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Eigenvalue inequalities for differences of means of Hilbert space operators

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ABSTRACT

We prove several eigenvalue inequalities for the differences of various means of two positive invertible operators A and B on a separable Hilbert space, under the assumption that $A - B$ is compact. Equality conditions of these inequalities are also obtained.

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1. Introduction

Let $\mathfrak{B}(\mathcal{H})$ denote the space of all bounded linear operators on a separable Hilbert space $\mathcal{H}$. For a compact positive operator $A \in \mathfrak{B}(\mathcal{H})$, let $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant 0$ denote the eigenvalues of A arranged in decreasing order and repeated according to multiplicity.

A useful characterization of compact operators (see, e.g., [6] or [13, p. 59]) says that

$$A \in \mathfrak{B}(\mathcal{H}) \text{ is compact if and only if } \langle Ae_n, e_n\rangle \to 0 \text{ as } n \to \infty \tag{1}$$

for every orthonormal set $\{e_n\}$ in $\mathcal{H}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product defined on $\mathcal{H}$. The characterization (1) implies the following fact:

$$\text{If } A, B \in \mathfrak{B}(\mathcal{H}) \text{ are positive such that } A \text{ is compact and } A \geqslant B, \text{ then } B \text{ is compact.} \tag{2}$$

The Weyl's monotonicity principle for compact positive operators (see, e.g., [2, p. 63] or [4, p. 26]) says that if $A, B \in \mathfrak{B}(\mathcal{H})$ are compact positive operators such that $A \geqslant B$, then $\lambda_j(A) \geqslant \lambda_j(B)$ for $j = 1, 2, \ldots$.

For $0 < \mu < 1$, the μ-weighted arithmetic mean of two positive operators $A, B \in \mathfrak{B}(\mathcal{H})$, denoted by $A\nabla_\mu B$, is the operator defined by

$$A\nabla_\mu B = (1-\mu)A + \mu B.$$

In addition, if A is invertible and $\mu > 0$, then an operator (defined earlier in [10]) called the μ-weighted geometric mean of A and B, denoted by $A\sharp_\mu B$, is defined by

$$A\sharp_\mu B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\mu A^{1/2}.$$

In particular, if $\mu = \frac{1}{2}$, the operators $A\nabla_{1/2}B$ and $A\sharp_{1/2}B$ are called the arithmetic mean and the geometric mean of A and B, respectively. One of the interesting properties of the μ-weighted geometric mean (see, e.g., [1, p. 35]) is that if $A, B \in \mathfrak{B}(\mathcal{H})$ are invertible positive operators, then

$$A\sharp_\mu B = B\sharp_{1-\mu}A \tag{3}$$

for $0 < \mu < 1$. Moreover, when A and B commute, we have $A\sharp_\mu B = A^{1-\mu}B^\mu$ for $\mu > 0$. It can be shown, as in the finite-dimensional case given in [9,11], that if $A, B \in \mathfrak{B}(\mathcal{H})$ are such that A is invertible and B is positive, then, for $0 < \mu < 1$,

$$A^*(A^{*-1}BA^{-1})^\mu A \leqslant (A^*A)\,\nabla_\mu B \tag{4}$$

with equality if and only if $A^*A = B$. In particular, if A is positive and invertible, then in the inequality (4), replacing A by $A^{1/2}$, we have

$$A\sharp_\mu B \leqslant A\nabla_\mu B. \tag{5}$$

with equality if and only if $A = B$.

For $0 < \mu < 1$ and for positive and invertible operators $A, B \in \mathfrak{B}(\mathcal{H})$, the arithmetic mean of the operators $A\sharp_\mu B$ and $A\sharp_{1-\mu}B$, denoted by $H_\mu(A, B)$, is called the μ-weighted Heinz mean of A and B, that is

$$H_\mu(A, B) = \frac{A\sharp_\mu B + A\sharp_{1-\mu}B}{2}.$$

It can be seen that $H_\mu(A, B) = H_{1-\mu}(A, B) = H_\mu(B, A)$. Moreover, the inequality (5) implies that

$$H_\mu(A, B) \leqslant \frac{A+B}{2} \tag{6}$$

with equality if and only if $A = B$.

It can be seen that if $A, B \in \mathfrak{B}(\mathcal{H})$ are positive invertible operators such that $A - B$ is compact, then the operator $A^{-1/2}(A-B)^2A^{-1/2}$ is compact. This follows from the fact that the space of compact operators is a two sided ideal in $\mathfrak{B}(\mathcal{H})$. Moreover, it follows from the spectral theorem applied in the

Calkin Algebra setting that the operators $A\nabla_\mu B - A\sharp_\mu B$ and $\frac{A+B}{2} - H_\mu(A, B)$ are also compact, for $\mu > 0$. To see this, let $\mathcal{K}(\mathcal{H})$ denote the closed two sided ideal of compact operators in $\mathfrak{B}(\mathcal{H})$, and let $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the Calkin algebra and $\pi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the canonical homomorphism of $\mathfrak{B}(\mathcal{H})$ onto $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. If $A, B \in \mathfrak{B}(\mathcal{H})$ are such that $A - B$ is compact, then $\pi(A) = \pi(B)$. Since the Calkin algebra is a C^*-algebra, and hence it can be represented as an operator algebra, it follows from the spectral theorem in this setting, that if $A, B \in \mathfrak{B}(\mathcal{H})$ are positive invertible operators such that $A-B$ is compact, then $\pi(A)\sharp_\mu\pi(B) = \pi(A)\nabla_\mu\pi(B)$, and so $\pi(A\sharp_\mu B) = \pi(A\nabla_\mu B)$. Hence, $A\nabla_\mu B - A\sharp_\mu B$ is compact. Using a similar argument, it can be shown that $\frac{A+B}{2} - H_\mu(A, B)$ is also compact.

Recently, in the finite dimensional Hilbert space setting, eigenvalue inequalities for the difference of the arithmetic mean and the geometric mean of two positive definite $n \times n$ matrices have been established. It has been shown in [5] that if A and B are $n \times n$ positive definite matrices such that $A \geqslant B > 0$, then

$$\lambda_j\left(\frac{A+B}{2} - A\sharp_{1/2}B\right) \geqslant \frac{1}{8}\lambda_j\left(A^{-1/2}(A-B)^2A^{-1/2}\right) \tag{7}$$

and

$$\lambda_j\left(\frac{A+B}{2} - A\sharp_{1/2}B\right) \leqslant \frac{1}{8}\lambda_j\left(B^{-1/2}(A-B)^2B^{-1/2}\right) \tag{8}$$

for $j = 1, 2, \ldots, n$. Moreover, recent operator inequalities for differences of means of Hilbert space operators have been given in [7,8].

In this paper we are interested in eigenvalue inequalities for differences of means of positive invertible operators in $\mathfrak{B}(\mathcal{H})$. In Section 2, we give eigenvalue inequalities for the difference of the μ-weighted arithmetic mean and the μ-weighted geometric mean of two positive invertible operators. Some of our results in Section 2 present natural generalizations of the inequalities (7) and (8). In Section 3, we present eigenvalue inequalities for the difference of the arithmetic mean and the μ-weighted Heinz mean of two positive invertible operators. In Section 4, we investigate the equality conditions of our inequalities given in Sections 2 and 3.

2. Eigenvalue inequalities for the difference of the weighted arithmetic mean and the weighted geometric mean

In this section we present upper and lower bounds for the eigenvalues of the operator $A\nabla_\mu B - A\sharp_\mu B$, where $A, B \in \mathfrak{B}(\mathcal{H})$ are positive invertible operators such that $A - B$ is compact.

For $x, y \in \mathbb{R}$, $y \neq -1$, let

$$K(x, y) = \frac{x(1-x)}{2(1+y)^2}.$$

It can be easily seen that

$$K(x, y) = K(1-x, y). \tag{9}$$

In our analysis, we need the following scalar inequalities. They are known as Bernoulli's inequalities (see, e.g., [12, p. 76]).

Lemma 1.

(a) *Let $x \in [-1, \beta]$, where $\beta \geqslant 0$. If $0 < \mu < 1$, then*

$$1 + \mu x - (1+x)^\mu \geqslant K(\mu, \beta)x^2 \tag{10}$$

with equality if and only if $x = 0$.

(b) *Let* $x \in [\gamma, \infty)$, *where* $-1 < \gamma \leqslant 0$. *If* $0 < \mu < 1$, *then*

$$1 + \mu x - (1 + x)^{\mu} \leqslant K(\mu, \gamma)x^2 \tag{11}$$

with equality if and only if $x = 0$.

Another result that we also need is the following. It involves a monotonicity property for operator functions (see, e.g., [3]).

Lemma 2. *Let* $X \in \mathfrak{B}(\mathcal{H})$ *be self-adjoint and let* f *and* g *be continuous functions such that* $f(t) \geqslant g(t)$ *for all* $t \in Sp(X)$ *(the spectrum of* X*). Then* $f(X) \geqslant g(X)$ *with equality if and only if* $f(t) = g(t)$ *for all* $t \in Sp(X)$.

Now, we present our first main result in this section.

Theorem 1. *Let* $A, B \in \mathfrak{B}(\mathcal{H})$ *be positive such that* $A \geqslant B > 0$ *and* $A - B$ *is compact. If* $0 < \mu < 1$, $\beta \geqslant 0$, *and* $-1 < \gamma \leqslant 0$, *then*

$$\lambda_j \left(A\nabla_{\mu}B - A\sharp_{\mu}B\right) \geqslant K(\mu, \beta)\lambda_j \left(A^{-1/2} (A - B)^2 A^{-1/2}\right) \tag{12}$$

for $j = 1, 2, \ldots$ *and*

$$\lambda_j \left(A\nabla_{\mu}B - A\sharp_{\mu}B\right) \leqslant K(\mu, \gamma)\lambda_j \left(B^{-1/2} (A - B)^2 B^{-1/2}\right) \tag{13}$$

for $j = 1, 2, \ldots$.

Proof. We prove the inequality (12). The proof of the inequality (13) is similar. Let $X = A^{-1/2}BA^{-1/2}$. Then $I \geqslant X > 0$ and so $Sp(X) \subseteq (0, 1]$. Applying Lemma 1, for $x = t - 1$, $t \in Sp(X)$, we have

$$\begin{aligned} 1 - \mu + \mu t - t^{\mu} &= 1 + \mu(t - 1) - t^{\mu} \\ &\geqslant K(\mu, \beta)\,(t - 1)^2 . \end{aligned} \tag{14}$$

It follows from the inequality (14) and Lemma 2 that

$$\begin{aligned} (1 - \mu)\, I + \mu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\mu} &= (1 - \mu)I + \mu X - X^{\mu} \\ &\geqslant K(\mu, \beta)\,(X - I)^2 \\ &= K(\mu, \beta)\left(A^{-1/2}BA^{-1/2} - I\right)^2 \end{aligned}$$

and so

$$\begin{aligned} A\nabla_{\mu}B - A\sharp_{\mu}B &= (1 - \mu)\, A + \mu B - A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\mu} A^{1/2} \\ &\geqslant K(\mu, \beta)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}. \end{aligned} \tag{15}$$

Since $A - B$ is compact, the operator $A\nabla_{\mu}B - A\sharp_{\mu}B$ is also compact, and since the operator $K(\mu, \beta)A^{1/2} \left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}$ is positive, it follows from the inequality (15), together with the fact (2), that the operator

$$K(\mu, \beta)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}$$

is compact. The Weyl's monotonicity principle for compact positive operators, together with the inequality (15), implies that

$$\lambda_j\left(A\nabla_\mu B - A\sharp_\mu B\right) \geqslant K(\mu,\beta)\lambda_j\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2A^{1/2}\right) \tag{16}$$

for $j = 1, 2, \ldots$. For $Y = \left(A^{-1/2}BA^{-1/2}-I\right)A^{1/2}$ it can be seen that

$$\begin{aligned}
&\lambda_j\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2A^{1/2}\right)\\
&= \lambda_j\left(Y^*Y\right)\\
&= \lambda_j\left(YY^*\right)\\
&= \lambda_j\left(\left(A^{-1/2}BA^{-1/2}-I\right)A\left(A^{-1/2}BA^{-1/2}-I\right)\right)\\
&= \lambda_j\left(A^{-1/2}\left(A-B\right)^2A^{-1/2}\right)
\end{aligned} \tag{17}$$

for $j = 1, 2, \ldots$. Now, the inequality (12) follows from the inequality (16) and the identity (17). □

A particular case of Theorem 1, for $\beta = \gamma = 0$, can be stated as follows. This result presents our promised natural generalization of the inequalities (7) and (8). In the finite-dimensional case, the compactness of $A - B$ can be deleted, because every operator in this case is compact.

Corollary 1. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive such that $A \geqslant B > 0$ and $A - B$ is compact. If $0 < \mu < 1$, then*

$$\lambda_j\left(A\nabla_\mu B - A\sharp_\mu B\right) \geqslant \frac{\mu\left(1-\mu\right)}{2}\lambda_j\left(A^{-1/2}\left(A-B\right)^2A^{-1/2}\right)$$

and

$$\lambda_j\left(A\nabla_\mu B - A\sharp_\mu B\right) \leqslant \frac{\mu\left(1-\mu\right)}{2}\lambda_j\left(B^{-1/2}\left(A-B\right)^2B^{-1/2}\right)$$

for $j = 1, 2, \ldots$. In particular, for $\mu = \frac{1}{2}$, we have

$$\lambda_j\left(A\nabla_{1/2} B - A\sharp_{1/2} B\right) \geqslant \frac{1}{8}\lambda_j\left(A^{-1/2}\left(A-B\right)^2A^{-1/2}\right)$$

and

$$\lambda_j\left(A\nabla_{1/2} B - A\sharp_{1/2} B\right) \leqslant \frac{1}{8}\lambda_j\left(B^{-1/2}\left(A-B\right)^2B^{-1/2}\right)$$

for $j = 1, 2, \ldots$.

An application of Lemma 1 can be stated as follows.

Lemma 3. *Let $0 < \mu < 1$.*

(a) *Let $x \in [-1, \beta]$, where $\beta \geqslant 0$. If $0 < \mu < 1$, then*

$$\left(1+\mu x\right)^2 - \left(1+x\right)^{2\mu} \geqslant \left(K(2\mu,\beta)+\mu^2\right)x^2 \tag{18}$$

with equality if and only if $x = 0$.

(b) *Let $x \in [\gamma, \infty)$, where $-1 < \gamma \leqslant 0$. If $0 < \mu < 1$, then*

$$(1+\mu x)^2 - (1+x)^{2\mu} \leqslant \left(K(2\mu,\gamma)+\mu^2\right) x^2 \tag{19}$$

with equality if and only if $x = 0$.

Proof. Since

$$(1+\mu x)^2 - (1+x)^{2\mu} - \mu^2 x^2 = 1 + 2\mu x - (1+x)^{2\mu},$$

the results follow from this relation and Lemma 1. □

Based on Lemma 3, we have the following result.

Theorem 2. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive such that $A \geqslant B > 0$ and $A - B$ is compact. If $0 < \mu < 1$, $\beta \geqslant 0$, and $-1 < \gamma \leqslant 0$, then*

$$\lambda_j\left((A\nabla_\mu B)\,A^{-1}\,(A\nabla_\mu B) - A\sharp_{2\mu}B\right) \geqslant \left(K(2\mu,\beta)+\mu^2\right)\lambda_j\left(A^{-1/2}\,(A-B)^2\,A^{-1/2}\right) \tag{20}$$

and

$$\lambda_j\left((A\nabla_\mu B)\,A^{-1}\,(A\nabla_\mu B) - A\sharp_{2\mu}B\right) \leqslant \left(K(2\mu,\gamma)+\mu^2\right)\lambda_j\left(B^{-1/2}\,(A-B)^2\,B^{-1/2}\right) \tag{21}$$

for $j = 1, 2, \ldots$.

Proof. We prove the inequality (20). The proof of the inequality (21) is similar. Let $X = A^{-1/2}BA^{-1/2}$. Then $I \geqslant X > 0$ and so $Sp(X) \subseteq (0, 1]$. Applying Lemma 1, for $x = t-1, t \in Sp(X)$, we have

$$(1-\mu+\mu t)^2 - t^{2\mu} \geqslant \left(K(2\mu,\beta)+\mu^2\right)(t-1)^2\,. \tag{22}$$

It follows from the inequality (22) and Lemma 3 that

$$\begin{aligned}
&\left((1-\mu)\,I + \mu A^{-1/2}BA^{-1/2}\right)^2 - \left(A^{-1/2}BA^{-1/2}\right)^{2\mu}\\
&= ((1-\mu)\,I + \mu X)^2 - X^{2\mu}\\
&\geqslant \left(K(2\mu,\beta)+\mu^2\right)(X-I)^2\\
&= \left(K(2\mu,\beta)+\mu^2\right)\left(A^{-1/2}BA^{-1/2}-I\right)^2
\end{aligned}$$

and so

$$\begin{aligned}
&(A\nabla_\mu B)\,A^{-1}\,(A\nabla_\mu B) - A\sharp_{2\mu}B\\
&= A^{1/2}\left[\left((1-\mu)I + \mu A^{-1/2}BA^{-1/2}\right)^2 - \left(A^{-1/2}BA^{-1/2}\right)^{2\mu}\right]A^{1/2}\\
&\geqslant \left(K(2\mu,\beta)+\mu^2\right)A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}.
\end{aligned} \tag{23}$$

Since $A - B$ is compact, the operator $(A\nabla_\mu B)\,A^{-1}\,(A\nabla_\mu B) - A\sharp_{2\mu}B$ is also compact, and since the operator $\left(K(2\mu,\beta)+\mu^2\right)A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}$ is positive, it follows from the inequality (23), together with the fact (2), that the operator $\left(K(2\mu,\beta)+\mu^2\right)A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}$ is compact. The Weyl's monotonicity principle for compact positive operators, together with the inequality (23), implies that

$$\begin{aligned}&\lambda_j\left(\left(A\nabla_\mu B\right)A^{-1}\left(A\nabla_\mu B\right)-A\sharp_{2\mu}B\right)\\&\quad\geqslant\left(K(2\mu,\beta)+\mu^2\right)\lambda_j\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2A^{1/2}\right)\end{aligned}\tag{24}$$

for $j=1,2,\ldots$. Now the result follows from the inequality (24) and the identity (17). □

Further results can be obtained using the following scalar inequalities.

Lemma 4.

(a) *If* $\frac{1}{2}\leqslant\mu<1, x\geqslant 1$ *or* $0<\mu\leqslant\frac{1}{2}, 0<x\leqslant 1$, *then*

$$1+\mu(x^2-1)-x^{2\mu}\geqslant 2\mu\,(1-\mu)\,(x-1)^2\tag{25}$$

with equality if and only if $x=1$.

(b) If $\frac{1}{2}\leqslant\mu<1, 0<x\leqslant 1$ *or* $0<\mu\leqslant\frac{1}{2}, x\geqslant 1$, *then*

$$1+\mu(x^2-1)-x^{2\mu}\leqslant 2\mu\,(1-\mu)\,(x-1)^2\tag{26}$$

with equality if and only if $x=1$.

Proof. We prove the inequality (25). The proof of the inequality (26) is similar. Consider the function $g:(0,\infty)\to\mathbb{R}$ defined by

$$g(x)=1-\mu+\mu x^2-x^{2\mu}-2\mu(1-\mu)(x-1)^2.$$

Then

$$\begin{aligned}g'(x)&=2\mu x-2\mu x^{2\mu-1}-4\mu(1-\mu)(x-1)\\&=2\mu\,(2\mu-1))\,x-2\mu x^{2\mu-1}+4\mu(1-\mu)\end{aligned}$$

and

$$g''(x)=2\mu\,(2\mu-1))\left(1-x^{2\mu-2}\right).$$

If $\frac{1}{2}\leqslant\mu<1$ and $x\geqslant 1$, then $g''(x)\geqslant 0$. Hence, g' is increasing on $[1,\infty)$, which implies that $g'(x)\geqslant g'(1)$ for $x\geqslant 1$. Since $g'(x)\geqslant 0$ for $x\geqslant 1$, we conclude that g is increasing on the interval $[1,\infty)$. This implies the inequality $g(x)\geqslant g(1)=0$, which is valid for $x\geqslant 1$.

Similarly, if $0<\mu\leqslant\frac{1}{2}$ and $0<x\leqslant 1$, then $g''(x)\geqslant 0$. Hence, g' is increasing on $(0,1]$, which implies that $g'(x)\geqslant g'(1)$ for $0<x\leqslant 1$. Since $g'(x)\geqslant 0$ for $0<x\leqslant 1$, we conclude that g is increasing on the interval $(0,1]$. This implies the inequality $g(x)\geqslant g(1)=0$, which is valid for $0<x\leqslant 1$. □

Based on Lemma 4, we have the following related result.

Theorem 3. *Let* $A,B\in B(H)$ *be positive such that* $A\geqslant B>0$ *and* $A-B$ *is compact.*

(a) *If* $0<\mu\leqslant\frac{1}{2}$, *then*

$$\lambda_j\left(A\nabla_\mu\left(BA^{-1}B\right)-A\sharp_{2\mu}B\right)\geqslant 2\mu\,(1-\mu)\,\lambda_j\left(A^{-1/2}\,(A-B)^2\,A^{-1/2}\right)\tag{27}$$

for $j = 1, 2, \ldots,$ *and if* $\frac{1}{2} \leqslant \mu < 1$, *then*

$$\lambda_j \left(A \nabla_\mu \left(BA^{-1}B \right) - A \sharp_{2\mu} B \right) \leqslant 2\mu \left(1-\mu\right) \lambda_j \left(A^{-1/2} \left(A-B\right)^2 A^{-1/2} \right) \tag{28}$$

for $j = 1, 2, \ldots.$

(b) *If* $0 < \mu \leqslant \frac{1}{2}$, *then*

$$\lambda_j \left(B \nabla_\mu \left(AB^{-1}A \right) - B \sharp_{2\mu} A \right) \leqslant 2\mu \left(1-\mu\right) \lambda_j \left(B^{-1/2} \left(A-B\right)^2 B^{-1/2} \right) \tag{29}$$

for $j = 1, 2, \ldots,$ *and if* $\frac{1}{2} \leqslant \mu < 1$, *then*

$$\lambda_j \left(B \nabla_\mu \left(AB^{-1}A \right) - B \sharp_{2\mu} A \right) \geqslant 2\mu \left(1-\mu\right) \lambda_j \left(B^{-1/2} \left(A-B\right)^2 B^{-1/2} \right) \tag{30}$$

for $j = 1, 2, \ldots.$

Proof. We prove the inequality (27). The proof of the inequalities (28)-(30) is similar. Since $A \geqslant B > 0$, we have $Sp(A^{-1/2}BA^{-1/2}) \subseteq (0, 1]$. So for $t \in Sp(A^{-1/2}BA^{-1/2})$, we have

$$\begin{aligned} 1 - \mu + \mu t^2 - t^{2\mu} &= 1 + \mu(t^2 - 1) - t^{2\mu} \\ &\geqslant 2\mu \left(1-\mu\right) \left(t-1\right)^2 \quad \text{(by the inequality (25))} \end{aligned}$$

Consequently,

$$(1-\mu)\, I + \mu (A^{-1/2}BA^{-1/2})^2 - (A^{-1/2}BA^{-1/2})^{2\mu} \geqslant 2\mu \left(1-\mu\right) \left(A^{-1/2}BA^{-1/2} - I \right)^2$$

and so

$$\begin{aligned} & A \nabla_\mu \left(BA^{-1}B \right) - A \sharp_{2\mu} B \\ &= (I - \mu)\, A + \mu A^{1/2} (A^{-1/2}BA^{-1/2})^2 A^{1/2} - A^{1/2} (A^{-1/2}BA^{-1/2})^{2\mu} A^{1/2} \\ &\geqslant 2\mu \left(1-\mu\right) A^{1/2} \left(A^{-1/2}BA^{-1/2} - I \right)^2 A^{1/2}. \end{aligned} \tag{31}$$

Since $A - B$ is compact, the operator $A \nabla_\mu \left(BA^{-1}B \right) - A \sharp_{2\mu} B$ is also compact, and since the operator $2\mu \left(1-\mu\right) A^{1/2} \left(A^{-1/2}BA^{-1/2} - I \right)^2 A^{1/2}$ is positive, it follows from the inequality (31), together with the fact (2), that the operator $2\mu \left(1-\mu\right) A^{1/2} \left(A^{-1/2}BA^{-1/2} - I \right)^2 A^{1/2}$ is compact. The Weyl's monotonicity principle for compact positive operators together with the inequality (31) implies that

$$\begin{aligned} & \lambda_j \left(A \nabla_\mu \left(BA^{-1}B \right) - A \sharp_{2\mu} B \right) \\ &\geqslant 2\mu \left(1-\mu\right) \lambda_j \left(A^{1/2} \left(A^{-1/2}BA^{-1/2} - I \right)^2 A^{1/2} \right) \\ &= 2\mu \left(1-\mu\right) \lambda_j \left(A^{-1/2} \left(A-B\right)^2 A^{-1/2} \right) \end{aligned}$$

for $j = 1, 2, \ldots.$ This proves the inequality (27). □

3. Eigenvalue inequalities for the difference of the arithmetic mean and the weighted geometric Heinz mean

In this section we employ some of our results given in Section 2 to obtain upper and lower bounds for the eigenvalues of the operator $\frac{A+B}{2} - H_\mu(A, B)$, where $A, B \in \mathfrak{B}(\mathcal{H})$ are positive invertible operators such that $A - B$ is compact.

Theorem 4. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive such that $A \geqslant B > 0$ and $A - B$ is compact. If $0 < \mu < 1$, $\beta \geqslant 0$, and $-1 < \gamma \leqslant 0$, then*

$$\lambda_j\left(\frac{A+B}{2} - H_\mu(A, B)\right) \geqslant K(\mu, \beta)\lambda_j\left(A^{-1/2}(A-B)^2 A^{-1/2}\right) \tag{32}$$

for $j = 1, 2, \ldots$ and

$$\lambda_j\left(\frac{A+B}{2} - H_\mu(A, B)\right) \leqslant K(\mu, \gamma)\lambda_j\left(B^{-1/2}(A-B)^2 B^{-1/2}\right) \tag{33}$$

for $j = 1, 2, \ldots$.

Proof. We prove the inequality (32). The proof of the inequality (33) is similar. In the inequality (15), replacing μ by $1 - \mu$, we have

$$\begin{aligned} A\nabla_{1-\mu}B - A\sharp_{1-\mu}B &\geqslant K(1-\mu, \beta)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2} \\ &= K(\mu, \beta)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}. \end{aligned} \tag{34}$$

Combining the inequalities (15) and (34) we have

$$\begin{aligned} \frac{A+B}{2} - H_\mu(A, B) &= \frac{A + B - (A\sharp_\mu B + A\sharp_{1-\mu}B)}{2} \\ &= \frac{(A\nabla_\mu B - A\sharp_\mu B) + (A\nabla_{1-\mu}B - A\sharp_{1-\mu}B)}{2} \\ &\geqslant K(\mu, \beta)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}. \end{aligned} \tag{35}$$

Since $A-B$ is compact, the operator $\frac{A+B}{2} - H_\mu(A, B)$ is also compact, and since the operator $K(\mu, \beta)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}$ is positive, it follows from the inequality (35), together with the fact (2), that the operator

$$2\mu(1-\mu)A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}$$

is compact. The Weyl's monotonicity principle for compact positive operators, together with the inequality (35), implies that

$$\lambda_j\left(\frac{A+B}{2} - H_\mu(A, B)\right) \geqslant K(\mu, \beta)\lambda_j\left(A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}\right) \tag{36}$$

for $j = 1, 2, \ldots$. Now the result follows from the inequality (36) and the identity (17). □

4. Equality conditions

In this section we study the equality conditions of our inequalities presented in Sections 2 and 3. Our analysis here is mainly based on the following lemma [4, p. 26].

Lemma 5. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be compact positive operators such that $A \geqslant B$. Then $A = B$ if and only if $\lambda_j(A) = \lambda_j(B)$ for $j = 1, 2, \ldots$.*

Based on Lemma 5 and the equality conditions of the inequalities (10) and (11), we get our first main result in this section.

Theorem 5. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive such that $A \geqslant B > 0$ and $A - B$ is compact. If $0 < \mu < 1$, $\beta \geqslant 0$, and $-1 < \gamma \leqslant 0$, then*

(a)

$$\lambda_j \left(A\nabla_\mu B - A\sharp_\mu B\right) = K(\mu, \beta)\lambda_j \left(A^{-1/2} (A-B)^2 A^{-1/2}\right) \tag{37}$$

for $j = 1, 2, \ldots$ if and only if $A = B$.

(b)

$$\lambda_j \left(A\nabla_\mu B - A\sharp_\mu B\right) = K(\mu, \gamma)\lambda_j \left(B^{-1/2} (A-B)^2 B^{-1/2}\right) \tag{38}$$

for $j = 1, 2, \ldots$ if and only if $A = B$.

Proof. We prove part (a). The proof of part (b) is similar. Suppose that

$$K(\mu, \beta)\lambda_j \left(A^{-1/2} (A-B)^2 A^{-1/2}\right) = \lambda_j \left(A\nabla_\mu B - A\sharp_\mu B\right)$$

for $j = 1, 2, \ldots$. Since

$$\lambda_j \left(A^{1/2} \left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}\right) = \lambda_j \left(A^{-1/2} (A-B)^2 A^{-1/2}\right)$$

for $j = 1, 2, \ldots$, we have

$$\lambda_j \left(A\nabla_\mu B - A\sharp_\mu B\right) = \lambda_j \left(A^{1/2} \left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}\right) \tag{39}$$

for $j = 1, 2, \ldots$. The equality (39), together with the inequality (15) and Lemma 5, implies that

$$A\nabla_\mu B - A\sharp_\mu B = A^{1/2} \left(A^{-1/2}BA^{-1/2} - I\right)^2 A^{1/2}$$

and so

$$(1-\mu) I + \mu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^\mu = K(\mu, \beta) \left(A^{-1/2}BA^{-1/2} - I\right)^2$$

which is equivalent to saying that

$$1 + \mu(t-1) - t^\mu = K(\mu, \beta) (t-1)^2$$

for all $t \in Sp(A^{-1/2}BA^{-1/2})$. It follows from the equality condition of Lemma 1 that $Sp(A^{-1/2}BA^{-1/2}) = \{1\}$. Since the operator $A^{-1/2}BA^{-1/2}$ is positive, it follows that $A^{-1/2}BA^{-1/2} = I$, and hence $A = B$.

The converse is trivial and the proof is complete. □

Based on the equality conditions of the inequalities in Lemma 1, and using an argument similar to that used in the proof of Theorem 5, we have the following result.

Theorem 6. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive such that $A \geqslant B > 0$ and $A - B$ is compact. If $0 < \mu < 1$, $\beta \geqslant 0$, and $-1 < \gamma \leqslant 0$, then*

(a)

$$\lambda_j \left(\left(A\nabla_\mu B\right) A^{-1} \left(A\nabla_\mu B\right) - A\sharp_{2\mu} B\right) = \left(K(2\mu, \beta) + \mu^2\right) \lambda_j \left(A^{-1/2} (A-B)^2 A^{-1/2}\right) \tag{40}$$

for $j = 1, 2, \ldots$ if and only if $A = B$.

(b)

$$\lambda_j\left((A\nabla_\mu B)\,A^{-1}\,(A\nabla_\mu B) - A\sharp_{2\mu}B\right) = \left(K(2\mu,\gamma)+\mu^2\right)\lambda_j\left(A^{-1/2}\,(A-B)^2\,A^{-1/2}\right) \quad (41)$$

for $j = 1, 2, \ldots$ *if and only if* $A = B$.

Using an argument similar to that used in the proof of Theorem 5, finally, we obtain equality conditions of the inequalities (27)–(30) and of the inequalities (32) and (33).

Theorem 7. *Let* $A, B \in \mathfrak{B}(\mathcal{H})$ *be positive such that* $A \geqslant B > 0$ *and* $A - B$ *is compact.*

(a) *If* $0 < \mu \leqslant \frac{1}{2}$, *then*

$$\lambda_j\left(A\nabla_\mu\left(BA^{-1}B\right) - A\sharp_{2\mu}B\right) = 2\mu\,(1-\mu)\,\lambda_j\left(A^{-1/2}\,(A-B)^2\,A^{-1/2}\right)$$

for $j = 1, 2, \ldots$ *if and only if* $A = B$

(b) *If* $\frac{1}{2} \leqslant \mu < 1$, *then*

$$\lambda_j\left(A\nabla_\mu\left(BA^{-1}B\right) - A\sharp_{2\mu}B\right) = 2\mu\,(1-\mu)\,\lambda_j\left(A^{-1/2}\,(A-B)^2\,A^{-1/2}\right)$$

for $j = 1, 2, \ldots$ *if and only if* $A = B$.

(c) *If* $0 < \mu \leqslant \frac{1}{2}$, *then*

$$\lambda_j\left(B\nabla_\mu\left(AB^{-1}A\right) - B\sharp_{2\mu}A\right) = 2\mu\,(1-\mu)\,\lambda_j\left(B^{-1/2}\,(A-B)^2\,B^{-1/2}\right)$$

for $j = 1, 2, \ldots$ *if and only if* $A = B$

(d) *If* $\frac{1}{2} \leqslant \mu < 1$, *then*

$$\lambda_j\left(B\nabla_\mu\left(AB^{-1}A\right) - B\sharp_{2\mu}A\right) = 2\mu\,(1-\mu)\,\lambda_j\left(B^{-1/2}\,(A-B)^2\,B^{-1/2}\right)$$

for $j = 1, 2, \ldots$ *if and only if* $A = B$.

Theorem 8. *Let* $A, B \in \mathfrak{B}(\mathcal{H})$ *be positive such that* $A \geqslant B > 0$ *and* $A - B$ *is compact. If* $0 < \mu < 1$, $\beta \geqslant 0$, *and* $-1 < \gamma \leqslant 0$, *then*

(a)

$$\lambda_j\left(\frac{A+B}{2} - H_\mu(A,B)\right) = K(\mu,\beta)\lambda_j\left(A^{-1/2}\,(A-B)^2\,A^{-1/2}\right)$$

for $j = 1, 2, \ldots$ *if and only if* $A = B$.

(b)

$$\lambda_j\left(\frac{A+B}{2} - H_\mu(A,B)\right) = K(\mu,\gamma)\lambda_j\left(B^{-1/2}\,(A-B)^2\,B^{-1/2}\right)$$

for $j = 1, 2, \ldots$ if and only if $A = B$.

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