = \frac{1}{5}$. But we can see from [13] that the semi-implicit Euler method is unstable when the implicit parameter $\alpha = 0.1$ and the stepsize $h = \frac{1}{5}$.

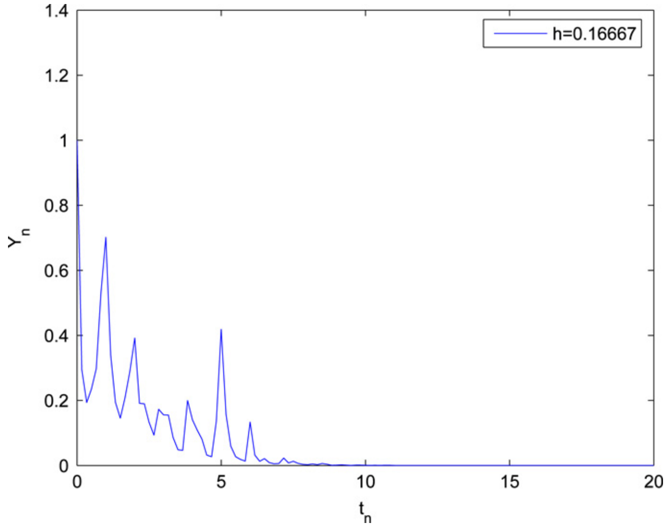

Fig. 5. Simulations with fixed stepsize $h = 1/6$ for **Case 3** of (25).

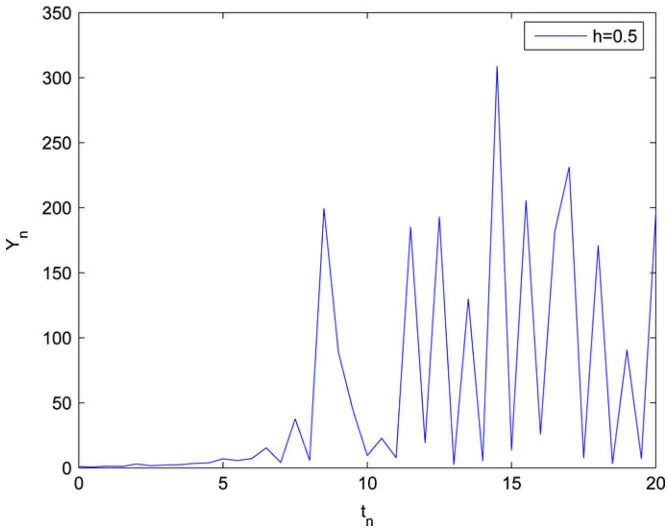

Fig. 6. Simulations with fixed stepsize $h = 0.5$ for **Case 3** of (25).

Remark 5.1. Cases 1 and 2 indicate that the restriction on stepsize h of the MS-stable SSBE method is less than that of the Euler–Maruyama method and that of the semi-implicit Euler method, respectively. On the other hand, we consider the test Eq. (25) with parameters $a = -8.5$, $b = 7$, $c = 1$ and $d = 0.5$. Using Lemma 2.1, we easily see that the exact solution is asymptotically stable in the mean-square sense. Theorem 5 in [13] shows that for any $\alpha \in [0, 0.9089]$, the semi-implicit Euler method is MS-stable if $h \in (0, 0.1176)$. By Theorem 4.1, however, the SSBE method is MS-stable if $h \in (0, 0.04596)$. In this case, the restriction on stepsize h of the MS-stable semi-implicit Euler method is less than that of the SSBE method.

Case 3. To discuss whether the SSBE scheme is stable or not when the stepsizes are not in the stable range, we choose the coefficients of the test Eq. (25) as $a = -8$, $b = 4$, $c = 1$ and $d = 1$. By Theorem 4.1, the SSBE method is MS-stable when the stepsize $h \in (0, \frac{1}{6})$. Figs. 5 and 6 illustrate the numerical simulation by the SSBE scheme when $h = \frac{1}{6}$ and $h = \frac{1}{2}$ respectively. Fig. 5 denotes the numerical solution of the SSBE method with the critical stepsize $h = \frac{1}{6}$ still kept stable. This implies that the mean-square stability bound that we obtained is maybe not optimal. From Fig. 6, we can see apparently that when $h = \frac{1}{2}$, the SSBE method cannot preserve the mean-square stability of the test equation. Hence, in order to let the SSBE scheme share the asymptotical stability in the mean-square sense of the linear stochastic delay differential system, we require the restrictions on stepsize.

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