Numerical Analysis of Singularly Perturbed Nonlinear Reaction-Diffusion Problems with Multiple Solutions

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Abstract—Semilinear reaction-diffusion two-point boundary value problems with multiple solutions are considered. Here the second-order derivative is multiplied by a small positive parameter and consequently these solutions can have boundary or interior layers. A survey is given of the results obtained in our recent investigations into the numerical solution of these problems on layer-adapted meshes. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Consider the singularly perturbed semilinear reaction-diffusion boundary-value problem

\[ F_\varepsilon u(x) \equiv -\varepsilon^2 u''(x) + b(x, u) = 0, \quad \text{for } x \in X := (0, 1), \]  
\[ u(0) = g_0, \quad u(1) = g_1, \]  

where \( \varepsilon \) is a small positive parameter, \( b \in C^\infty(X \times \mathbb{R}^1) \), and \( g_0 \) and \( g_1 \) are given constants. Related problems arise in the modelling of many biological processes [1, Section 14.7]. Under the hypotheses that we assume below, (1.1) may have multiple solutions that exhibit boundary- and/or interior-layer behaviour.

Asymptotic analyses of similar problems can be found in [2–6]. These analyses deal fully with the case of boundary layers, but are incomplete when interior layers are present, since it is then
It is quite difficult to construct tight upper and lower solutions to bound each solution of (1.1). Thus it is no surprise that no satisfactory numerical analysis of the interior-layer case has been carried out up to now. Numerical analyses of the boundary-layer case appear in [7], where a graded mesh is used to obtain almost first-order convergence in the discrete maximum norm, and [8], where a Shishkin mesh yields almost second-order convergence in the same norm.

The novelty of our approach lies in the introduction of a dynamical systems flavour into the analysis. While this has been used in the asymptotic literature (e.g., [6]), it has not until now filtered through into the numerical analysis of these problems. The dynamical systems framework enables us to simplify the complicated analysis of [8] and, more significantly, provides a mechanism by which one can rigorously prove convergence of a finite difference method in the case where the solution of (1.1) has interior layers. No result of this type was previously known.

This paper will describe the nature of solutions to (1.1), then outline the convergence results for numerical methods given in [9], where boundary layers were considered, and in [10], where the more difficult case of interior layers is dealt with.

2. HYPOTHESES

The reduced problem associated with (1.1) is defined by formally setting \( \varepsilon = 0 \) in (1.1a),

\[ b(x, \varphi) = 0, \quad \text{for } x \in X. \tag{2.1} \]

In the literature it is often assumed that \( b_u(x, u) > \gamma^2 > 0 \) for all \( (x, u) \in X \times \mathbb{R} \) and some positive constant \( \gamma \), which implies that the reduced problem has a unique solution \( u_0 \in C^\infty(X) \).

Nevertheless the assumption is unnatural and restrictive since it is imposed even at points that are far from any solution of (1.1). Consequently weaker local hypotheses will be imposed that permit (2.1) to have multiple solutions.

We shall consider (1.1) under hypotheses that lead to solutions having only boundary layers; then we impose additional hypotheses that produce solutions with interior and boundary layers. First, like previous authors [2-4,7,8], assume that

(i) equation (1.1) has a stable reduced solution, i.e., there exists a \( C^\infty \) solution \( u_0 \) of (2.1) such that

\[ b_u(x, u_0) > \gamma^2 > 0, \quad \text{for all } x \in X; \tag{2.2a} \]

(ii) equation (1.1) has stable boundary layers, i.e., the stable reduced solution \( u_0 \) of (i) satisfies

\[ \int_0^v b(0, u_0(0) + s) \, ds > 0, \quad \text{for all } v \in (0, g_0 - u_0(0)]', \tag{2.2b} \]

and

\[ \int_0^v b(1, u_0(1) + s) \, ds > 0, \quad \text{for all } v \in (0, g_1 - u_0(1)]'; \tag{2.2c} \]

here the notation \( (0, a]' \) is defined to be \( (0, a] \) when \( a > 0 \) and \( [a, 0) \) when \( a < 0 \). Then the boundary-value problem (1.1) has a solution that, away from the boundary, is essentially the same as \( u_0(x) \); it has no interior layer but may exhibit boundary layers. The analysis of Sections 3 and 4 below deals with this case.

Now impose the additional hypotheses that the reduced problem (2.1) has three simple roots \( \varphi = \varphi_k \in C^\infty(X) \), for \( k = 1, 2, 3 \), such that

\[ \varphi_1(x) < \varphi_2(x) < \varphi_3(x), \quad \text{for } x \in X; \tag{2.3a} \]

\[ b(x, \varphi_k(x)) = 0, \quad \text{for } k = 1, 2, 3 \text{ and } x \in X; \tag{2.3b} \]

\[ b_u(x, \varphi_k(x)) > \gamma_k^2 > 0, \quad \text{for } k = 1, 3 \text{ and } x \in X; \quad b_u(x, \varphi_2) < 0, \quad \text{for } x \in X; \tag{2.3c} \]

\[ \int_{\varphi_1(T_0)}^{\varphi_3(T_0)} b(T_0, v) \, dv = 0 \quad \text{and} \quad \int_{\varphi_1(T_0)}^{\varphi_3(T_0)} b_x(T_0, v) \, dv < 0. \tag{2.4} \]

Essentially the same hypotheses appear in [5, Section 4.15.4] and [6, Section 2.3.2].
If we also impose a condition analogous to (2.2b) on $\varphi_1$ and a condition analogous to (2.2c) on $\varphi_3$, then it can be shown that (1.1) has a solution that, roughly speaking, is near $\varphi_1(x)$ for $x \in (0, T_0)$ and near $\varphi_3(x)$ for $x \in (T_0, 1)$; there is an interior layer located approximately at $x = T_0$ and in general there are boundary layers at $x = 0$ and $x = 1$. As boundary and interior layers are handled independently in our approach, in the case where (2.3) and (2.4) are taken as hypotheses we simplify the presentation without loss of generality by assuming also that

$$
\varphi_1(0) = g_0 \quad \text{and} \quad \varphi_3(1) = g_1, \quad (2.5)
$$

to exclude boundary layers. This case is dealt with in Sections 5 and 6.

**Example 2.1.** Consider the following equation of Howes [3, p. 60]:

$$
-e^2 u'' + u(u-1)(u-x-3/2) = 0, \quad \text{for } x \in X. \quad (2.6)
$$

Here $b(x, u) = u(u-1)(u-x-3/2)$ and the reduced problem $b(x, \varphi) = 0$ has three solutions: $\varphi_1(x) \equiv 0$, $\varphi_2(x) \equiv 1$, and $\varphi_3(x) = x + 3/2$. A calculation shows that $\varphi_1$ and $\varphi_3$, but not $\varphi_2$, are stable reduced solutions, and (2.4) is satisfied with $T_0 = 1/2$. With boundary conditions such as $u(0) = 3/2$ and $u(1) = 1/2$, multiple solutions can be computed using the method of Section 6 below; see Figure 1.

### 3. Boundary Layers: Asymptotic Analysis

Consider first (1.1) under hypotheses (2.2). As advertised earlier, the key feature of our approach is a dynamical systems interpretation of the problem. Thus, rewrite (1.1) as the system

$$\varepsilon u' = U, \quad \varepsilon U' = b(x, u).$$
We seek a solution \( u \) of (1.1) that, away from the endpoints \( x = 0 \) and \( x = 1 \), is close to the stable reduced solution \( u_0 \). To begin, take \( u_0 \) as the zero-order smooth component in an asymptotic expansion of \( u \).

Define the stretched variable \( \xi := x/\varepsilon \), where \( 0 < \xi < \infty \). A dot will be used to denote differentiation with respect to \( \xi \). The autonomous nonlinear system for the zero-order boundary-layer term \( v(x) \equiv \tilde{v}(\xi) \) associated with the endpoint \( x = 0 \) is

\[
\dot{\tilde{v}} = \tilde{V}, \quad \tilde{V} = \tilde{b}(\tilde{v}) := b(0, u_0(0) + \tilde{v}), \quad \text{for } 0 < \xi < \infty,
\]

with boundary conditions \( \tilde{v}(0) = g_0 - u_0(0), \tilde{v}(\infty) = \tilde{V}(\infty) = 0 \).

To see that this system has a solution, consider the associated \((\tilde{v}, \tilde{V})\) phase plane. A solution of the system (if it exists) is given by a trajectory that leaves the point \((\tilde{v}(0), \tilde{V}(0))\), where \( \tilde{V}(0) \) is unknown, and enters \((0,0)\) as \( \xi \to \infty \). Now \( b(x, u_0(x)) \equiv 0 \) implies that the system has a fixed point at \((0,0)\). One can easily check that the eigenvalues of the Jacobian matrix associated with the right-hand side of the system are \( \pm \sqrt{b_\xi} \). But \( b_\xi(x, u_0) > \gamma^2 > 0 \), so the eigenvalues are real and of opposite sign at \((0,0)\), i.e., \((0,0)\) is a saddle point. Thus four separatrices meet at \((0,0)\) and two of them enter this saddle point as \( \xi \to \infty \). A necessary and sufficient condition for one of these two separatrices to be a solution curve of the system is that the straight line \( \tilde{v}(0) = g_0 - u_0(0) \) intersects that separatrix.

The separatrices entering \((0,0)\) satisfy the boundary condition \( \tilde{v}(\infty) = \tilde{V}(\infty) = 0 \). Now \( \tilde{V} \, d\tilde{V} = \tilde{b}(\tilde{v}) \, d\tilde{v} \), so one can integrate and solve to obtain \( \tilde{V} = \pm \sqrt{2 \int_0^\xi b(s) \, ds} \). Here the appropriate sign of \( \tilde{V} = \dot{\tilde{v}} \) should be chosen; e.g., if \( \tilde{v}(0) > 0 \), then to get a trajectory that leaves \((\tilde{v}(0), \tilde{V}(0))\) and decays to \((0,0)\) as \( \xi \to \infty \), choose the negative root.

The significance of the stable boundary condition (2.2b) now emerges: it ensures that \( \tilde{V} \) is defined along this separatrix from \( \tilde{v}(0) \) to \( \tilde{v}(\infty) \). That is, the dynamical system does have a solution \((\tilde{v}, \tilde{V})\).

The behaviour of this solution is needed in Section 4. Set \( \gamma_0 = \sqrt{b_\xi(0, u_0(0))} \). One can show that for each \( \delta \in (0, \gamma_0) \), there exists a positive constant \( C_\delta \) such that \( |\tilde{v}^{(k)}(\xi)| \leq C_\delta e^{-(\gamma_0 - \delta)\xi} \) for \( 0 \leq \xi < \infty \) and \( k = 0, 1, \ldots, 4 \). The proof of this fact [9] uses the stable manifold theorem of dynamical systems for the cases \( k = 0, 1 \); then the other cases follow from the system that defines \( v \).

The higher-order terms in the standard asymptotic expansion of \( u \) are solutions of linear problems and are easily analysed. One can then show [2] that for sufficiently small \( \varepsilon \), there exists a solution \( u(x) \) of the original problem that is unique in a neighbourhood of the zero-order asymptotic expansion \( u_0(x) + v(x) \). Furthermore, from [2] we get

\[
|u(x) - [u_0(x) + v(x) + \varepsilon v_1(x)]| \leq C\varepsilon^2, \quad \text{for all } x \in [0,1], \tag{3.1}
\]

where \( v_1 \) is the next term in the asymptotic expansion of \( u \).

One can extend the above ideas to construct approximate lower and upper asymptotic expansions \( \alpha \) and \( \beta \) that satisfy

\[
\alpha(x) \leq u_0(x) + v(x) + \varepsilon v_1(x) \leq \beta(x), \quad \text{for } 0 \leq x \leq 1. \tag{3.2}
\]

These functions are vital for the analysis of the error in our numerical method in Section 4: they will serve as discrete sub- and super-solutions of our discrete problem.

4. BOUNDARY LAYERS: NUMERICAL ANALYSIS

Let the mesh \( 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1 \) be arbitrary. Set \( h_i = x_i - x_{i-1} \) and \( h_i = (h_i + h_{i+1})/2 \) for each \( i \). Use the three-point difference scheme

\[
P^N (u^N)_i := -\varepsilon^2 \delta^2 u_i^N + b(x_i, u_i^N) = 0, \quad \text{for } i = 1, \ldots, N - 1,
\]
$u_0^N = g_0, u_1^N = g_1$, where

$$
\delta^2 u_i^N = \frac{1}{h_i} \left( \frac{u_{i+1}^N - u_i^N}{h_{i+1}} - \frac{u_i^N - u_{i-1}^N}{h_i} \right)
$$

is the standard difference approximation of $u''(x_i)$.

An operator $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a $Z$-field if for all $i \neq j$ the mapping

$$
x_j \mapsto (H(x_0, x_1, \ldots, x_N))_i
$$

is a monotonically decreasing function when $x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ are fixed.

**Lemma 4.1.** (See [11].) Let $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be continuous and a $Z$-field. Let $r \in \mathbb{R}^{n+1}$ be given. Assume that there exist $\alpha^{n+1}, \beta^{n+1} \in \mathbb{R}^{n+1}$ such that $\alpha^{n+1} \leq \beta^{n+1}$ and $H\alpha^{n+1} \leq r \leq H\beta^{n+1}$. (The inequalities are understood to hold true component-wise.) Then the equation $Hy = r$ has a solution $y \in \mathbb{R}^{n+1}$ with $\alpha^{n+1} \leq y \leq \beta^{n+1}$.

**Remark 4.1.** Lemma 4.1 is needed to show that the discrete system has at least one solution $u^N$ and to determine the accuracy of this solution, since the weak hypotheses (2.2) do not guarantee that the difference scheme satisfies a discrete maximum principle.

Using (3.1), (3.2), and Lemma 4.1 with the discrete mesh functions $\alpha(x_i)$ and $\beta(x_i)$, an intricate calculation yields the following.

**Theorem 4.1.** Let $w(x) = v(x) + \varepsilon v_1(x)$. Let the mesh be such that

$$\left| -\varepsilon^2 \left[ \delta^2 w_i - w_i'' \right] \right| \leq \frac{p_1 \varepsilon^2}{2}, \quad \text{for } i = 1, \ldots, N-1. \tag{4.1}$$

Assume that $\varepsilon \leq N^{-1}$. Suppose also that $C_2(\varepsilon^2 + p_1^2) \leq p_1 \gamma^2 / 2$ where $C_2$ is a certain constant. Then the discrete system has a solution $u^N$, and for $N$ sufficiently large,

$$|u(x_i) - u^N_i| \leq C(p_1 + N^{-2}), \quad \text{for all } i.$$

Here and subsequently $C$ (sometimes subscripted) is a generic constant that is independent of $\varepsilon$ and the mesh.

This result is valid on a very general class of layer-resolving meshes; it is now applied to two specific families of meshes that can be shown [9] to satisfy (4.1).

On the Bakhvalov mesh, the mesh points $x_i$ are $x_i = x(t_i)$, with $t_i = i/N$, where the mesh-generating function $x(t) \in C[0, 1]$ is defined by

$$x(t) = \begin{cases} 
- \left(\frac{2}{\gamma}\right) \varepsilon \ln(1 - 4t), & \text{for } 0 \leq t \leq \theta, \\
\frac{1}{2} - \frac{d}{2} \left(1 - t\right), & \text{for } \theta < t \leq \frac{1}{2},
\end{cases}$$

with $\theta = 1/4 - C_3 \varepsilon$ for some constant $C_3$, and $x(t) = 1 - x(1-t)$ for $1/2 < t \leq 1$. Here $d = [1/2 + (2/\gamma)\varepsilon \ln(1 - 4\theta)]/(1/2 - \theta)$ is chosen so that $x(t)$ is continuous at $t = \theta$.

To construct a Shishkin mesh, let $N$ be an even integer. Set $\sigma = \min\{((2\varepsilon/\gamma) \ln N, 1/2\}$. Divide the intervals $[0, \sigma]$, $[\sigma, 1 - \sigma]$, and $[1 - \sigma, 1]$ into $N/4$, $N/2$, and $N/4$ equidistant subintervals, respectively.

**Theorem 4.2.** Assume that $\varepsilon \leq N^{-1}$. There exists a discrete solution $u^N$ on the Bakhvalov mesh such that for $N$ sufficiently large,

$$|u(x_i) - u^N_i| \leq CN^{-2}, \quad \text{for } i = 0, \ldots, N.$$

There exists a discrete solution $\tilde{u}^N$ on the Shishkin mesh such that for $N$ sufficiently large,

$$|u(x_i) - \tilde{u}^N_i| \leq CN^{-2} \ln^2 N, \quad \text{for } i = 0, \ldots, N.$$

**Proof.** In both cases use Theorem 4.1; take $p_1 = CN^{-2}$ for the Bakhvalov mesh and $p_1 = CN^{-2} \ln^2 N$ for the Shishkin mesh.

Numerical results in [9] illustrate the sharpness of these bounds. While [8] contains a result similar to Theorem 4.2 for the Shishkin mesh, the Z-field technique used here is much simpler than the topological degree theory invoked in [8].
5. INTERIOR LAYERS: ASYMPTOTIC ANALYSIS

Now assume that (2.3)-(2.5) hold true. As in (3.2) we seek to construct sub- and super-solutions of a solution $u$ of (1.1), but the presence of an S-shaped interior layer near $T_0$ greatly complicates the analysis. Thus only an outline of the construction is given here; for full details see [10].

Let us define the interior-layer transition point as the point $T \in (0, 1)$ such that

$$u(T) = \varphi_2(T_0).$$

Note that in general $T \neq T_0$. Instead, $T$ and the stretched variable $\xi$ satisfy

$$T = T_0 + \varepsilon T_1, \quad x = T + \varepsilon \xi,$$

where $T_1 = T_1(\varepsilon)$ will be defined later.

We shall modify an argument of Fife [12]. Consider equation (1.1a) on the separate intervals $(0, T)$ and $(T, 1)$ with (2.5) and the additional boundary condition (5.1). Clearly $u(x)$ is continuous on $[0, 1]$. If we can choose $T$ such that, in addition, $u(x)$ is differentiable at $x = T$, then this $u(x)$ is a solution to our original problem on $[0, 1]$.

On each of the intervals $(0, T)$ and $(T, 1)$ we have a boundary-value problem; the solutions of these problems have boundary layers at $x = T^-$ and $x = T^+$, respectively. Construct their first-order asymptotic expansions: $\varphi_1(x) + v^-_0(\xi) + \varepsilon v^-_1(\xi)$ for $x \in [0, T]$, $\varphi_3(x) + v^+_0(\xi) + \varepsilon v^+_1(\xi)$ for $x \in [T, 1]$, as described in Table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|l|}
\hline
$f(\xi)$ & Domain & Differential Equation & $f(0)$ \\
\hline
$v^-_0$ & $(-\infty, 0]$ & $-\left(v^-_0\right)_{\xi} + b\left(T_0, \varphi_1(T_0) + v^-_0\right) = 0$ & $\varphi_2(T_0) - \varphi_1(T_0)$ \\
$v^+_0$ & $[0, \infty)$ & $-\left(v^+_0\right)_{\xi} + b\left(T_0, \varphi_3(T_0) + v^+_0\right) = 0$ & $\varphi_2(T_0) - \varphi_3(T_0)$ \\
$v^-_1$ & $(-\infty, 0]$ & $-\left(v^-_1\right)_{\xi} + b_u(T_0, v^-_0)v^-_1 = -(\xi + T_1)F_{1,x}\left(T_0, v^-_0\right)$ & $-T_1\varphi_1(T_0)$ \\
v^+_1 & $[0, \infty)$ & $-\left(v^+_1\right)_{\xi} + b_u(T_0, v^+_0)v^+_1 = -(\xi + T_1)F_{3,x}\left(T_0, v^+_0\right)$ & $-T_1\varphi_3(T_0)$ \\
\hline
\end{tabular}
\end{table}

Here the auxiliary functions $F_k(x, t) := b(x, \varphi_k(x) + t)$ for $k = 1, 2, 3$, and we used a new function $V_0$ that is defined in columns 2 and 3 of Table 2.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$f(\xi)$ & $f$ for $\xi \leq 0$ & $f$ for $\xi \geq 0$ & $f(0)$ & $f''(0)$ \\
\hline
$V_0$ & $v^-_0 + \varphi_1(T_0)$ & $v^+_0 + \varphi_3(T_0)$ & $\varphi_2(T_0)$ & 0 \\
\hline
\end{tabular}
\end{table}

Column 4 in Table 2 follows from combining columns 2 and 3 with Table 1.

Note that these asymptotic expansions are slightly nonstandard insofar as the equations are expanded around the point $T_0$ instead of the more usual boundary point $T$. This choice enables us to use the integral properties (2.4) of the function $b(x, u)$ at $x = T_0$.

By a dynamical systems argument, we can prove the following.

**Lemma 5.1.** There exist solutions $v^-_0(\xi)$ and $v^+_0(\xi)$ to the boundary problems of lines 1 and 2 in Table 1. For $-\infty < \xi < \infty$ and $k = 0, 1, \ldots, 4$ we have

$$\left| \frac{d^k}{d\xi^k} v^-_0(\xi) \right| \leq C_\delta e^{-(\gamma_1 - \delta)\xi}, \quad \left| \frac{d^k}{d\xi^k} v^+_0(\xi) \right| \leq C_\delta e^{-(\gamma_3 - \delta)\xi},$$

where $\delta \in (0, \min(\gamma_1, \gamma_3))$ is arbitrary. Furthermore, the function $V_0$ lies in $C^\infty(-\infty, \infty)$, $V'_0(\xi) > 0$ for all $\xi$, and $V''_0(0) = 0$.

Imitating an argument of Fife [2, Section 2] then yields the following.
Lemma 5.2. There exist solutions \(v_1^- (\xi)\) and \(v_1^+ (\xi)\) to the boundary problems of lines 3 and 4 in Table 1. Furthermore, for \(-\infty < \xi < \infty\) and \(k = 0, 1, \ldots, 4\) we have

\[
\left| \frac{d^k}{d\xi^k} v_1^- (\xi) \right| \leq C_\delta e^{-(\gamma_1 - \delta)\xi}, \quad \left| \frac{d^k}{d\xi^k} v_1^+ (\xi) \right| \leq C_\delta e^{-(\gamma_2 - \delta)\xi},
\]

where \(\delta \in (0, \min\{\gamma_1, \gamma_2\})\) is arbitrary.

Set \(\Phi(T_1) := \epsilon u''(T^-) - \epsilon u''(T^+).\) Our goal is to choose \(T_1\) so that \(\Phi(T_1) = 0,\) as then \(u(x)\) is differentiable at \(x = T\) as desired.

A careful calculation based on the properties of the terms in our asymptotic expansion leads to the following.

Lemma 5.3. For each fixed \(\epsilon,\) there exists \(T_1 = T_1(\epsilon) = O(1)\) such that

\[
\Phi(T_1(\epsilon)) = 0.
\]

Remark 5.1. Fife [12] also breaks the original problem into two boundary-value problems posed on \([0, T]\) and \([T, 1]\), then uses an implicit function theorem to prove existence of \(T\) such that \(u''(T-) = u''(T^+),\) but does not quantify \(|T - T_0|\) as precisely as we do. Vasil’eva and Butuzov [5,6] develop a zero-order asymptotic expansion whose accuracy is proved only away from the interior layer but they do not indicate clearly how the transition point can be determined.

We have now shown that there exists \(T = T_0 + O(\epsilon)\) such that the combination of solutions of equation (1.1a) on \((0, T)\) and \((T, 1),\) with boundary conditions (2.5) and (5.1), yields an interior-layer solution \(u(x)\) of (1.1). Furthermore, this interior transition layer is a juxtaposition of two standard exponential boundary layers each of width \(O(\epsilon [\ln \epsilon]).\) These facts enable us to prove an analogue of (3.2), but a simple combination of sub- and super-solutions of boundary-layer type fails to produce sub- and super-solutions for the interior-layer problem; one must [10] introduce certain extra terms in a neighbourhood of \(T\) and these terms are significant in Section 6 when constructing discrete sub- and super-solutions.

6. INTERIOR LAYERS: NUMERICAL ANALYSIS

In Section 5, we saw that the interior layer is in theory located at the point \(T\) specified in (5.1). Unfortunately, when computing a numerical solution, the value of \(T\) is unknown a priori. Nevertheless it turns out that one can design a simple and successful numerical method by centering the fine mesh at the neighbouring point \(T_0\) (whose value we do know).

As in Section 4, discretize (1.1a) using the classical three-point finite-difference scheme. To resolve the interior transition layer, we shall use a Shishkin-type mesh [13]. For brevity we describe this only for Example 2.1, in which \(T_0 = 1/2.\)

Let \(N\) be an integer divisible by 4. Set \(\sigma = \min\{\alpha \epsilon \ln N, 1/4\},\) where \(\alpha > 2/\min\{\gamma_1, \gamma_3\}.\) Divide the intervals \([0, 1/2 - \sigma],\) \([1/2 - \sigma, 1/2 + \sigma],\) and \([1/2 + \sigma, 1]\) into \(N/4, N/2,\) and \(N/4\) equidistant subintervals, respectively. In practice one usually has \(\sigma \ll 1,\) so the mesh is fine on \([1/2 - \sigma, 1/2 + \sigma]\) and coarse otherwise.

Theorem 6.1. (See [10].) Assume that \(N^{-2} \ln^2 N \leq \epsilon \leq N^{-1}.\) There exists a discrete solution \(u^N\) on the Shishkin mesh such that for \(N\) sufficiently large,

\[
|u(x_i) - u_i^N| \leq CN^{-2} \ln^2 N, \quad \text{for } i = 0, \ldots, N.
\]

The proof of this result uses Lemma 4.1 as in Section 4, but the difficult part is the construction of discrete sub- and super-solutions based on the asymptotic expansion of Section 5. A weaker result holds true in the case \(\epsilon \leq N^{-2} \ln^2 N;\) see [10].
Table 3. Computational rates \( r \) in \( (N^{-1}\ln N)^r \) and maximum nodal errors.

<table>
<thead>
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<th>( N )</th>
<th>( \varepsilon = 2^{-4} )</th>
<th>( \varepsilon = 2^{-6} )</th>
<th>( \varepsilon = 2^{-8} )</th>
<th>( \varepsilon = 2^{-12} )</th>
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Numerical results for the interior-layer solution of Example 2.1 with boundary conditions \( u(0) = 0, u(1) = 5/2 \) are presented in Table 3. Newton’s method is used to solve the discrete nonlinear system of equations, with initial guess \( \varphi_1(x_i) \) for \( i < N/2 \), \( \varphi_3(x_i) \) for \( i > N/2 \), and \( (\varphi_1(x_i) + \varphi_3(x_i))/2 \) for \( i = N/2 \). We took \( \alpha = 5.5/\sqrt{2} \) (the value of \( \alpha \) is larger than the reader might expect in order to ensure that the fine portion of our \( T_0 \)-centered mesh comfortably encompasses the actual transition point \( T \)). A similar approach (with different boundary conditions) yielded the final solution in Figure 1.

The exact solution \( u(x) \) is unknown, so we assumed that \( u_i^N - u_i \approx C(N^{-1}\ln N)^r \) and proceeded as in [9, Section 4.2]. Computed approximate values of the rates of convergence \( r \) are presented in the upper part of Table 1, and the computed approximate values of the discrete maximum norm errors \( \|u_i^N - u_i\| \) are presented in the lower part.

REFERENCES